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Spaces of continuous functions defined on Mrówka spaces

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Abstract

We prove that for a maximal almost disjoint family $\mathcal{A}$ on ω, the space $C_p(\Psi(\mathcal{A}), 2^\omega)$ of continuous Cantor-valued functions with the pointwise convergence topology defined on the Mrówka space $\Psi(\mathcal{A})$ is not normal. Using CH we construct a maximal almost disjoint family $\mathcal{A}$ for which the space $C_p(\Psi(\mathcal{A}), 2)$ of continuous $\{0,1\}$-valued functions defined on $\Psi(\mathcal{A})$ is Lindelöf. These theorems improve some results due to Dow and Simon in [Spaces of continuous functions over a Ψ-space, Preprint]. We also prove that this space $C_p(\Psi(\mathcal{A}), 2) = X$ is a Michael space; that is, X^n is Lindelöf for every $n \in \mathbb{N}$ and neither X^ω nor $X \times \omega^\omega$ are normal. Moreover, we prove that for every uncountable almost disjoint family $\mathcal{A}$ on ω and every compactification $b\Psi(\mathcal{A})$ of $\Psi(\mathcal{A})$, the space $C_p(b\Psi(\mathcal{A}), 2^\omega)$ is not normal.
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0. Introduction

All spaces considered in this article will be Tychonoff. For spaces X and E, $C_p(X, E)$ denotes the space of all continuous functions defined on X and with values in E with the topology of pointwise convergence; that is, the topology of $C_p(X, E)$ is inherited from the Tychonoff product E^X. As usual, we write $C_p(X)$ instead of $C_p(X, \mathbb{R})$. We are going to use the symbol $L(X)$ for the *Lindelöf number* of space X (the minimum infinite cardinal number τ such that every open cover of X has a subcover of cardinality $\leqslant \tau$), and $e(X)$ is the *extent* of X (the supremum of the cardinalities of all the closed and discrete subspaces of X).

Some of the most interesting topics in spaces $C_p(X, E)$ are related with their normality, Lindelöf degree and extent, and the relation between them. Next, we give some fundamental results about the foregoing.

0.1 (Reznichenko [17]). *If $e(C_p(X)) > \aleph_0$, then $C_p(X)$ is not normal.*

0.2 (Reznichenko [17]). *$C_p(X)$ is normal if and only if $C_p(X)$ is collectionwise normal.*

As every $C_p(X)$ has cellularity $\leqslant \aleph_0$ and every paracompact space with cellularity $\leqslant \aleph_0$ is Lindelöf, we have:

0.3. *A space $C_p(X)$ is paracompact iff $C_p(X)$ is Lindelöf.*

0.4 (Tkachuk [18]). *If $C_p(X)$ is normal, then $C_p(X)$ is countably paracompact.*

0.5 (Tkachuk [18]). *The space $C_p(X)$ is hereditarily normal iff $C_p(X)$ is perfectly normal.*

0.6 (Baturov [2]). *Let X be a Lindelöf Σ-space. Then for every subspace Y of $C_p(X)$, the extent $e(Y)$ of Y is equal to the Lindelöf number $L(Y)$ of Y.*

As a corollary of 0.1 and 0.6, we obtain that if X is a Lindelöf Σ-space, normality, countable extent and Lindelöf property coincide in $C_p(X)$. However, if X is the one-point Lindelöfication $L(\omega_1) = \omega_1 \cup \{*\}$ of the discrete space of cardinality ω_1, then $C_p(X)$ is normal (then $e(C_p(X)) = \aleph_0$), but it is not Lindelöf. It is of general interest to specify classes of spaces for which countable extent, normality and the Lindelöf property are well correlated.

Just, Sipacheva and Szeptycki proved in [9] that the space $X = L(\omega_1) \times (\omega + 1) \setminus \{(*, \omega)\}$ has countable extent and $C_p(X)$ is not normal. This space X is monolithic and of character ω_1. They also construct, using the combinatorial principle $\diamondsuit$, a separable and first-countable space Y such that $C_p(Y)$ is not normal and has countable extent. This space Y is a Mrówka space $\Psi(\mathcal{A})$ where $\mathcal{A}$ is an almost disjoint family built along an $(\omega_1–p)$-ultrafilter on ω.

Most of the known results about normality or the Lindelöf number in spaces $C_p(X)$ are of the following type: if $C_p(X)$ is normal or Lindelöf, then X must satisfy certain topological properties. So, a natural problem is to find some classes of spaces X for which

$C_p(X)$ is normal or Lindelöf. In this direction, we know that if X is an Eberlein compact space or if X contains a countable collection of subsets $\mathcal{N}$ such that every open subset of X is the union of a subcollection of $\mathcal{N}$ (in particular, if X is separable and metrizable), then $C_p(X)$ is Lindelöf.

Recently, Buzyakova [3] discovered that for every ordinal α, $C_p(X)$ is Lindelöf if $X = \alpha \setminus \{\beta < \alpha\colon \operatorname{cf}(\beta) > \omega\}$.

Motivated by [3], Dow and Simon [6] analyzed the spaces $C_p(\Psi(\mathcal{A}))$ where $\mathcal{A}$ is an almost disjoint family on ω and $\Psi(\mathcal{A})$ is the Mrówka space related to $\mathcal{A}$, and answered several questions posed in [3]. They proved:

(1) for every maximal almost disjoint family $\mathcal{A}$, $C_p(\Psi(\mathcal{A}))$ is not Lindelöf;
(2) assuming $\diamondsuit$, they constructed a mad family $\mathcal{A}$ such that $C_p(\Psi(\mathcal{A}), \{0,1\})$ is Lindelöf. This $\mathcal{A}$ has the characteristic that the Stone–Čech compactification of $\Psi(\mathcal{A})$ coincides with its one-point compactification;
(3) assuming $\mathfrak{b} > \omega_1$, $C_p(\Psi(\mathcal{A}), 2)$ is not Lindelöf for every mad family $\mathcal{A}$.

In this article, we also analyze Lindelöf property and normality in spaces of continuous functions over a Mrówka space. We prove that if $\mathcal{A}$ is a quasi-maximal almost disjoint family (in particular, if $\mathcal{A}$ is a mad family), $C_p(\Psi(\mathcal{A}))$ is not normal (Section 3). Moreover, we construct in Section 4, using CH, a Mrówka mad family $\mathcal{A}$ such that, for $X = C_p(\Psi(\mathcal{A}), \{0,1\})$, X^n is Lindelöf and X^ω and $X \times \omega^\omega$ are not normal. We also construct from CH a Luzin gap $\mathcal{A}$ such that $C_p(\Psi(\mathcal{A}))$ has countable extent. In Section 2 we prove that for every compactification $b\Psi(\mathcal{A})$ of an uncountable almost disjoint family $\mathcal{A}$, $C_p(b\Psi(\mathcal{A}))$ is not normal. Section 1 is devoted to some basic definitions and basic results about normality of spaces $\Psi(\mathcal{A})$.

The concepts, terminology and notations used and not defined in this article can be found in [1,8,10].

1. Preliminaries

The set of all natural numbers is denoted by ω, $\mathbb{N}$ is the set of positive integers, and $\mathbb{R}$, $\mathbb{Q}$ and $\mathbb{P}$ (or ω^ω) are the spaces of real, rational and irrational numbers with the natural topology. By I we denote the unit closed interval $[0,1] \subset \mathbb{R}$.

We have already mentioned, in the Introduction, what the Lindelöf degree and the extent of a space X mean. Another topological cardinal invariant that we are going to deal with is the *cellularity* of a space X, which is denoted by $c(X)$. This is the supremum of the cardinalities of all collections of open and pairwise disjoint subsets of X.

