Applied General Topology
© Universidad Politécnica de Valencia
Volume 10, No. 2, 2009
pp. 207-219

More on ultrafilters and topological games

R. A. González-Silva and M. Hrušák*

Abstract. Two different open-point games are studied here, the $\mathcal{G}$-game (of Bouziad [4]) and the $\mathcal{G}_p$-game (introduced in [11]), defined for each $p \in \omega^*$. We prove that for each $p \in \omega^*$, there exists a space in which none of the players of the $\mathcal{G}_p$-game has a winning strategy. Nevertheless a result of P. Nyikos, essentially shows that it is consistent, that there exists a countable space in which all these games are undetermined.
We construct a countably compact space in which player II of the $\mathcal{G}_p$-game is the winner, for every $p \in \omega^*$. With the same technique of construction we built a countably compact space X, such that in $X \times X$ player II of the $\mathcal{G}$-game is the winner. Our last result is to construct ω_1-many countably compact spaces, with player I of the $\mathcal{G}$-game as a winner in any countable product of them, but player II is the winner in the product of all of them in the $\mathcal{G}$-game.

2000 AMS Classification: Primary 54A20, 91A05: secondary 54D80, 54G20

Keywords: open-point game, ultrafilter, $\mathcal{G}$-space, $\mathcal{G}_p$-space, countably compact

1. Introduction and preliminaries

In [15] G. Gruenhage introduced a local game on topological spaces, so called *open-point* game (here denoted as the W-game). Given a topological space X and a point $x \in X$, the rules of the open-point game are as follows: Two players I and II play infinitely many innings, in the n-th inning player I choosing a neighborhood U_n of x and player II responding with a point $x_n \in U_n$. After ω-many innings we declare a winner, using the sequence $(x_n)_{n<\omega}$ of the moves

*The first listed author gratefully acknowledges support received from PROMEP grant no. 103.5/07/2636. The second author was supported partially by DGAPA grant no. IN108802 and partially by GAČR grant 201/00/1466.

of the second player. We say that player I wins the $W(x,X)$-game if the sequence $(x_n)_{n<\omega}$ converges to x, otherwise player II is declared a winner.

This game and its variations (see [4], [11] and [17]) have proved useful in studying local and convergence properties of topological spaces. These variants have the same rules and only differ from the W-game in the way a winner is declared. Following A. Bouziad [4], we say that player I wins the $\mathcal{G}(x,X)$-*game* if $\{x_n : n < \omega\}$ has an accumulation point in X, otherwise, player II is the winner.

Here we are mainly concerned with an ultrafilter version of the open-point game as introduced and studied in [11] and [12]. Recall the definition of the p-limit of a sequence (R. A. Bernstein [2]). Let p be a free filter on ω. A point x of a space X is said to be the p-limit of a sequence $(x_n)_{n<\omega}$ in X ($x = p\text{-}\lim x_n$) if for every neighborhood U of x, $\{n < \omega : x_n \in V\} \in p$.

Now, we are ready to define the $\mathcal{G}_p$-game, a parametrized version of the above mentioned $\mathcal{G}$-game. Let p be a free ultrafilter on ω. We say that player I wins the $\mathcal{G}_p(x,X)$-*game* if $p\text{-}\lim x_n$ exists (in X). Otherwise, player II wins.

In what follows we are mostly concerned with the question as to whether either player has a winning strategy in one of the above mentioned games. A strategy for one of the players is an algorithm that specifies each move of the player in every possible situation. More precisely, a *strategy for player* I in the open-point game is any sequence of functions $\sigma = \{\sigma_n : \mathcal{N}(x)^n \times X^n \to \mathcal{N}(x) : n < \omega\}$. A sequence $(x_n)_{n<\omega}$ in X is called a σ-*sequence* if $x_{n+1} \in \sigma_{n+1}(\langle x_0, ..., x_n\rangle; \langle V_0, ..., V_n\rangle) = V_{n+1}$, for each $n < \omega$. A strategy σ for player I is a *winning strategy* in the $\mathcal{G}(x,X)$-game (respect. $W(x,X)$-game, $\mathcal{G}_p(x,X)$-game), if each σ-sequence has an accumulation point in X (respect. $x_n \to x$, or there exist $y \in X$ such that $p\text{-}\lim x_n = y$). A space X is called a $\mathcal{G}$-*space* (respect. W-*space*, $\mathcal{G}_p$-*space*) if player I has a winning strategy in the $\mathcal{G}(x,X)$-game (resp. $W(x,X)$-game, $\mathcal{G}_p(x,X)$-game), for every $x \in X$.

Similarly, one defines a *strategy for player* II. It is a sequence of functions $\rho = \{\rho_n : X^n \times \mathcal{N}(x)^{n+1} \to X : n < \omega\}$, such that $\rho_n(\langle x_0, ..., x_{n-1}\rangle; \langle V_0, ..., V_n\rangle) \in V_n$, for each $n < \omega$. A sequence $\langle (V_n, x_n) : n < \omega\rangle$ where $V_n \in \mathcal{N}(x)$ and $x_n \in V_n$ is called a ρ-*sequence*, if $x_n = \rho_n(\langle x_0, ..., x_{n-1}\rangle; \langle V_0, ..., V_n\rangle) \in V_n$, for each $n < \omega$. A strategy ρ for player II is a *winning strategy* in the $\mathcal{G}(x,X)$-game (respect. $W(x,X)$-game, $\mathcal{G}_p(x,X)$-game), if for each ρ-sequence, $\langle (V_n, x_n) : n < \omega\rangle$, the set $\{x_n : n < \omega\}$ does not have cluster point in X (resp. $x_n \not\to x$, or the p-limit of the sequence $\{x_n\}$ does not exist).

