Open Problems in Topology II – Edited by E. Pearl

Completely separable MAD families

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An infinite family $\mathcal{A} \subseteq [\omega]^\omega$ is *almost disjoint* if any two of its distinct elements have finite intersection. A family $\mathcal{A}$ is *maximal almost disjoint (MAD)* if it is almost disjoint and for every $X \in [\omega]^\omega$ there is an $A \in \mathcal{A}$ such that $A \cap X$ is infinite.

There are almost disjoint (hence also MAD) families of cardinality $\mathfrak{c}$ and many MAD families with special combinatorial and/or topological properties can be constructed using set-theoretic assumptions like CH, MA or $\mathfrak{b} = \mathfrak{c}$. However, special MAD families are notoriously difficult to construct in ZFC alone. The reason being the lack of a device ensuring that a recursive construction of a MAD family would not prematurely terminate, an object that would serve a similar purpose as independent linked families do for the construction of special ultrafilters (see [**15**]). The notion of a completely separable MAD family is a candidate for such a device and, moreover, is an interesting notion in its own right.

MAD families provide a powerful tool for topological constructions. Not only for the study of $\beta\mathbb{N} \setminus \mathbb{N}$, the Čech–Stone remainder of the discrete countable space ([**1**, **2**]) but also, typically via the corresponding Ψ-space [**8**], the study of convergence properties of topological spaces [**6**, **21**].

1. The main problem

The notion of completely separable MAD family was introduced by S.H. Hechler [**9**] in 1971:

Definition. An infinite MAD family $\mathcal{A}$ on ω is *completely separable* if for every subset $M \subseteq \omega$ either there is an $A \in \mathcal{A}$ with $A \subseteq M$ or there is a finite subfamily $\mathcal{B} \subseteq \mathcal{A}$ with $M \subseteq \bigcup \mathcal{B}$.

A year later, P. Erdős and S. Shelah asked the central problem of this article:

Problem 1. *Does there exist a completely separable MAD family in* ZFC*?* 405?

A MAD family $\mathcal{A}$ on ω is of *true cardinality* $\mathfrak{c}$ if for every subset $M \subseteq \omega$ the set $\{A \in \mathcal{A} : |M \cap A| = \omega\}$ is either finite or of size $\mathfrak{c}$. It is easily seen that every completely separable MAD family is of true cardinality $\mathfrak{c}$. On the other hand, the existence of a MAD family of true cardinality $\mathfrak{c}$ readily implies the existence of a completely separable MAD family.

An almost disjoint family $\mathcal{A}$ is *nowhere MAD* if for every $X \subseteq \omega$ either $X \subseteq^* \bigcup \mathcal{B}$ for some finite $\mathcal{B} \subseteq \mathcal{A}$ or there is a $B \in [X]^\omega$ almost disjoint from all elements of $\mathcal{A}$. Given a cardinal number κ, a MAD family $\mathcal{A}$ is *κ-partitionable* if $\mathcal{A}$ can be

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partitioned into κ subfamilies $\{\mathcal{A}_\xi : \xi < \kappa\}$ such that $\mathcal{A} \setminus \mathcal{A}_\xi$ is nowhere MAD for every $\xi < \kappa$.

Note that a MAD family $\mathcal{A}$ is $\mathfrak{c}$-partitionable if and only if it is of true cardinality $\mathfrak{c}$. This motivated A. Dow to ask how close can we get to constructing a $\mathfrak{c}$-partitionable MAD family:

406? **Problem 2.** *For which κ is there a κ-partitionable MAD family?*

In [**21**] it is shown that 2-partitionable families exist in ZFC. This was later extended by E. van Douwen to show that ω-partitionable MAD familes exist. In fact, one can, in ZFC, construct a $\mathfrak{t}$-partitionable MAD family (personal communication by A. Dow), but it is not known whether there is a $\mathfrak{b}$-partitionable MAD family.

Problem 1 has a close connection to the disjoint refinement property. Given two familes $\mathcal{M}, \mathcal{A} \subseteq [\omega]^\omega$, we say that the family $\mathcal{A}$ *refines* the family $\mathcal{M}$, if for each $M \in \mathcal{M}$ there is an $A \in \mathcal{A}$ such that $A \subseteq M$. We say that a family $\mathcal{M} \subseteq [\omega]^\omega$ has an *almost disjoint refinement* if there is an almost disjoint family $\mathcal{A} \subseteq [\omega]^\omega$ which refines $\mathcal{M}$. Clearly, not every $\mathcal{M} \subseteq [\omega]^\omega$ has an almost disjoint refinement, $\mathcal{M} = [\omega]^\omega$ being an example. It is known that every uniform ultrafilter on ω has an almost disjoint refinement [**3**]; to present other examples, we need a definition. Given $\mathcal{R} \subseteq [\omega]^\omega$, let

$$\mathcal{I}^+(\mathcal{R}) = \{M \subseteq \omega : |\{R \in \mathcal{R} : |M \cap R| = \omega\}| \geq \omega\}.$$

If $\mathcal{R}$ is an infinite partition of ω into infinite sets, then $\mathcal{I}^+(\mathcal{R})$ has an almost disjoint refinement [**3**].