Recall that a collection $\mathcal{A}$ of subsets of the natural numbers ω is an *almost disjoint family* if each A in $\mathcal{A}$ is infinite, and for two different elements $A, B \in \mathcal{A}$, $|A \cap B| < \aleph_0$. A *maximal almost disjoint family* (mad family) is a maximal element in the family of all the almost disjoint families with the containment order.

A topological space X is a *Mrówka space* (a *Mrówka–Isbell space* or a *Ψ-space*, see [7, Problem 5I]) if it has the form $\omega \cup \mathcal{A}$, where $\mathcal{A}$ is an almost disjoint family, and its topology is generated by the following base: each $\{n\}$ is open for every $n \in \omega$, and an open canonical neighborhood of $A \in \mathcal{A}$ is of the form $\{A\} \cup B$ where $B \subset \omega$ and $A \setminus B$ is finite. In this case, we denote X by $\Psi(\mathcal{A})$. This kind of spaces was introduced by Mrówka in [13]. For every almost disjoint family $\mathcal{A}$, $\Psi(\mathcal{A})$ is a 0-dimensional locally compact first countable space, $\mathcal{A}$ is a closed discrete subspace of $\Psi(\mathcal{A})$ and ω is dense. Moreover, $\Psi(\mathcal{A})$ is pseudocompact if and only if $\mathcal{A}$ is maximal. So, $\Psi(\mathcal{A})$ is not normal if $\mathcal{A}$ is an infinite mad family.

The following result is obvious.

1.1. Proposition. *Let $\mathcal{A}$ be an almost disjoint family on ω. Then, $\Psi(\mathcal{A})$ is collectionwise normal if and only if $|\mathcal{A}| \leqslant \aleph_0$.*

The normality of $\Psi(\mathcal{A})$ can be expressed in several ways:

1.2. Proposition. *For an almost disjoint family $\mathcal{A}$ the following statements are equivalent*:

(1) *$\Psi(\mathcal{A})$ is normal.*
(2) *Every function $\phi : \mathcal{A} \to \{0, 1\}$ has a full extension*; *that is, there exist a continuous function $\tilde{\phi} : \Psi(\mathcal{A}) \to \{0, 1\}$ which extends ϕ.*
(3) *For every $\mathcal{B} \subset \mathcal{A}$, there is a partitioner $C \subseteq \omega$ of $\mathcal{B}$*; *that is, $A \subset^* C$ for all $A \in \mathcal{B}$, and $|A \cap C| =^* \emptyset$ for all $A \in \mathcal{A} \setminus \mathcal{B}$.*

So, if $2^{\omega} < 2^{\omega_1}$, the space $\Psi(\mathcal{A})$ is not normal for every uncountable $\mathcal{A}$. Moreover, Martin Axiom plus $\neg$CH implies that there are spaces $\Psi(\mathcal{A})$ which are normal. Indeed, for each subset X of the Cantor set 2^{ω}, we take the collection $\mathcal{A}_X = \{A_f \colon f \in X\}$ where $A_f = \{f \restriction n \colon n \in \omega\}$. $\mathcal{A}_X$ is an almost disjoint family of subsets of the countable set $2^{<\omega} = \{f \restriction n \colon f \in 2^{\omega}, n \in \omega\}$, and $\Psi(\mathcal{A}_X)$ is normal if and only if X is a Q-set in 2^{ω}.

We will call an almost disjoint family $\mathcal{A}$ *Mrówka* if the one-point compactification $\alpha\Psi(\mathcal{A})$ of $\Psi(\mathcal{A})$ coincides with its Stone–Čech compactification $\beta\Psi(\mathcal{A})$. This kind of almost disjoint families are maximal and exist in ZFC (see [14]). An almost disjoint family $\mathcal{A}$ is Mrówka iff $\beta\Psi(\mathcal{A})$ is 0-dimensional and one of the sets $f^{-1}(0) \cap \mathcal{A}$, $f^{-1}(1) \cap \mathcal{A}$ is finite for each $f \in C(\Psi(\mathcal{A}), 2)$.

We are going to frequently use the following well-known facts.

1.3. Lemma.

(1) *If the extent of a normal space X is countable, then X is collectionwise normal.*
(2) *If X is a collectionwise Hausdorff space and $c(X) \leqslant \aleph_0$, then the extent of X is countable.*
(3) *If Z is dense in a Tychonoff product E^X and E is separable, then $c(Z) \leqslant \aleph_0$.*

2. $C_p(b\Psi(\mathcal{A}), 2^{\omega})$ is not normal for every compactification $b\Psi(\mathcal{A})$ of $\Psi(\mathcal{A})$

The following is a generalization of a result due to Corson [5].

2.1. Theorem. *Let $X = \prod\{X_\alpha\colon \alpha \in A\}$ be the product of separable metric spaces, $Y \subset X$, Y everywhere dense in X, and let the space Z be a continuous image of Y. If $Z \times Z$ is normal, then Z is collectionwise normal.*

As a consequence of Theorem 2.1, we have:

2.2. Corollary. *Let X be a 0-dimensional space. If $C_p(X, 2^\omega)$ is normal, then it is collectionwise normal.*

Proof. $C_p(X, 2^\omega)$ is a dense subset of the product of $|X|$ copies of the separable metric space 2^ω. We have that $C_p(X, 2^\omega) \cong C_p(X, 2)^\omega \cong C_p(X, 2)^\omega \times C_p(X, 2)^\omega \cong C_p(X, 2^\omega) \times C_p(X, 2^\omega)$. So, if $C_p(X, 2^\omega)$ is normal, then $C_p(X, 2^\omega) \times C_p(X, 2^\omega)$ is normal. Therefore, by Theorem 2.1, $C_p(X, 2^\omega)$ is collectionwise normal. □

A well-known problem which has not been solved asks if normality of $C_p(X, 2)$ (respectively, $C_p(X, \omega)$) implies that $C_p(X, 2)$ (respectively, $C_p(X, \omega)$) is collectionwise normal for every topological space X. In our context we can modify this question as follows:

2.3. Problems. Is it true that for every almost disjoint family $\mathcal{A}$, $C_p(\Psi(\mathcal{A}), 2)$ (respectively, $C_p(\Psi(\mathcal{A}), \omega)$) is normal implies that $C_p(\Psi(\mathcal{A}), 2)$ (respectively, $C_p(\Psi(\mathcal{A}), \omega)$) is collectionwise normal?

The following result was proved in [4, Theorem 3.2].

2.4. Proposition. *Let X be a 0-dimensional space. Then, the space $C_p(X, 2)$ is countably compact if and only if X is a P-space.*

2.5. Proposition. *If X is a 0-dimensional space which is not a P-space, and if $C_p(X, 2) \times \omega^\omega$ contains a closed, discrete subspace of cardinality $> \aleph_0$, then $C_p(X, 2^\omega)$ is not normal.*

Proof. $C_p(X, 2^\omega)$ is homeomorphic to $C_p(X, 2) \times C_p(X, 2)^\omega$. Since X is not a P-space, $C_p(X, 2)$ has a closed copy of ω (Proposition 2.4), then $C_p(X, 2)^\omega$ contains a closed copy of the irrationals ω^ω. Since $e(C_p(X, 2) \times \omega^\omega) > \aleph_0$, then the extent of $C_p(X, 2) \times C_p(X, 2)^\omega$ is also an uncountable cardinal number. But the cellularity of $C_p(X, 2^\omega)$ is countable, so $C_p(X, 2^\omega)$ cannot be collectionwise normal (Lemma 1.3(2)), and so $C_p(X, 2^\omega)$ is not normal (Corollary 2.2). □

The following result is a consequence of a theorem of R. Pol and D.P. Baturov. A proof can be found in [1, p. 166].