We denote the fact that player I has a winning strategy in the $\mathcal{G}(x,X)$-game, by $I \uparrow \mathcal{G}(x,X)$. If he does not have a winning strategy we write $I \downarrow \mathcal{G}(x,X)$. When $I \uparrow \mathcal{G}(x,X)$ for every $x \in X$, this is denoted by $I \uparrow \mathcal{G}(X)$. The meaning of $II \uparrow \mathcal{G}(x,X)$, $II \downarrow \mathcal{G}(x,X)$ is defined analogously with the same notation used for the W-game or $\mathcal{G}_p$-game.

The following implications are easy consequences from definitions, $I \uparrow W(X) \Longrightarrow I \uparrow \mathcal{G}_p(X) \Longrightarrow I \uparrow \mathcal{G}(X)$. They can not be reversed in general, as shown for the spaces ω^*, $\beta(\omega) \setminus \{q \in \omega^* : q \leq_{RF} p\})$, but they are equivalent to first countability if X is a countable space (see Proposition 2.5). Dually, $II \uparrow W(X) \Longleftarrow II \uparrow \mathcal{G}_p(X) \Longleftarrow II \uparrow \mathcal{G}(X)$. These implications are also strict, the same examples work. In the next diagram, one can see relationships of these games with other concepts of general topology (for more details, see [13]).

$$
\begin{array}{ccccc}
 & & X \text{ is compact} & & \\
 & & \downarrow & & \\
X \text{ is first countable} & & X \text{ is } p\text{-compact} & \longrightarrow & X \text{ is countably compact} \\
\downarrow & & \downarrow & & \downarrow \\
I \uparrow W(X) & \longrightarrow & I \uparrow \mathcal{G}_p(X) & \longrightarrow & I \uparrow \mathcal{G}(X)
\end{array}
$$

Sharma proved in [23] that X is strongly Frechet $\Longleftrightarrow II \downarrow W(X)$, where a space X is called *strongly Fréchet* iff for every point $x \in X$, and every sequence $(A_n)_{n<\omega}$ of subsets of X with $x \in \overline{A_n}$ for each $n < \omega$, there exists a sequence $\{x_n\}$ such that $x_n \in A_n$ for every $n \in \omega$ and $x_n \to x$.

The notation used here is mostly standard. The Stone-Čech compactification $\beta\omega$ of the countable discrete space ω is identified with the set of all ultrafilters on ω and its remainder $\omega^* = \beta\omega \setminus \omega$ denotes the set of all free ultrafilters on ω. If $f : \omega \to X$ is a function into a compact space X, $\hat{f}$ denotes its (unique) extension to $\beta\omega$. Two ultrafilters are said to be of the same type (in $\beta\omega$) if there is a permutation f of ω such that $\hat{f}$ takes one to the other. The set of ultrafilters of the same type as a fixed ultrafilter p, is denoted by $T(p)$. For $p, q \in \omega^*$, $p \leq_{RK} q$ denotes that p is *Rudin-Keisler* bellow q and means that there is $f : \omega \to \omega$ such that $\hat{f}(q) = p$. The relation $p \leq_{RF} q$ is the *Rudin-Frolík* order and it means that there is an embedding $f : \omega \to \beta\omega$ such that $\hat{f}(p) = q$.

2. Indeterminacy of the games $\mathcal{G}_p$, W and $\mathcal{G}$

We say that a game is *determined* on a space X if for every point of X one of the players (not the same for all points) has a winning strategy, otherwise, the game is *undetermined*. For nice definable spaces the games are typically determined. However, they are not determined in general. In this section we are going to work with the indeterminacy of the games $\mathcal{G}_p$, $\mathcal{G}$ and W. For this, let us introduce the following notation.

Let Y be a set. A subset $\mathbb{T}$ of $Y^{<\omega}$ is a *tree* if whenever $t \in \mathbb{T}$ and $s \in Y^{<\omega}$ with $s \subseteq t$, then $s \in \mathbb{T}$. Let t be an element of the tree $\mathbb{T}$, the set of successors of t, $\{y \in Y : t^\frown y \in \mathbb{T}\}$ is denoted by $succ_{\mathbb{T}}(t)$. A function $f : \omega \to Y$, is said to be a *branch* of $\mathbb{T}$, if $f\restriction_n \in \mathbb{T}$ for every $n < \omega$. The set of branches of $\mathbb{T}$ is denoted by $[\mathbb{T}]$.

Next we will show that for every $p \in \omega^*$, there is a countably compact space such that no player of the $\mathcal{G}_p$-game has a winning strategy. To that end the following lemmas will be useful.

The following fact is a standard reformulation of the existence of a winning strategy for player II (see e.g [17]).

Lemma 2.1. *Suppose that X is a topological space, $x \in X$ and $p \in \omega^*$. Then the following are equivalent:*

(1) $II \uparrow \mathcal{G}_p(x, X)$.
(2) *II has a wining strategy ρ' in the $\mathcal{G}_p(x, X)$-game such that $x \notin rng(\rho')$*
(3) *There exists a tree $\mathbb{T}$ such that*
  i. *For every $t \in \mathbb{T}$, $x \in \overline{succ_{\mathbb{T}}(t) \setminus \{x\}}$.*
  ii. *For every $f \in [\mathbb{T}]$, p-$\lim f(n)$ does not exist in X.*

Proof. 1$\Longrightarrow$2. Let $\rho = \{\rho_n : n < \omega\}$ be a winning strategy for player II in the $\mathcal{G}_p(x, X)$-game. We say that a sequence $\langle V_0, y_0, V_1, y_1, ..., V_n, y_n\rangle$ is ρ-legal, if the $V_0, ..., V_n$ are neighborhoods of x, and for each $i \in \{0, ..., n\}$, we have $\rho_i(\langle y_0, ..., y_{i-1}\rangle, \langle V_0, ..., V_i\rangle) = y_i \in V_i$.

We will recursively define a winning strategy ρ' such that:

(a) $x \notin rng(\rho')$ and
(b) For every ρ'-legal sequence $\langle V_0, x_0, V_1, x_1, ..., V_n, x_n\rangle$, there is a unique ρ-legal sequence $\langle V_0, y_0, V_1, y_1, ..., V_n, y_n\rangle$ such that $y_i = x_i$ whenever $y_i \neq x$.

If $n = 0$, let $\rho'_0(V_0)$ be equal to $\rho_0(V_0)$ if $\rho_0(V_0) \neq x$ otherwise $\rho'_0(V_0)$ is any element of $V_0 \setminus \{x\}$.

For the inductive step, let $\langle V_0, x_0, V_1, x_1, ..., x_{n-1}, V_n\rangle$ be sequence of moves where the x_i are played according to the strategy ρ'. Consider $\langle V_0, x_0, V_1, x_1, ..., V_{n-1}, x_{n-1}\rangle$. By the inductive hypothesis there is a unique ρ-legal sequence $\langle V_0, y_0, V_1, y_1, ..., V_{n-1}, y_{n-1}\rangle$ such that $y_i = x_i$ whenever $y_i \neq x$. Define $\rho'_n(\langle x_0, ..., x_{n-1}\rangle; \langle V_0, ..., V_n\rangle)$ as follows: It is equal to $\rho_n(\langle y_0, ..., y_{n-1}\rangle; \langle V_0, ..., V_n\rangle)$ if $\rho_n(\langle y_0, ..., y_{n-1}\rangle; \langle V_0, ..., V_n\rangle) \neq x$, otherwise is any point of $V_n \setminus \{x\}$.

It is clear that (a) holds and that ρ' is a strategy.

Now lets see that (b) holds. Let $\langle V_0, x_0\rangle$ be a ρ'-legal, then we have two cases, x_0 is equal to $\rho_0(V_0)$ or not, in any case (b) holds. Now suppose that the statement (b) is true for any ρ'-legal sequence of length n and let $\langle V_0, x_0, V_1, x_1, ..., V_n, x_n\rangle$ be ρ'-legal sequence, so the subsequence $\langle V_0, x_0, V_1, x_1, ..., V_{n-1}, x_{n-1}\rangle$ holds (b), hence there is a unique ρ-legal sequence $\langle V_0, y_0, V_1, y_1, ..., V_{n-1}, y_{n-1}\rangle$ fulling (b), and $\rho_n(\langle y_0, ..., y_{n-1}\rangle; \langle V_0, ..., V_n\rangle) = y_n$, so $\langle V_0, y_0, V_1, y_1, ..., V_n, y_n\rangle$ is the unique ρ-legal sequence.

Finally lets see that ρ' is a winning strategy for player II, for this, let $(x_n)_{n<\omega}$ be a sequence of moves of player II according to strategy ρ'. Then there is exists a unique sequence $(y_n)_{n<\omega}$ which is constructed by segments of $(x_n)_{n<\omega}$; the difference between $(x_n)_{n<\omega}$ and $(y_n)_{n<\omega}$ are the points y_n which are x. Since the p-lim y_n is not in X then the p-lim x_n is not in X, so ρ' is a winning strategy.

2⟹3. Let $\mathbb{T}'' = \{l \in (\mathcal{N}(x) \times X)^{<\omega} : l\ is\ a\ \rho'\text{–}legal\ sequence\}$ and define $\mathbb{T}' = \{g_{\restriction n} : g \in [\mathbb{T}'']\ and\ g\ is\ infinite\}$. Note that each $f \in [\mathbb{T}']$ is a $\mathcal{G}_p$-play a cording to strategy ρ', hence $\mathbb{T}' \neq \varnothing$ and if $s^f = (x^f_n)_{n<\omega}$ is the subsequence generated by the points of f, then this sequence does not have a p-limit in X. Set $\mathbb{T} = \{s^f\restriction_n : f \in [\mathbb{T}']\}$, with $s^f\restriction_n \subseteq s^g\restriction_m$ if and only if $f\restriction_n \subseteq g\restriction_m$.

To see that i holds, pick $t \in \mathbb{T}$ and a neighborhood U of x. From the construction of $\mathbb{T}$, choose a branch $f \in [\mathbb{T}']$ such that $t = s^f\restriction_n$. Let $(V^f_n)_{n<\omega}$ be the subsequence generated by the neighborhoods of f. Then $\rho'_{|t|}(\langle t(0), t(1), ..., t(|t| - 1)\rangle; \langle V^f_0, V^f_1, ..., V^f_{|t|-1}, U\rangle) \in U$, hence $U \cap (succ_{\mathbb{T}}(t) \setminus \{x\}) \neq \varnothing$. Finally, if $g \in [\mathbb{T}]$, then $g = s^f$ for some $f \in [\mathbb{T}']$, so p-lim $g(n)$ does not exist in X, this fulfilling condition ii.

3⟹1. Take a tree $\mathbb{T}$ fulfilling clauses i and ii. For each $n \in \omega$, define $\rho_n : X^n \times \mathcal{N}(x)^{n+1} \to X$, such that

$$\rho_n(\langle x_0, ..., x_{n-1}\rangle; \langle V_0, ..., V_n\rangle) \in V_n \cap succ_{\mathbb{T}}(\langle x_0, ..., x_{n-1}\rangle).$$

Let $\rho = \{\rho_n : n < \omega\}$. It is straightforward to see that ρ is a winning strategy for player II in the $\mathcal{G}_p(x, X)$-game, as in any play the resulting sequence is a branch of the tree $\mathbb{T}$, and by ii, it does not have a p-limit in X. □

The next result, due to Z. Frolík, is used in the proof of Lemma 2.3 and also later on in the text.