2.6. Theorem. *Let X be an uncountable separable scattered compactum whose ω_1th derived set is empty. Then $C_p(X, 2) \times \omega^\omega$ contains an uncountable closed discrete subspace.*

As a consequence of this result, we obtain the main result of this section.

2.7. Theorem. *Let $E \in \{I, \mathbb{R}, \mathbb{P}, 2^\omega\}$. For every uncountable almost disjoint family $\mathcal{A}$ and every compactification $b\Psi(\mathcal{A})$ of $\Psi(\mathcal{A})$, the space $C_p(b\Psi(\mathcal{A}), E)$ is not normal.*

Proof. It is sufficient to prove this theorem when $E = 2^\omega$. The function $f : b\Psi(\mathcal{A}) \to \alpha\Psi(\mathcal{A})$ defined by $f \restriction \Psi(\mathcal{A})$ is the identity function, and $f(x) = p$ for all $x \in b\Psi(\mathcal{A}) \setminus \Psi(\mathcal{A})$ where p is the point which compactifies $\Psi(\mathcal{A})$, is an onto closed continuous function. Let $f^\# : C_p(\alpha\Psi(\mathcal{A}), 2^\omega) \to C_p(b\Psi(\mathcal{A}), 2^\omega)$ defined by $f^\#(g) = g \circ f$. Then, $f^\#[C_p(\alpha\Psi(\mathcal{A}), 2^\omega)]$ is homeomorphic to $C_p(\alpha\Psi(\mathcal{A}), 2^\omega)$ and it is a closed subset of $C_p(b\Psi(\mathcal{A}), 2^\omega)$. But $\alpha\Psi(\mathcal{A})$ is a space that satisfies the conditions in Theorem 2.6; so, $C_p(\alpha\Psi(\mathcal{A}), 2^\omega)$ is not normal because of Proposition 2.5. Therefore, since $C_p(\alpha\Psi(\mathcal{A}), 2^\omega)$ can be consider as a closed subset of $C_p(b\Psi(\mathcal{A}), 2^\omega)$, this last one is not normal. $\square$

Observe that the previous result is true for E equal to $\mathbb{P}$ or 2^ω even if $b\Psi(\mathcal{A})$ is not 0-dimensional. On the other hand, Pol gave in [16], using CH, an example of an almost disjoint family $\mathcal{A}$ such that $C_p(\alpha\Psi(\mathcal{A}), 2)$ is Lindelöf.

For $k < \omega$, we will denote by $C_{p,k}(X, E)$ the space $C_p(C_{p,k-1}(X, E), E)$ where $C_{p,0}(X, E) = X$. For an uncountable almost disjoint family $\mathcal{A}$, the space $\Psi(\mathcal{A})$ is a closed subset of $C_{p,2n}(\Psi(\mathcal{A}), 2^\omega)$. If the space $C_{p,2n}(\Psi(\mathcal{A}), 2^\omega)$ were normal, it would be collectionwise normal (Corollary 2.2); then, $\Psi(\mathcal{A})$ would be collectionwise normal as well. But this would mean that $|\mathcal{A}| \leqslant \aleph_0$ (Proposition 1.1); a contradiction. Therefore, for $E \in \{I, \mathbb{R}, \mathbb{P}, 2^\omega\}$, $C_{p,2n}(\Psi(\mathcal{A}), E)$ is not normal for every $n \in \mathbb{N}$.

Moreover, it is known that if X and $C_p(X, I)$ are normal, then each closed discrete subset of X has to be countable. So, for an uncountable almost disjoint family $\mathcal{A}$ such that $\Psi(\mathcal{A})$ is normal, $C_{p,n}(\Psi(\mathcal{A}), E)$ is not normal for every $n \in \mathbb{N}$, where $E \in \{I, \mathbb{R}\}$. This is the case for a canonical almost disjoint family $\Psi(\mathcal{A}_X)$ defined by a Q-set X.

3. $C_p(\Psi(\mathcal{A}))$ is not normal when $\mathcal{A}$ is a mad family

From now on we are going to use the following standard notations. For spaces X and E, $n \in \mathbb{N}$, points $x_1, x_2, \ldots, x_n$ of X and subsets $A_1, \ldots, A_n$ of E, the symbol $[x_1, \ldots, x_n; A_1, \ldots, A_n]$ will represent the set $\{f \in E^X\colon\ f(x_i) \in A_i\ \forall i \in \{1, \ldots, n\}\}$. If $A_i = A \subset E$ for all $i \in \{1, \ldots, n\}$, we will write $[x_1, \ldots, x_n; A]$ instead of $[x_1, \ldots, x_n; A, \ldots, A]$.

Let $\mathcal{A}$ be a mad family. For each $A \in \mathcal{A}$, we take the characteristic function of $\{A\} \cup A$ in $\Psi(\mathcal{A})$, $\tilde{\chi}_A : \Psi(\mathcal{A}) \to \{0, 1\}$ ($\tilde{\chi}_A(x) = 1$ iff $x = A$ or $x \in A$), and the characteristic function of A in ω, $\chi_A : \omega \to \{0, 1\}$ ($\chi_A(x) = 1$ iff $x \in A$). Now, we consider the set $D = \{(\tilde{\chi}_A, \chi_A)\colon\ A \in \mathcal{A}\}$ as a subspace of the product $Z = C_p(\Psi(\mathcal{A}), 2) \times \mathcal{T}$, where $\mathcal{T}$ is equal to $\{f \in 2^\omega\colon\ |f^{-1}(1)| = \aleph_0\}$ and has the topology inherited by the Tychonoff product 2^ω.

3.1. Claim. *The set D is a closed and discrete subset of $Z = C_p(\Psi(\mathcal{A}), 2) \times \mathcal{T}$ of cardinality $|\mathcal{A}|$.*

Proof. For each $A \in \mathcal{A}$, $V = [A; \{1\}] \times \mathcal{T} = \{(f, g) \in Z\colon\ f(A) = 1\}$ is an open set containing $(\tilde{\chi}_A, \chi_A)$, and $V \cap D = \{(\tilde{\chi}_A, \chi_A)\}$. So, D is discrete.

Assume now that $(f, g) \in \mathrm{cl}_Y D$ where $Y = C_p(\Psi(\mathcal{A}), 2) \times 2^\omega$. If for some $n \in \omega$, $f(n) \neq g(n)$, then $W = [n; \{f(n)\}] \times [n; \{g(n)\}]$ is an open subset of Y, $(f, g) \in W$ and $W \cap D = \emptyset$. This is not possible; hence, $f \restriction \omega = g$.

If $(f, g) \in \mathrm{cl}_Y D \setminus D$, then $f \restriction \mathcal{A} \equiv 0$. In fact, if $A, B \in \mathcal{A}$ with $A \neq B$ and $f(A) = 1 = f(B)$, then $[A, B; \{1\}] \times 2^\omega$ is an open subset of Y which contains (f, g) and which does not intersect D. Now, if f takes the value 1 only in one element of $\mathcal{A}$, say A, then, since $f \restriction \omega = g$ and $(f, g) \notin D$, either there is $n \notin A$ such that $f(n) = 1$ or there is $n \in A$ for which $f(n) = 0$. So, $W = [A, n; \{1\}] \times 2^\omega$ in the first case, or $W = [A, n; \{1\}, \{0\}]$ in the second case, is an open set in Y, $(f, g) \in W$ and $W \cap D = \emptyset$, which is not possible. We conclude that $f \restriction \mathcal{A} \equiv 0$. But this means (since $\mathcal{A}$ is a mad family) that $(f \restriction \omega)^{-1}(1)$ is finite. Therefore $(f, g) \notin Z$. $\square$

3.2. Claim. *The space $\mathcal{T}$ is homeomorphic to ω^ω.*

Proof. In fact, $\mathcal{T}$ is dense in 2^ω, its complement $2^\omega \setminus \mathcal{T}$ is equal to $F = \bigcup_{n<\omega} F_n$ where $F_n = \{f \in 2^\omega\colon\ |\{s < \omega\colon\ f(s) = 1\}| \leqslant n\}$. So, F is dense and F_σ in 2^ω. We conclude that $\mathcal{T}$ is homeomorphic to the irrational numbers (see [8, p. 370]). $\square$

So, the space $C_p(\Psi(\mathcal{A}), 2) \times \omega^\omega$ contains a closed and discrete subspace of cardinality $|\mathcal{A}|$. Since $\Psi(\mathcal{A})$ is not a P-space, $C_p(\Psi(\mathcal{A}), 2)$ has a closed copy of ω (Proposition 2.4). (The set $\{\chi_n\colon\ n < \omega\}$ where χ_n is the characteristic function of $\{0, \ldots, n\}$ in $\Psi(\mathcal{A})$, is a closed and discrete subspace of $C_p(\Psi(\mathcal{A}), 2)$.) Thus, $C_p(\Psi(\mathcal{A}), 2) \times \omega^\omega$ is a closed subspace of $C_p(\Psi(\mathcal{A}), 2^\omega)$, $C_p(\Psi(\mathcal{A}), I)$ and $C_p(\Psi(\mathcal{A}))$. So, we have $|\mathcal{A}| \leqslant e(C_p(\Psi(\mathcal{A}), 2^\omega)) \leqslant e(C_p(\Psi(\mathcal{A}), I)) \leqslant e(C_p(\Psi(\mathcal{A}))) \leqslant w(C_p(\Psi(\mathcal{A}))) = |\mathcal{A}| \leqslant L(C_p(\Psi(\mathcal{A}), 2^\omega)) \leqslant L(C_p(\Psi(\mathcal{A}), I)) \leqslant L(C_p(\Psi(\mathcal{A}))) \leqslant w(C_p(\Psi(\mathcal{A}))) = |\mathcal{A}|$, where $w(C_p(\Psi(\mathcal{A})))$ is the weight of space $C_p(\Psi(\mathcal{A}))$. That is:

3.3. Claim. *Let $\mathcal{A}$ be a mad family. Then, $e(C_p(\Psi(\mathcal{A}), 2^\omega)) = e(C_p(\Psi(\mathcal{A}), I)) = e(C_p(\Psi(\mathcal{A}))) = L(C_p(\Psi(\mathcal{A}), 2^\omega)) = L(C_p(\Psi(\mathcal{A}), I)) = L(C_p(\Psi(\mathcal{A}))) = |\mathcal{A}|$.*

Besides, if X is collectionwise normal and $c(X) \leqslant \aleph_0$, then the extent of X is countable. Therefore, we conclude:

3.4. Theorem. *Let $\mathcal{A}$ be an infinite maximal almost disjoint family on ω. Then, the spaces $C_p(\Psi(\mathcal{A}), 2^\omega)$, $C_p(\Psi(\mathcal{A}), \omega^\omega)$, $C_p(\Psi(\mathcal{A}), I)$, $C_p(\Psi(\mathcal{A}))$ are not normal, and their extent and Lindelöf number are all equal to $|\mathcal{A}|$.*

Proof. In fact, the cellularity of $C_p(\Psi(\mathcal{A}), 2^\omega)$ is equal to $\aleph_0$. If $C_p(\Psi(\mathcal{A}), 2^\omega)$ were normal, it would be collectionwise normal (Corollary 2.2), and, by Lemma 1.3, its extent must be countable, contrary to Claim 3.3. The last assertion of this theorem is Claim 3.3. $\square$

It is easy to prove from Theorem 3.4 that for every almost disjoint family $\mathcal{A}$ such that there is a mad family $\mathcal{B} \supset \mathcal{A}$ with $|\mathcal{B} \setminus \mathcal{A}| < \aleph_0$, the spaces $C_p(\Psi(\mathcal{A}), 2^\omega)$, $C_p(\Psi(\mathcal{A}), \omega^\omega)$,

$C_p(\Psi(\mathcal{A}), I)$ and $C_p(\Psi(\mathcal{A}))$ are not normal, and their extents coincide with their Lindelöf degrees and they are all equal to $|\mathcal{A}|$. In the case $\mathcal{A}$ has a countable infinite difference with a mad family, we cannot further use the same techniques, but they have the same properties as we are going to prove next. In order to obtain our purpose we are going to use general results. We decided to present Theorems 3.4 and 3.9 below and their proofs separately because for mad families we were able to give a more constructive proof, which shows the nature of space $C_p(\Psi(\mathcal{A}))$ more clearly.

Given a topological space X and a subspace Y of X, we denote by $\chi(Y, X)$ the *character of Y in X*; that is, $\chi(Y, X) = \min\{|\mathcal{B}|\colon \mathcal{B}$ is a base of Y in $X\}$, where $\mathcal{B}$ is a base of Y in X means that each element in $\mathcal{B}$ is open in X, and for each open set A of X containing Y, there is $B \in \mathcal{B}$ such that $Y \subseteq B \subseteq A$.

3.5. Definition. An almost disjoint family $\mathcal{A}$ of subsets of ω is *quasi-maximal* if there is a maximal almost disjoint family $\mathcal{B}$ containing $\mathcal{A}$ and such that $|\mathcal{B} \setminus \mathcal{A}| \leqslant \aleph_0$.

Obviously, every maximal almost disjoint family is quasi-maximal and, since every almost disjoint family with cardinality $\aleph_0$ is not maximal, every quasi-maximal almost disjoint family has cardinality not equal to $\aleph_0$.

3.6. Proposition. *Let $\mathcal{A}$ be an almost disjoint family on ω. Then, $\chi(\mathcal{A}, \Psi(\mathcal{A})) = \aleph_0$ if and only if $\mathcal{A}$ is quasi-maximal.*

Proof. Assume that $\chi(\mathcal{A}, \Psi(\mathcal{A})) = \aleph_0$ and $|\mathcal{A}| \geqslant \aleph_0$. Let $\mathcal{M} = \{M_n\colon n \in \omega\} \subseteq \mathcal{P}(\omega)$ be a countable collection of subsets of ω which are closed in $\Psi(\mathcal{A})$ and such that $\mathcal{B} = \{\Psi(\mathcal{A}) \setminus M\colon M \in \mathcal{M}\}$ is a base of $\mathcal{A}$ in $\Psi(\mathcal{A})$. Let $\mathcal{D} = \{M \in \mathcal{M}\colon |M| = \aleph_0\}$. Let $\{L_n\colon n \in \omega\}$ be an enumeration of $\mathcal{D}$ in such a way that if $\mathcal{D}$ is finite, then $L_0, \ldots, L_{n_0}$ are all different, $\mathcal{D} = \{L_0, \ldots, L_{n_0}\}$ and $L_n = L_{n_0}$ for all $n \geqslant n_0$, and if $\mathcal{D}$ is infinite, $L_n \neq L_m$ if $n \neq m$. Now we take $S_0 = L_0$, $S_1 = L_1 \setminus L_0, \ldots, S_{n+1} = L_{n+1} \setminus \bigcup_{i \leqslant n} L_i, \ldots,$ and $\mathcal{S} = \{S_n\colon n < \omega\}$. It happens that the new collection $\mathcal{A} \cup \{S \in \mathcal{S}\colon |S| = \aleph_0\}$ is a maximal almost disjoint family.