Lemma 2.2 (Frolík). *If $f, g : \omega \to \omega^*$ are embeddings and $p \in \omega^*$. Then, $\hat{f}(p) = \hat{g}(p)$ if and only if $\{n < \omega : f(n) = g(n)\} \in p$.*

Lemma 2.3. *Let $p \in \omega^*$ and $\mathbb{T} \subseteq (\omega^*)^{<\omega}$ be a countable tree, such that*

1. *For each $t \in \mathbb{T}$, $|succ_{\mathbb{T}}(t)| \geq 2$.*
2. *For each $f \in [\mathbb{T}]$, f is an embedding in ω^*.*
3. *If $f, g \in [\mathbb{T}]$, $f \neq g$, then $|f \cap g| < \aleph_0$.*

Then, $\hat{f}(p) \neq \hat{g}(p)$ for any two elements $f, g \in [\mathbb{T}]$, and in particular the set $p[\mathbb{T}] = \{p\text{-lim} f(n) : f \in [\mathbb{T}]\}$ has cardinality $\mathfrak{c}$.

Proof. Follows from clauses 2 and 3, and Lemma 2.2. □

The idea to construct a space X in which the $\mathcal{G}_p$-game is undetermined (for $p \in \omega^*$ fixed), is to construct recursively a space $X \subset \omega^*$, diagonalizing across all the possible strategies for players I and II. There are two obvious obstacles to doing this. If we don't know X, then we can't say too much about the strategies. Another obstacle, is that there are going to be at least $2^{|X|}$ possible strategies. Fortunately Lemma 2.3 can be used to overcome both obstacles. The space X is going to be constructed in $\mathfrak{c}$-many steps, so the cardinality of $\{\mathbb{T} \subseteq X^{<\omega} : \mathbb{T}$ satisfies the conditions of Lemma 2.3 $\}$ is at most $\mathfrak{c}$.

Theorem 2.4. *For each $p \in \omega^*$, there exists a countably compact space X such that for every $x \in X$, $I \downarrow \mathcal{G}_p(x, X)$ and $II \downarrow \mathcal{G}_p(x, X)$.*

Proof. Fix a bijection $\Phi : \mathfrak{c} \to \mathfrak{c} \times \mathfrak{c}$ such that, for $\Phi(\alpha) = (\Phi_0(\alpha), \Phi_1(\alpha))$, we have $\Phi_0(\alpha), \Phi_1(\alpha) \leq \alpha$, for each $\alpha < \mathfrak{c}$. By recursion we are going to construct for each $\nu < \mathfrak{c}$, spaces X_ν, Y_ν and a sequence of trees $\{\mathbb{T}^\nu_\alpha : \alpha < \mathfrak{c}\}$, such that

(1) $X_0 \subset \omega^*$ is countable and dense in itself, and $Y_0 = \varnothing$.
(2) $X_\eta \subset X_\mu$ y $Y_\eta \subset Y_\mu$, for all $\eta < \mu < \nu$.
(3) $|X_\mu| \leq |\mu + \omega|$ y $|Y_\mu| \leq |\mu|$, for all $\mu < \nu$.
(4) $X_\mu \cap Y_\eta = \varnothing$, for all $\eta < \mu < \nu$.
(5) $\{\mathbb{T}^\nu_\alpha : \alpha < \mathfrak{c}\}$ is an enumeration of all trees in $X_\nu^{<\omega}$ satisfying the conditions of Lemma 2.3.
(6) If $\mu + 1 < \nu$, then $X_{\mu+1} \cap p[\mathbb{T}^{\Phi_0(\mu)}_{\Phi_1(\mu)}] \neq \varnothing$ and $Y_{\mu+1} \cap p[\mathbb{T}^{\Phi_0(\mu)}_{\Phi_1(\mu)}] \neq \varnothing$.

The construction of the space X_0 can be done using Theorem 1.4.7 of [18]. For a limit ordinal ν, define $X_\nu = \bigcup_{\mu<\nu} X_\mu$ and $Y_\nu = \bigcup_{\mu<\nu} Y_\mu$. When $\nu = \mu + 1$, define $X_\nu = X_\mu \cup \{p_\mu\}$ and $Y_\nu = Y_\mu \cup \{q_\mu\}$, where $p_\mu, q_\mu \in \omega^*$ have the property that $p_\mu \neq q_\mu$ and

$$p_\mu, q_\mu \in p[\mathbb{T}^{\Phi_0(\mu)}_{\Phi_1(\mu)}] \setminus (X_\mu \cup Y_\mu).$$

Let $X = \bigcup_{\nu<\mathfrak{c}} X_\nu$. Note that if $\mathbb{T} \subseteq X^{<\omega}$ is a tree which satisfies the condition of Lemma 2.3, then there exists a $\nu < \mathfrak{c}$ such that $\mathbb{T} \subseteq X_\nu^{<\omega}$, hence there exists $\alpha < \mathfrak{c}$ such that $\mathbb{T}^\nu_\alpha = \mathbb{T}$. And by the fact that Φ is onto, then there is $\gamma < \mathfrak{c}$ with $\mathbb{T}^{\Phi_0(\gamma)}_{\Phi_1(\gamma)} = \mathbb{T}$.

Let's see that X is countably compact. Take a countable subset Y of X, without loss of generality we can assume that it is discrete. It is easy to construct a tree $\mathbb{T}$ contained in $Y^{<\omega}$ with the properties of Lemma 2.3. Hence by the observation before, there exists $\gamma < \mathfrak{c}$ with $\mathbb{T}^{\Phi_0(\gamma)}_{\Phi_1(\gamma)} = \mathbb{T}$. So p_γ is cluster point of Y, which is in X.