For the converse implication assume that $\mathcal{A}$ is an almost disjoint family and $\mathcal{B}$ is a mad family such that $\mathcal{A} \subset \mathcal{B}$ and $|\mathcal{B} \setminus \mathcal{A}| \leqslant \aleph_0$. Let $\mathcal{C} = \mathcal{B} \setminus \mathcal{A}$ and $\mathcal{H} = \{\Psi(\mathcal{A}) \setminus \bigcup \mathcal{K}\colon \mathcal{K} \subset [\omega]^{<\omega} \cup \mathcal{C}$ and $|\mathcal{K}| < \aleph_0\}$. Of course, $\mathcal{H}$ is countable. Without loss of generality, we can assume that the elements in $\mathcal{C}$ are pairwise disjoint. It is not difficult now to verify that $\mathcal{H}$ is a base for $\mathcal{A}$ in $\Psi(\mathcal{A})$. □

The following result is a generalization of Proposition IV.7.4 in [1] and its proof requires a slight modification to that given for it in [1].

3.7. Theorem. *Let X be a 0-dimensional space with an open, countable and dense subset M such that the set A of isolated points in $F = X \setminus M$ is not countable and is dense in F. If moreover $\chi(F, X) \leqslant \aleph_0$, then $C_p(X, 2) \times \omega^\omega$ contains a closed, discrete subspace of cardinality $|A|$.*

3.8. Theorem. *Let $\mathcal{A}$ be an infinite quasi-maximal almost disjoint family on ω. Then, the spaces $C_p(\Psi(\mathcal{A}), 2^\omega)$, $C_p(\Psi(\mathcal{A}), \omega^\omega)$, $C_p(\Psi(\mathcal{A}), I)$, $C_p(\Psi(\mathcal{A}))$ are not normal.*

Proof. Because of Proposition 3.6 and Theorem 3.7, $C_p(\Psi(\mathcal{A}), 2) \times \omega^\omega$ contains a closed and discrete subset of cardinality $|\mathcal{A}| > \aleph_0$. Now, we use Proposition 2.5 in order to conclude that $C_p(\Psi(\mathcal{A}), 2^\omega)$ is not normal. Since $C_p(\Psi(\mathcal{A}), 2^\omega)$ is a closed subset of $C_p(\Psi(\mathcal{A}), \omega^\omega)$, $C_p(\Psi(\mathcal{A}), I)$ and $C_p(\Psi(\mathcal{A}))$, they are also not normal. □

3.9. Theorem. *Let $\mathcal{A}$ be a quasi-maximal almost disjoint family on ω. Then, the extent of spaces $C_p(\Psi(\mathcal{A}), 2^\omega)$, $C_p(\Psi(\mathcal{A}), \omega^\omega)$, $C_p(\Psi(\mathcal{A}), I)$, $C_p(\Psi(\mathcal{A}))$ coincide with their Lindelöf degree and they are all equal to $|\mathcal{A}|$.*

Proof. This is a consequence of Theorem 3.8 and some similar arguments to those given before Claim 3.3. □

Proposition 0.3 and Theorem 3.8 induce us to ask if there is a maximal almost disjoint family $\mathcal{A}$ for which $C_p(\Psi(\mathcal{A}), 2^\omega)$ is countably paracompact. Following some argumentations in [19] it is possible to prove that $V = L$ implies that every countably paracompact space of character $\leqslant 2^{\aleph_0}$ is collectionwise Hausdorff. So, since $\chi(C_p(\Psi(\mathcal{A}), 2^\omega)) \leqslant 2^{\aleph_0}$ and $c(C_p(\Psi(\mathcal{A}), 2^\omega)) \leqslant \aleph_0$, we obtain the following result (see Lemma 1.3(2) and Theorem 3.9).

3.10. Theorem ($V = L$). *For every quasi-maximal almost disjoint family $\mathcal{A}$, the space $C_p(\Psi(\mathcal{A}), 2^\omega)$ is not countably paracompact.*

3.11. Problem. Can Theorem 3.10 be proved in ZFC without any additional set theoretical axiom?

4. A Lindelöf $C_p(\Psi(\mathcal{A}), 2)$ from CH

In this section we present the construction of a maximal almost disjoint family $\mathcal{A} \subseteq [\omega]^\omega$ such that $C_p(\Psi(\mathcal{A}), 2)$ is Lindelöf. We assume CH.

For an almost disjoint family $\mathcal{A}$ and $i \in \{0, 1\}$, we denote by $\sigma_n^i(\mathcal{A})$ the closed subspace $\{f \in C_p(\Psi(\mathcal{A}), 2)\colon |f^{-1}(i) \cap \mathcal{A}| \leqslant n\}$ of $C_p(\Psi(\mathcal{A}), 2)$. If $\mathcal{A}$ is Mrówka (that is, if the one-point compactification of $\Psi(\mathcal{A})$ coincides with its Stone–Čech compactification), then $C_p(\Psi(\mathcal{A}), 2) = \bigcup_{n\in\omega, i\in\{0,1\}} \sigma_n^i(\mathcal{A})$. For every $n < \omega$, $\sigma_n^0(\mathcal{A})$ is homeomorphic to $\sigma_n^1(\mathcal{A})$. We are going to write $\sigma_n(\mathcal{A})$ instead of $\sigma_n^1(\mathcal{A})$. Thus,

4.1. Theorem. *If $\mathcal{A}$ is a Mrówka mad family, then $C_p(\Psi(\mathcal{A}), 2)$ is Lindelöf if and only if $\sigma_n(\mathcal{A})$ is Lindelöf for each $n \in \omega$.*

To characterize when $\sigma_n(\mathcal{A})$ is Lindelöf, we need certain terminology and notation. For an almost disjoint family $\mathcal{A}$, $\mathcal{A}^\perp$ is the ideal $\{b \subset \omega\colon |b \cap a| < \aleph_0\ \forall a \in \mathcal{A}\}$; and for $a, b \in \mathcal{P}(\omega)$, $a\Delta b$ will denote their symmetric difference; that is $a\Delta b = (a \cup b) \setminus (a \cap b)$.

For a subset a of ω, we will distinguish between the characteristic function of a in 2^ω and the characteristic function of a in $2^{\Psi(\mathcal{A})}$ by denoting as χ_a the former and $\hat{\chi}_a$ the latter. Given an almost disjoint family $\mathcal{A}$ and $\mathcal{Y} \subseteq \mathcal{P}(\omega)$, we will say that $\mathcal{A}^n$ is *concentrated* on $\mathcal{Y}$, if for each open subset U of the Cantor set 2^ω containing $\chi_{\mathcal{Y}} = \{\chi_y\colon\ y \in \mathcal{Y}\}$, there is a countable $\mathcal{B} \subseteq \mathcal{A}$ such that $\chi_{\bigcup x} \in U$ for all $x \in [\mathcal{A} \setminus \mathcal{B}]^n$. And we will say that $\mathcal{A}^n + \mathcal{A}^\perp$ is *concentrated* on $\mathcal{Y}$, if for each open set $U \supseteq \mathcal{Y}$, there is a countable $\mathcal{B} \subseteq \mathcal{A}$ such that $\chi_{(\bigcup x)\Delta b} \in U$ for all $x \in [\mathcal{A} \setminus \mathcal{B}]^n$ and all $b \in \mathcal{A}^\perp$. We now state a theorem which characterizes when $\sigma_n(\mathcal{A})$ is Lindelöf, for an almost disjoint family $\mathcal{A}$.

4.2. Theorem. *Suppose that $\mathcal{A}$ is an almost disjoint family and $n > 0$. Then $\sigma_n(\mathcal{A})$ is Lindelöf if and only if $\mathcal{A}^k + \mathcal{A}^\perp$ is concentrated on $\mathcal{A}^\perp$ for each $k \leqslant n$.*

Before we prove this theorem, we note one corollary:

4.3. Corollary. *Suppose that $\mathcal{A} = \{a_\alpha\colon\ \alpha < \omega_1\}$ is mad. Then $\sigma_n(\mathcal{A})$ is Lindelöf if and only if $\mathcal{A}^k$ is concentrated on $[\omega]^{<\omega}$ for all $k \leqslant n$.*

Proof. Here $\mathcal{A}^\perp$ is precisely $[\omega]^{<\omega}$, so by the theorem it suffices to show that $\mathcal{A}^k$ is concentrated on $[\omega]^{<\omega}$ if and only if $\mathcal{A}^k + [\omega]^{<\omega}$ is concentrated on $[\omega]^{<\omega}$. One direction is trivial, for the other direction, assume that $\mathcal{A}^k$ is concentrated on $[\omega]^{<\omega}$. Fix an open neighborhood U of $\chi_{[\omega]^{<\omega}} = \{\chi_s\colon\ s \in [\omega]^{<\omega}\}$. For each $s \in [\omega]^{<\omega}$, let $U_s = \{f + \chi_s\colon\ f \in U\}$ be the translate of U by χ_s. We have that, $\chi_a \in U_s$ if and only if $\chi_{a\Delta s} \in U$. Each U_s is an open neighborhood of $\chi_{[\omega]^{<\omega}}$, and there are only countably many such translates. It follows that there is a countable subset $\mathcal{B}$ of $\mathcal{A}$ such that for all $x \in [\mathcal{A} \setminus \mathcal{B}]^k$ and all $s \in [\omega]^{<\omega}$, $\chi_{\bigcup x} \in U_s$. That is, $\chi_{(\bigcup x)\Delta s} \in U$. $\square$

Proof of the theorem. By induction on n. Note first that $\sigma_0(\mathcal{A}) = \{\hat{\chi}_b\colon\ b \in \mathcal{A}^\perp\}$ is homeomorphic to the subset $\{\chi_b\colon\ b \in \mathcal{A}^\perp\}$ of 2^ω, so $\sigma_0(\mathcal{A})$ is Lindelöf. Suppose $n \geqslant 1$ and that for all $k \leqslant n$, $\mathcal{A}^k + \mathcal{A}^\perp$ is concentrated on $\mathcal{A}^\perp$. By induction assume that $\sigma_{n-1}(\mathcal{A})$ is Lindelöf. Fix a cover $\mathcal{U}$ of $\sigma_n(\mathcal{A})$ constituted by canonical open subsets of $C_p(\Psi(\mathcal{A}), 2)$. By the inductive hypothesis, there is a countable $\mathcal{V} \subseteq \mathcal{U}$ such that $\sigma_{n-1}(\mathcal{A}) \subseteq \bigcup \mathcal{V}$. For each $x \in [\mathcal{A}]^n$, let $F_x = \{f \in \sigma_n(\mathcal{A})\colon\ f^{-1}(1) \cap \mathcal{A} = x\}$. Each F_x is homeomorphic to a subset of 2^ω; so it is covered by a countable subset $\mathcal{U}_x$ of $\mathcal{U}$. Thus it suffices to prove the following lemma:

4.4. Lemma. $D = \{x \in [\mathcal{A}]^n\colon\ F_x$ *is not covered by* $\mathcal{V}\}$ *is countable.*

Proof. If D is not countable, choose an uncountable set $\{x_\alpha\colon\ \alpha \in \omega_1\} \subseteq [\mathcal{A}]^n$ and $f_\alpha \in F_{x_\alpha}$ such that $f_\alpha \notin \bigcup \mathcal{V}$. By going to a subset we may assume that the x_α's form a Δ-system with root r. So, for each α, there is a member b_α of $\mathcal{A}^\perp$ such that $f_\alpha \restriction \omega$ is the characteristic function of $(\bigcup x_\alpha)\Delta b_\alpha$.

Consider F_r. It is covered by $\mathcal{V}$. Let

$$W = \bigcup \{V \cap 2^\omega\colon\ V \in \mathcal{V} \text{ and } V \cap F_r \neq \emptyset\}.$$

Let W_r be the translate of W by $\bigcup r$: $W_r = \{f + \chi_{\bigcup r}\colon f \in W\}$. That is, for $a \subset \omega$, $\chi_a \in W_r$ if and only if $\chi_{a \Delta (\bigcup r)} \in W$. First, note that W_r is a neighborhood of $\mathcal{A}^\perp$. To see this, fix $x \in \mathcal{A}^\perp$. Thus the characteristic function of $x \Delta (\bigcup r)$ extends to a continuous function $f \in F_r$. And since $\mathcal{V}$ covers F_r, there is a $V \in \mathcal{V}$ with $f \in V$. So, $\chi_{x \Delta (\bigcup r)} \in V \cap 2^\omega$. Therefore, $\chi_x \in W_r$ as required.

By changing the sets b_α on a finite set, we may assume that $f_\alpha \restriction \omega$ is the characteristic function of $\bigcup r \Delta (\bigcup (x_\alpha \setminus r) \cup b_\alpha)$. By our assumption of concentration, we may fix β so that $(\bigcup x_\alpha \setminus r) \cup b_\alpha \in W_r$ for all $\alpha > \beta$. Thus $f_\alpha \restriction \omega \in W$ for all $\alpha > \beta$. If we choose $\alpha > \beta$ large enough so that the supports of all $V \in \mathcal{V}$ lie below α we get that f_α is covered by $\mathcal{V}$. Contradiction. This finishes the proof of the lemma; hence, we have demonstrated the necessity of 4.2. □

Now we give the proof of the sufficiency of Theorem 4.2. Suppose that $\mathcal{A}^k + \mathcal{A}^\perp$ is not concentrated on $\mathcal{A}^\perp$ for some $k \leqslant n$. So, we may fix an open $U \subseteq 2^\omega$, a disjoint family $\{y_\alpha\colon \alpha < \omega_1\} \subseteq [\mathcal{A}]^k$, and $b_\alpha \in \mathcal{A}^\perp$ such that

(1) $\chi_{\mathcal{A}^\perp} = \{\chi_b\colon b \in \mathcal{A}^\perp\} \subset U$, and
(2) $g_\alpha = \chi_{(\bigcup y_\alpha) \Delta b_\alpha} \notin U$ for each $\alpha < \omega_1$.