Claim 1: For each $x \in X$, $I \downarrow \mathcal{G}_p(x, X)$.

Fix $x \in X$ and suppose that player I has a winning strategy $\sigma = \{\sigma_n : n < \omega\}$ at x in the $\mathcal{G}_p$-game. For each $s \in 2^{<\omega}$ inductively pick $x_s \in X$ and a clopen neighborhood W_n of x, such that

$$\begin{aligned}
&x_s \in \sigma_{|s|}(\langle x_{s|_0}, x_{s|_1}, ..., x_{s||s|-1}\rangle; \langle V_0, ..., V_{|s|-1}\rangle) = V_{|s|},\\
&x_s \notin \{x_r : r \in 2^{\leq |s|} \text{ and } r \neq s\},\\
&x_s \in W_n \text{ for all } s \in 2^n \text{ and}\\
&x_s \notin W_n \text{ for all } s \in 2^{<n}.
\end{aligned}$$

Define $t_s = \langle x_{s|_0}, x_{s|_1}, x_{s|_2}, ..., x_s\rangle$ and $\mathbb{T} = \{t_s : s \in 2^{<\omega}\}$. From our construction it follows that $\mathbb{T}$ is a tree which satisfies the premises of Lemma 2.3. Hence there exists $\gamma < \mathfrak{c}$ with $\mathbb{T}^{\Phi_0(\gamma)}_{\Phi_1(\gamma)} = \mathbb{T}$. And $f \in [\mathbb{T}]$ such that p-$\lim f(n) = q_\gamma$. Note that $rng(f)$ is a σ-sequence which does not have a p-limit in X. So the strategy σ is not winning.

Claim 2: For each $x \in X$, $II \downarrow \mathcal{G}_p(x, X)$.

Suppose that there exist $x \in X$ and a tree $\mathbb{T} \subseteq X^{<\omega}$ with the properties of Lemma 2.1. We are going to construct a countable subtree $\overline{\mathbb{T}}$ of $\mathbb{T}$ which is going to satisfy the conditions of Lemma 2.3. Fix $t \in \mathbb{T}$. For each $s \in 2^{<\omega}$ and $n > 0$, pick inductively points $x_{s^\frown 0} \neq x_{s^\frown 1}$ in X and a clopen neighborhood V_n of x with the followings properties:

$$\begin{aligned}
&x_{s^\frown 0}, x_{s^\frown 1} \in succ_T(t^\frown x_{s|_1}{}^\frown x_{s|_2}{}^\frown ...{}^\frown x_s),\\
&x_{s^\frown 0}, x_{s^\frown 1} \notin \{x_r : r \in 2^{\leq |s|+1} \setminus \{s^\frown 0, s^\frown 1\}\},\\
&x_{s^\frown 0}, x_{s^\frown 1} \in V_n, \text{ for each } s \in 2^{n-1} \text{ and } n-1 \geq 0,\\
&x_{s^\frown 0}, x_{s^\frown 1} \notin V_n, \text{ for each } s \in 2^{<n-1} \text{ and } n-1 > 0.
\end{aligned}$$

Let $t_s = t^\frown \langle x_{s|_1}, x_{s|_2}, ..., x_s \rangle$, and define $\overline{\mathbb{T}} = \{t_s : s \in 2^{<\omega}\}$. So $\overline{\mathbb{T}}$ is a subtree of $\mathbb{T}$ like Lemma 2.3. Hence there exists $\gamma < \mathfrak{c}$ with $\mathbb{T}^{\Phi_0(\gamma)}_{\Phi_1(\gamma)} = \overline{\mathbb{T}}$. However from this fact, there is a branch $f \in [\overline{\mathbb{T}}]$ with $p\text{-}\lim f(n) \in X$. □

The proof of the following fact is analogous to the proof given in [15, Theorem 3.3] for the W-game. We have already mentioned that, for a countable space X, the existence of a winning strategy for player I in the $\mathcal{G}$-game on X is equivalent to X being first countable.

Proposition 2.5. *In a Tychonoff countable space X, the following statements are equivalent for a fixed element x in X:*

(1) $\chi(x, X) = \aleph_0$.
(2) $I \uparrow \mathcal{G}(x, X)$.

Proof. $1 \Longrightarrow 2$. It is straightforward to define a winning strategy for player I using a countable local base.

$2 \Longrightarrow 1$. Suppose that $\chi(x, X) > \aleph_0$. Let σ be any strategy for player I. Enumerate the range of σ as $\{V_n : n < \omega\}$. As X is zero-dimensional, we can get for each $n < \omega$, a clopen subset U_n such that

(1) $U_{n+1} \subset U_n$, for every $n < \omega$.
(2) $\bigcap_{n<\omega} U_n = \{x\}$.
(3) $U_n \subset V_n$, for every $n < \omega$.

Since $\chi(x, X) > \aleph_0$, there exists a neighborhood V of x such that $|U_n \setminus V| = \aleph_0$ for each $n < \omega$. Take $x_n \in U_n \setminus V$ for each $n < \omega$. Then $x \notin \overline{\{x_n : n < \omega\}}$. Now, if $y \in X \setminus \{x\}$, then there exist $n < \omega$ with $y \notin U_n$, hence $X \setminus U_n \in \mathcal{N}(y)$, so $|(X \setminus U_n) \cap \{x_n : n < \omega\}| < \aleph_0$, i.e. $y \notin \overline{\{x_n : n < \omega\}}$. Hence the sequence $\{x_n : n < \omega\}$ does not have cluster points. It is easy to see that it contains a subsequence which is σ-sequence without cluster points. Therefore the strategy σ is not winning. □

Theorem 1.12 of [21] essentially says that it is consistent that there exist countable dense-in-themselves spaces on which our three games are undetermined. We will need the following version of this result.