Each g_α extends naturally to a continuous $f_\alpha : \Psi(\mathcal{A}) \to 2$ such that $f_\alpha(a) = 1$ if and only if $a \in y_\alpha$. Since $\{y_\alpha\colon \alpha < \omega_1\}$ is a disjoint family, any complete accumulation point of the f_α's must be in $\sigma_0(\mathcal{A})$. Moreover, since U contains $\chi_{\mathcal{A}^\perp} = \sigma_0(\mathcal{A})$, there is a neighborhood V of $\sigma_0(\mathcal{A})$ such that $f \restriction \omega \in U$ for each $f \in V$. Thus, $f_\alpha \notin V$ for all $\alpha < \omega_1$. This means that $\{f_\alpha\colon \alpha < \omega_1\}$ has no complete accumulation point in $\sigma_n(\mathcal{A})$. □

4.5. Theorem. *Assume* CH. *There is a Mrówka maximal almost disjoint family $\mathcal{A}$ such that $C_p(\Psi(\mathcal{A}), 2)$ is Lindelöf.*

Proof. Let $\{U_\alpha\colon \omega \leqslant \alpha \in \omega_1\}$ enumerate all open sets in 2^ω that contain $[\omega]^{<\omega}$. For each β, let $\mathcal{U}_\beta$ be a family of canonical basic open sets in 2^ω such that $\bigcup \mathcal{U}_\beta = U_\beta$. Let $\{x_\alpha\colon \omega \leqslant \alpha < \omega_1\}$ enumerate all infinite co-infinite subsets of ω. We will construct $\{a_\alpha\colon \alpha < \omega_1\}$ recursively, so that it is a Mrówka mad family $\mathcal{A}$ satisfying $\mathcal{A}^n$ is concentrated on $[\omega]^{<\omega}$ for each n. To begin the construction, let $\{a_n\colon n \in \omega\}$ be any partition of ω into infinite sets.

Assume that $\{a_\beta\colon \beta < \alpha\}$ has been chosen so that:

(a) For each $\beta \in [\omega, \alpha)$ and for each $x \in [a_\gamma\colon \beta \leqslant \gamma < \alpha]^{<\omega}$, $\chi_{(\bigcup x) \Delta s} \in U_\beta$ for every $s \in [\omega]^{<\omega}$.

(b) $\{a_\beta\colon \beta < \alpha\}$ is an almost disjoint family.

(c) For each $\beta \in [\omega, \alpha)$, a_β has infinite intersection with x_β and with $\omega \setminus x_\beta$ (unless one of these sets is covered by a finite union of a_ξ's with $\xi < \beta$).

If x_α or $\omega \setminus x_\alpha$ is covered by a finite set from $\{a_\beta\colon \beta < \alpha\}$, we do nothing at stage α (or just choose a_α almost disjoint from previous a_β arbitrary). Otherwise, to construct a_α, enumerate as $(\mathcal{V}_n, y_n)$ all pairs $(\mathcal{U}'_\beta, y)$ where $\beta \in [\omega, \alpha)$, $y \in [a_\gamma\colon \beta \leqslant \gamma < \alpha]^{<\omega}$ and $\mathcal{U}'_\beta$ is a finite translate of $\mathcal{U}_\beta$ ($\mathcal{U}'_\beta = \{U + \chi_s\colon U \in \mathcal{U}_\beta\}$ for a $s \in [\omega]^{<\omega}$ where $U + \chi_s = \{f + \chi_s\colon f \in U\}$). Note that (a) can be equivalently formulated as for each such x, χ_x is in every finite translate of U_β. Thus, by (a), we have that $\chi_{s \Delta \bigcup y_n}$ is in

$\bigcup \mathcal{V}_n$ for every $s \in [\omega]^{<\omega}$. Also enumerate $\{a_\beta\colon \beta < \alpha\}$ as $\{b_n\colon n \in \omega\}$. We will construct a_α as the union of finite sets s_n by recursion on n as follows: having chosen s_m and integers k_m for $m < n$ so that $s_m \subseteq k_m$ and $s_m \cap k_i = s_i$ for each $i < m < n$, we consider the pair $(\mathcal{V}_n, y_n)$. Note that the characteristic function of $s_{n-1} \cup \bigcup y_n$ is of the form $\chi_{s \Delta \bigcup y_n}$ for a $s \in [\omega]^{<\omega}$. Thus by (a), we have that $\chi_{s_{n-1} \cup \bigcup y_n} \in \bigcup \mathcal{V}_n$. So, there is $V_n = [t_0, \ldots, t_k; \{\varepsilon_0\}, \ldots, \{\varepsilon_k\}] \in \mathcal{V}_n$ ($\varepsilon_i \in \{0, 1\}$) such that $\chi_{s_{n-1} \cup \bigcup y_n} \in V_n$. Take $k'_n > \max\{t_0, \ldots, t_k, k_{n-1}\}$. Now choose $j_0 \in x_\alpha$ and $j_1 \notin x_\alpha$ such that $j_i > k'_n$ and such that $j_i \notin \bigcup\{b_i\colon i \leqslant n\}$. Let $s_n = s_{n-1} \cup \{j_0, j_1\}$, and let $k_n > \max\{j_0, j_1\}$. This completes the recursive construction of a_α. Clearly, by construction, (b) and (c) are preserved. To see that (a) is preserved, suppose that $\beta \in [\omega, \alpha)$ and $x \in [a_\gamma\colon \beta \leqslant \gamma \leqslant \alpha]^{<\omega}$, and fix a finite set C. Consider the translate $\chi_{\bigcup x} + \chi_C$ of $\chi_{\bigcup x}$. If $a_\alpha \notin x$ then there is nothing to show. So, suppose that $a_\alpha \in x$. Then, $(\mathcal{U}_\beta + \chi_C, x \setminus \{a_\alpha\})$ is enumerated as $(\mathcal{V}_n, y_n)$ in the construction of a_α, where $\mathcal{U}_\beta + \chi_C = \{U + \chi_C\colon U \in \mathcal{U}_\beta\}$. Recall that $\chi_{s_{n-1} \cup \bigcup y_n}$ is an element of the basic open set $[t_0, \ldots, t_k; \{\varepsilon_0\}, \ldots, \{\varepsilon_k\}]$. By the construction we have that $\chi_{a_\alpha \cup \bigcup y_n}(t_i) = \chi_{s_{n-1} \cup \bigcup y_n}(t_i) = \varepsilon_i$. Thus $\chi_{\bigcup x} \in \bigcup \mathcal{V}_n$. Hence, by definition of $\mathcal{V}_n$, we have that $\chi_{\bigcup x} + \chi_C \in U_\beta$ as required.

This completes the construction of the almost disjoint family $\mathcal{A} = \{a_\alpha\colon \alpha \in \omega_1\}$. By (b) and (c) $\mathcal{A}$ is a Mrówka mad family. And by (a) $\mathcal{A}^k$ is concentrated on $[\omega]^{<\omega}$ for each k as required. □

4.6. Corollary. *For the mad family $\mathcal{A}$ constructed in Theorem* 4.5, *the space $C_p(\beta\Psi(\mathcal{A}), 2)$ is Lindelöf.*

Proof. It is sufficient to observe that the function $\phi_n : \sigma_n \to \{f \in C_p(\beta\Psi(\mathcal{A}), 2)\colon |f^{-1}(1)| \leqslant n\}$ defined by $\phi_n(f)$ equal to the continuous extension $\tilde{f}$ of f to $\beta\Psi(\mathcal{A})$, is a continuous function for all $n \in \mathbb{N}$. □

The space $C_p(\Psi(\mathcal{A}), 2)$ where $\mathcal{A}$ is the Mrówka mad family constructed in Theorem 4.5, provides us, in CH, with a nice example of a *Michael space* (see [11,12]). Indeed,

4.7. Theorem. *Let $\mathcal{A}$ be the Mrówka almost disjoint family constructed in Theorem* 4.5, *and let X be the space $C_p(\Psi(\mathcal{A}), 2)$. Then we have*:

(1) *X^n is Lindelöf for every $n \in \mathbb{N}$ and X^ω is not normal.*
(2) *$X \times \omega^\omega$ is not normal.*

Proof. By Claim 3.1 and Theorem 3.4, $X \times \omega^\omega = C_p(\Psi(\mathcal{A}), 2) \times \omega^\omega$ and $X^\omega \cong C_p(\Psi(\mathcal{A}), 2^\omega)$ are not normal.

Furthermore, $C_p(\Psi(\mathcal{A}), 2)^k \cong C_p(\Psi(\mathcal{A}), 2^k)$, and

$$C_p\big(\Psi(\mathcal{A}), 2^k\big) = \bigcup_{n<\omega} \bigcup_{i \in \{0,1,\ldots,2^k-1\}} \sigma_n^i(\mathcal{A}).$$

But, each $\sigma_n^i(\mathcal{A})$ is Lindelöf (Theorem 4.5), so $C_p(\Psi(\mathcal{A}), 2)^k$ is Lindelöf. □

We could ask about the possibility of constructing an almost disjoint family $\mathcal{A}$ for which $C_p(\Psi(\mathcal{A}), 2)$ is σ-compact. But this is in vain; in fact, Paniagua proved in [15] that for every uncountable almost disjoint family $\mathcal{A}$, $C_p(\Psi(\mathcal{A}), 2)$ is not σ-compact.

A classical problem in C_p-theory questions whether Lindelöfness of $C_p(X)$ implies that $C_p(X) \times C_p(X)$ is Lindelöf. We do not know the answer even for a Mrówka space X yet.

4.8. Problem. Let $\mathcal{A}$ be an almost disjoint family, and assume that $C_p(\Psi(\mathcal{A}))$ is Lindelöf. Then, is $C_p(\Psi(\mathcal{A}))^2$ Lindelöf?

An almost disjoint family $\mathcal{A}$ is *separable*, if for each countable $\mathcal{B} \subseteq \mathcal{A}$, $\mathcal{B}$ can be separated from $\mathcal{A} \setminus \mathcal{B}$. That is, there is $X \subseteq \omega$ such that $A \subseteq^* X$ for each $A \in \mathcal{B}$ and $A \cap X =^* \emptyset$ for each $A \in \mathcal{A} \setminus \mathcal{B}$. An almost disjoint family $\mathcal{A}$ is a *Luzin gap* if no disjoint uncountable $\mathcal{B}, \mathcal{C} \subseteq \mathcal{A}$ can be separated in this way. If an almost disjoint family $\mathcal{A} = \{A_\alpha\colon \alpha < \omega_1\}$ has the property that for each α and for each n, $\{\beta < \alpha\colon A_\beta \cap A_\alpha \subseteq n\}$ is finite, then $\mathcal{A}$ is a Luzin gap. Any such $\mathcal{A}$ will be called a *standard* Luzin gap.

In the paper [9], a separable Luzin gap $\mathcal{A}$ such that $C_p(\Psi(\mathcal{A}))$ is not normal but has countable extent, was constructed using $\diamondsuit$. In the same paper the authors asked whether $C_p(\Psi(\mathcal{A}))$ has countable extent for every separable Luzin gap. Here we construct a standard Luzin gap using CH such that $\sigma_1(\mathcal{A})$ has uncountable extent. We do not know if it can be made separable.

Example. Assuming CH there is a standard Luzin gap $\mathcal{A}$ such that $\sigma_1(\mathcal{A})$ has uncountable extent. Moreover, it has the property that $\mathcal{A}$ is not concentrated on $\mathcal{A}^\perp$.

Proof. We first construct a perfect tree $T \subseteq 2^{<\omega}$ as follows. Let $X \subseteq \omega$ consist of all elements k_n of the form

$$k_n = \left(\sum_{i=0}^{n} 2^i\right) + n.$$

Suppose that $n \in \omega$ and $T \cap 2^{\leqslant k_n+1}$ has been defined so that $T \cap 2^{k_n+1}$ has exactly 2^{n+1} elements $\{s_j\colon j < 2^{n+1}\}$. For each $j < 2^{n+1}$, let t_j be the unique extension of s_j such that $\mathrm{dom}(t_j) = k_n + 1 + 2^{n+1}$, $t_j(k_n + j + 1) = 1$ and t_j has value 0 at all other new coordinates. Let $T \cap 2^{k_{n+1}} = \{t_j\colon j < 2^{n+1}\}$ and let

$$T \cap 2^{k_{n+1}+1} = \{t_j{}^\frown i\colon j < 2^{n+1},\ i \in 2\}.$$

This completes the recursive definition of T. If f is a maximal branch through T, we denote by $a_f = f^{-1}(1)$. let $[T]$ denote the set of all such a_f. Note that this is a perfect subset of 2^ω. Note also that T has the following key properties

(a) For any $a \in [T]$, $a \setminus X$ is infinite.
(b) For any subset $Y \subseteq X$, there is $a \in [T]$ such that $a \cap X = Y$.
(c) If a and b are distinct elements of $[T]$, then $a \cap b \cap (\omega \setminus X)$ is finite.

We now construct an almost disjoint family $\mathcal{A}$ by recursion. The point of the construction is (a) to make $\mathcal{A}$ a Luzin gap, (b) to make sure that the open set $2^\omega \setminus [T]$ contains all elements of $\mathcal{A}^\perp$, and finally, (c) to make sure that $[T] \cap \mathcal{A}$ is uncountable. If we do this, it will follow that $\mathcal{A}$ will not be concentrated on $\mathcal{A}^\perp$, thus completing the proof.

To do all this we fix an enumeration $\{y_\alpha\colon \alpha \in \omega_1\}$ of $[T]$. Having defined an almost disjoint family $\mathcal{A}_\alpha = \{a^i_\beta\colon \beta < \alpha,\ i \in 2\}$ for some $\alpha < \omega_1$, so that

(1) for each $\beta < \alpha$, if $y_\beta \in \mathcal{A}^\perp_\beta$ then $a^0_\beta \cap y_\beta$ is infinite. Moreover, in this case, $a^0_\beta = y_\beta$ or $a^0_\beta = y_\beta \cup z_\beta$ for some other $z_\beta \in [T]$;

(2) for $\beta < \alpha$, $a^1_\beta \in [T]$.

If $y_\alpha \in \mathcal{A}^\perp$, enumerate $\mathcal{A}_\alpha$ as $\{b_n\colon n \in \omega\}$. Construct by recursion $Y \subseteq X$ so that Y is almost disjoint from each b_n and so that $Y \cap b_n \not\subseteq n$. Let $a \in [T]$ be such that $a \cap X = Y$. This is possible by property (b) of T. The branch f of T that determines a is distinct from all the branches that determine the sets in $\mathcal{A}_\alpha$, thus, by property (c), a is almost disjoint from all elements of $\mathcal{A}_\alpha$. Let $a^0_\alpha = a \cup y_\alpha$. In the case that $y_\alpha \notin \mathcal{A}^\perp$, proceed as above, and let $a^0_\alpha = a$. To define a^1_α repeat the construction using $\mathcal{A}_\alpha \cup \{a^0_\alpha\}$ in place of $\mathcal{A}_\alpha$. This completes the construction of $\mathcal{A}$. It follows by construction that $\mathcal{A}$ is a standard Luzin gap. Also, by choice of the A^1_α, $\mathcal{A} \cap [T]$ is uncountable. Finally, it also follows by our choice of a^0_α, that no $y_\alpha \notin \mathcal{A}^\perp$ so that $\mathcal{A}^\perp \subseteq 2^\omega \setminus [T]$ as required. $\square$

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