Theorem 2.6 (P. Nyikos)**.** *Assume $\mathfrak{p} > \omega_1$. If D is a countable dense subset of 2^{ω_1}, then $I \downarrow \mathcal{G}(D)$ and $II \downarrow W(D)$.*

From this Theorem and the implications between the games W, $\mathcal{G}_p$ and $\mathcal{G}$, we have the next corollary.

Corollary 2.7 ($\mathfrak{p} > \omega_1$)**.** *There exists a topological countable group G such that the games W, $\mathcal{G}$ and $\mathcal{G}_p$ are undetermined in G.*

3. PLAYER II AND COUNTABLE COMPACTNESS

If X is countably compact, player I has a (trivial) winning strategy in the $\mathcal{G}$-game. This is no longer true for the $\mathcal{G}_p$-game. In fact, it is easy to construct (for a fixed $p \in \omega^*$) a countably compact space X such that $II \uparrow \mathcal{G}_p(X)$. Now, we will construct a countably compact space X such that $II \uparrow \mathcal{G}_p(X)$ for every $p \in X$ and then show that there is a countably compact space X such that $II \uparrow \mathcal{G}(X \times X)$, which is a strengthening of results of Novak and Terasaka's examples (see [24, Lemma 3.1]).

Recall the definition of the *relative type*, introduced by Z. Frolík. Let $Y \in [\omega^*]^\omega$ be discrete and $p \in Y^* = \overline{Y}^{\beta\omega} \setminus Y$. The relative type of p with respect to Y is $T(\hat{h}(p))$, where $h : Y \to \omega$ is an embedding. It is going to be denote by $T(p, Y)$. Now, for a subset S of $\beta\omega$ and $p \in \omega^*$, define $T[p, S] = \{T(p, Y) : Y \in [S]^\omega$ *and Y is homeomorphic to ω*$\}$. Frolík proved that $T[p, \omega^*]$ has cardinality $\mathfrak{c}$.

Theorem 3.1. *There exists a countably compact space X such that $II \uparrow \mathcal{G}_p(x, X)$ for every $p \in \omega^*$ and $x \in X$.*

Proof. The space X is going to be the union of $\{X_\nu : \nu < \omega_1\}$, where each X_ν is constructed recursively. Suppose that for each $\mu < \nu < \omega_1$ we have X_μ such that

(1) $X_0 \subseteq \omega^*$ is countable and dense in it self.
(2) X_0 is a dense subset of X_μ, for each $\mu < \nu$.
(3) $|X_\mu| \leq \mathfrak{c}$, for each $\mu < \nu$.
(4) $X_\eta \subset X_\mu$, if $\eta < \mu < \nu$.
(5) If $\mu + 1 < \nu$, then every countable discrete subset of X_μ has a cluster point in $X_{\mu+1}$.
(6) For each $x \in X_\mu \setminus X_0$, $\{y \in X_\mu : T[x, X_0] \cap T[y, X_0] \neq \varnothing\} = \{x\}$.

We can assume the existence of the space X_0, using Theorem 1.4.7 of [18]. Now, we show how to construct X_ν. When ν is a limit ordinal, define $X_\nu = \bigcup_{\mu<\nu} X_\mu$. If ν is a successor ordinal, say $\nu = \mu + 1$ then we have from clause 3, that the set of all embeddings from ω to X_μ has size $\mathfrak{c}$, let $\{f_\alpha : \alpha < \mathfrak{c}\}$ be an enumeration of this set. For each $\alpha < \mathfrak{c}$, pick a point $p_\alpha \in \overline{f_\alpha[\omega]}^{\beta(\omega)}$ such that for all $y \in X_\mu$, $T[p_\alpha, X_0] \cap T[y, X_0] = \varnothing$, and also $T[p_\alpha, X_0] \cap T[p_\beta, X_0] = \varnothing$, for all $\beta < \alpha$. Define $X_{\mu+1} = X_\mu \cup \{p_\alpha : \alpha < \mathfrak{c}\}$.

Notice that our space $X = \bigcup_{\nu<\omega_1} X_\nu$ is countably compact and also for each $p \in \omega^*$, $|\{y \in X \setminus X_0 : T(p) \in T[y, X_0]\}| \leq 1$. Therefore, $|\{y \in X : T(p) \in T[y, X_0]\}| \leq \omega$.

Fix $p \in \omega^*$ and $x \in X$. Let's see that $II \uparrow \mathcal{G}_p(x, X)$. It follows from the previous observation that the set $A = \{q \in X : T(p) \in T[q, X_0]\}$ is countable. Enumerate it as $\{q_i : i < \omega\}$. For each $i < \omega$ fix an embedding $f_i : \omega \to X_0$ such that $\hat{f}_i(p) = q_i$. The strategy of player II is to choose in the n-th step $g(n) \in X_0 \setminus \{f_0(n), f_1(n), ..., f_n(n)\}$ such that the function $g : \omega \to X_0$ defined in this way is an embedding. From Lemma 2.2, we have $\hat{g}(p) \notin A$. And hence $T(p) \in T[\hat{g}(p), X_0]$, then $\hat{g}(p) \notin X$. So this is a winning strategy for player II in the $\mathcal{G}_p(x, X)$-game. □

In the construction of the next example, we use a space which is countable, dense in itself and extremally disconnected. This space is defined for a fixed ultrafilter $p \in \omega^*$ and it is denoted by $Seq(p)$, its underlying set is $\omega^{<\omega}$, the set of all finite sequences in ω. A set $U \subset \omega^{<\omega}$ is open if and only if for every $t \in U$, $\{n < \omega : t^\frown n \in U\} \in p$ (see [7], [20], [5], [25]).

Lemma 3.2. *There exists a countable dense-in-itself space $X \subset \omega^*$ such that, for any $x \in X$ there exists a sequence $\{V_n : n < \omega\} \subset \mathcal{N}(x)$ with the following property: if $\{x_n : n < \omega\} \subset X$ and $x_n \in \bigcap_{m \leq n} V_m$ for each $n < \omega$ then $x \notin \overline{\{x_n : n < \omega\}}$.*

Proof. Let $p \in \omega^*$ be not a P-point and consider the space $Seq(p)$. Using Theorem 1.4.7 of [18], we can take an homeomorphic copy of $Seq(p)$ inside of ω^*. So, now it is sufficient to prove that $Seq(p)$ is the desired space.

Since p is not a P-point, there exists a sequence $\{U_n : n < \omega\} \subset p$ without pseudointersection in p. Take $x \in Seq(p)$ and define $V_n = \{t \in Seq(p) : x \subseteq t$ and $t(|x|) \in U_n\}$. If $(x_n)_{n<\omega}$ is a sequence such that $x_n \in \bigcap_{m \leq n} V_m$, then $x_n(|x|) \in U_m$, for every $n > m$. Hence $W = \{x_n(|x|) : n < \omega\} \notin p$. So $U = \omega \setminus W \in p$, this implies that $V = \{t \in X : x \subseteq t$ and $t(|x|) \in U\}$ is a neighborhood of x, disjoint from $\{x_n : n < \omega\}$. □

It is easy to see that the product of at most ω-many W-spaces ($\mathcal{G}_p$-spaces), is also a W-space ($\mathcal{G}_p$-space). However, this is not true for $\mathcal{G}$-spaces, as we will see in the next example. An application of the following example, is the existence of a countably compact space whose product is not countably compact.

Example 3.3. There exists a countably compact space X such that $II \uparrow \mathcal{G}(X \times X)$.

Proof. Let X be the space constructed in Theorem 3.1, with the condition that, the space X_0 is homeomorphic to $Seq(p)$ where the free ultrafilter p is not a

P-point. Let's see that $II \uparrow \mathcal{G}((x,y), X \times X)$, for a fixed point $(x,y) \in X \times X$. By Δ we denote the set $\{(x,x) : x \in X_0\}$.

Case (i): $(x,y) \in X \times X \setminus (X_0 \times X_0)$. Let $\{(x_n, y_n) : n < \omega\}$ be an enumeration of all the points in $X_0 \times X_0$. For each $n < \omega$, let $W_n \in \mathcal{N}((x_n,y_n)) \setminus \mathcal{N}((x,y))$ clopen such that $X_0 \times X_0 \setminus \bigcup_{m \le n} W_m$ is infinite for every $n < \omega$ and also $(x_m, y_m) \notin W_n$ for every $m < n$ (this is possible because the space X_0 is a subspace of ω^* dense in it self). Let V_0 be the first move of player I, player II responds with a point $(g(0), h(0)) \in V_0 \cap (X_0 \times X_0)$, and at the same time he chooses clopen sets $A_0 \in \mathcal{N}(g(0)) \setminus \mathcal{N}(x)$ and $B_0 \in \mathcal{N}(h(0)) \setminus \mathcal{N}(y)$, such that

$$(X_0 \times X_0) \setminus [(A_0 \times X_0) \cup (X_0 \times B_0) \cup \Delta)] \text{ is infinite.}$$

Inductively players I and II produce a sequence of points in $X_0 \times X_0$, $\{(g(n), h(n)) : n < \omega\}$, and sequences of clopen sets $\{A_n : n < \omega\}$ and $\{B_n : n < \omega\}$, such that, if the moves of player I are denoted by ${V_n}'s$ then:

(1) $(g(0), h(0)) \in V_0$,
(2) $(g(n), h(n)) \in V_n \cap (X_0 \times X_0 \setminus [\bigcup_{m \le n} W_m \cup \bigcup_{m<n} (A_m \times X_0) \cup (X_0 \times B_m) \cup \Delta])$, for all $n < \omega$,
(3) $A_n \in \mathcal{N}(g(n)) \setminus \mathcal{N}(x)$, for all $n < \omega$,
(4) $B_n \in \mathcal{N}(h(n)) \setminus \mathcal{N}(y)$, for all $n < \omega$, and
(5) $X_0 \times X_0 \setminus [\bigcup_{m \le n} W_m \cup \bigcup_{m \le n} (A_m \times X_0) \cup (X_0 \times B_m) \cup \Delta]$ is infinite, for all $n < \omega$.

From the construction of the space X, it is possible that player II play inductively in this way, choosing the ${A_n}'s$ and ${B_n}'s$, fulfilling the previous conditions.

So at the end the resulting sequence $S(g,h) = \{(g(n), h(n)) : n < \omega\}$ is discrete and also the functions in each coordinate, $g, h : \omega \to X_0$ are embeddings. Now, since $(x_n, y_n) \in W_n$ for each $n < \omega$, no element of $X_0 \times X_0$ is a cluster point of the sequence $S(g,h)$. Note that if $(a,b) \in X \times X \setminus X_0 \times X_0$ is a cluster point of $S(g,h)$, then $T[a, X_0] \cap T[b, X_0] \ne \varnothing$ but from the construction of X, this implies that $a = b$ while $S(g,h) \cap \Delta = \varnothing$, so $S(g,h)$ is closed and discrete.

Case (ii): $(x,y) \in X_0 \times X_0$. Let $\{(x_n, y_n) : n < \omega\}$ be an enumeration of all the points in $X_0 \times X_0 \setminus \{(x,y)\}$ and let $\{W_n : n < \omega\}$ a sequence of neighborhoods as in Case (i). Let $\{U_n^x : n < \omega\}$ and $\{U_n^y : n < \omega\}$ sequence of neighborhoods of x and y respectively, like in Lemma 3.2. If V_n is the n-th move of player I, then player II is going to play as before, but with the clause 2 strengthened as:

$$(g(n), h(n)) \in V_n \cap (\bigcap_{m \le n} U_m^x \times U_m^y) \cap (X_0 \times X_0 \setminus [\bigcup_{m \le n} W_m \cup \bigcup_{m<n} (A_m \times X_0) \cup (X_0 \times B_m) \cup \Delta]).$$

Then as before, player II gets a discrete set $S(g,h)$, with g, h embeddings. And in this case, the only possible cluster point is (x,y), but from the choice of the sequences g and h, and the properties of the points of X_0, it follows that (x,y) is not a cluster point of $S(g,h)$. □

Question 3.4. *Is there for each $n \geq 2$, a space X such that X^n is countably compact and $II \uparrow \mathcal{G}(X^{n+1})$?.*

The next example will show a family of countably compact spaces, such that the product of countably many of them is a $\mathcal{G}$-space but in the product of all them, player II of the $\mathcal{G}$-game, has a winning strategy. The example shows that the converse of Theorem 2.2 of the paper [12] is not true which establishes that, if player I has a winning strategy in the $\mathcal{G}$-game in the product of ω_1-many spaces, then all but countably many of them are countably compact.

To make this example, we generalize the space from [11, Theorem 2.3]. For $p \in \omega^*$, let $R(p) = \{\hat{f}(p) : f : \omega \to \omega \text{ is strictly increasing}\}$.

Let $\varnothing \neq M \subseteq \omega^*$. Put $M_0 = \omega$ and $M_1 = \bigcup_{p \in M} R(p)$. Let $\nu < \omega_1$. If ν is limit ordinal, then $M_\nu = \bigcup_{\mu<\nu} M_\mu$. If $\nu = \mu + 1$, then we define

$$M_\nu = \{\hat{f}(p) : f : \omega \to M_\mu \text{ is an embedding}, f|_{A_f} \text{ is strictly increasing and } p \in M\}$$

where $A_f = \{n < \omega : f(n) \in \omega\}$ for a function $f : \omega \to \beta\omega$. Then define $\Omega(M) = \bigcup_{\nu<\omega_1} M_\nu$. By using arguments similar to those used in the paper [11], we can prove that, for $\varnothing \neq M \subseteq \omega^*$, the space $\Omega(M)$ is a countably compact $\mathcal{G}_p$-space, for all $p \in M$.

Example 3.5. There is a family $\{X_\nu : \nu < \omega_1\}$ of countably compact spaces such that $I \uparrow \mathcal{G}(\prod_{\nu<\mu} X_\nu)$, for every $\mu < \omega_1$, but $II \uparrow \mathcal{G}(\prod_{\nu<\omega_1} X_\mu)$.

Proof. We start fixing a family $\{p_\nu : \nu < \omega_1\}$ of free ultrafilters on ω which are pairwise RK-incomparable (see [22]). For $\nu < \omega_1$, define $X_\nu = \Omega(\{p_\mu : \mu \geq \nu\})$. We know that X_ν is a countably compact space and $I \uparrow \mathcal{G}_{p_\nu}(X_\nu)$, for all $\nu < \omega_1$. In fact $I \uparrow \mathcal{G}_{p_\mu}(X_\nu)$, for all $\mu > \nu$. Then by Theorem 2.6 of [12], we obtain that $I \uparrow \mathcal{G}_{p_\mu}(\prod_{\nu<\mu} X_\nu)$, this shows the first part of the theorem. Notice that from the linearity of the RF-order and the properties of the ultrafilters p_ν's, it follows that $\bigcap_{\nu<\omega_1} X_\nu = \omega$. Now, fix $x \in \prod_{\nu<\omega_1} X_\nu$, we will show that $II \uparrow \mathcal{G}(x, \prod_{\nu<\omega_1} X_\nu)$. Indeed, assume that player I has chosen at the n-th step $V_n = \bigcap_{\alpha \in F_n} [\alpha, V_\alpha]$, where $F_n \in [\omega_1]^{<\omega}$, $V_\alpha \in \mathcal{N}(x(\alpha))$ for each $\alpha \in F_n$ and $[\alpha, V_\alpha] = \{f \in \prod_{\nu<\omega_1} X_\nu : f(\alpha) \in V_\alpha\}$. The strategy of player II is to choose at the n-th step $x_n \in \prod_{\nu<\omega_1} X_\nu$ such that

$$x_n(\alpha) = \begin{cases} x(\alpha) & \text{if} \quad \alpha \in F_n, \\ n & \text{if} \quad \alpha \in \omega_1 \setminus (\bigcup_{m \leq n} F_m). \end{cases}$$

From the fact that $\bigcap_{\nu<\omega_1} X_\nu = \omega$, we have that for $\beta = sup(\bigcup_{n<\omega} F_n)$ the set $\{x_n|_{[\beta,\omega_1)} = n : n < \omega\}$ does not have a cluster point. So $\{x_n : n < \omega\}$ is close and discrete and hence player II wins. $\square$

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Received December 2008

Accepted September 2009

R. A. González-Silva (rgonzalez@culagos.udg.mx)
Departamento de Ciencias Exactas y Tecnológicas (UdG) Enrique Díaz de León 1144, Col. Paseos de la Montaña, 47460, Lagos de Moreno Jalisco, México

M. Hrušák (mhrusak@matmor.unam.mx)
Instituto de Matemáticas (UNAM) A.P. 61-3 Xangari, 58089 Morelia, Michoacán, México