Hence, a MAD family $\mathcal{A}$ is completely separable if an only if $\mathcal{A}$ is an almost disjoint refinement of $\mathcal{I}^+(\mathcal{A})$. However, the following problem is open, too:

407? **Problem 3.** *Let $\mathcal{A}$ be an infinite MAD family on ω. Does there exist an almost disjoint refinement of $\mathcal{I}^+(\mathcal{A})$?*

Note that this is the strongest possible formulation of an almost disjoint refinement property: Given an arbitrary family $\mathcal{M} \subseteq [\omega]^\omega$, which has an almost disjoint refinement $\mathcal{B}$, it is easy to find a MAD family $\mathcal{A}$ with $\mathcal{M} \subseteq \mathcal{I}^+(\mathcal{A})$; simply replace each $B \in \mathcal{B}$ by infinitely many disjoint subsets of it and extend to a maximal almost disjoint family.

If Problem 3 has an positive answer, then so does Problem 1, in a very strong sense:

Theorem ([**2**]). *The following statements are equivalent:*

(1) *For every MAD family $\mathcal{A}$ on ω, $\mathcal{I}^+(\mathcal{A})$ has an almost disjoint refinement.*
(2) *There is a set $\mathcal{S} \subseteq [\omega]^\omega$ satisfying* (a) *each infinite $M \subseteq \omega$ contains a member of $\mathcal{S}$ and* (b) *every infinite MAD family $\mathcal{A} \subseteq \mathcal{S}$ is completely separable.*

In particular, if $\mathcal{I}^+(\mathcal{A})$ has an almost disjoint refinement for every MAD family $\mathcal{A}$, then there is a completely separable MAD family. It is another open problem, whether this implication can be reversed.

There are consistent affirmative answers to Problem 3. Each of the following assumptions on cardinal invariants

$$\mathfrak{a} = 2^{\omega}, \quad \mathfrak{b} = \mathfrak{d}, \quad \mathfrak{d} \leq \mathfrak{a}, \quad \mathfrak{s} = \omega_1$$

implies that Problem 3 has a positive answer [**2**, **23**].

Also, if one relaxes maximality, then one can find an infinite AD family $\mathcal{A}$ which is completely separable in the sense that every set belonging to $\mathcal{I}^+(\mathcal{A})$ contains an element of $\mathcal{A}$ [**2**].

In conversation, A. Dow remarked that most applications of completely separable MAD familes require that they can be recursively constructed, rather then that they just exist, which seems to be similar to Problem 3:

Problem 4. *Suppose that $\mathcal{I}^+(\mathcal{A})$ has an almost disjoint refinement for every MAD family $\mathcal{A}$. Can every nowhere MAD family be extended to a completely separable MAD family?* 408?

2. Topological connection

An equivalent formulation of Problem 3 has been asked also in a purely topological language. The space $\beta\mathbb{N} \setminus \mathbb{N}$ is not extremally disconnected, so it must contain a point which belongs to the intersection of closures of two disjoint open sets. In 1967, R.S. Pierce asked in [**19**], whether it is possible to show, without using the Continuum Hypothesis, that there are 3-*points* in $\beta\mathbb{N} \setminus \mathbb{N}$, i.e points which lie simultaneously in the closure of three pairwise disjoint open sets. N. Hindman [**11**] then showed that there are not only 3-points, but even $\mathfrak{c}$-points in $\beta\mathbb{N} \setminus \mathbb{N}$ in 1969, and finally B. Balcar and P. Vojtáš [**3**] proved that every point in $\beta\mathbb{N} \setminus \mathbb{N}$ is a $\mathfrak{c}$-point in 1980.

Meanwhile, S.H. Hechler started to consider nowhere dense sets instead of points and showed that under MA, if S is any nowhere dense subset of $\beta\mathbb{N} \setminus \mathbb{N}$, then there exists a family of $\mathfrak{c}$ pairwise disjoint open sets each of which contains S in its closure [**10**]. Call such a set a $\mathfrak{c}$-*set*. It is easy to show that the following is nothing but a topological reformulation of Problem 3:

Problem 5. *Is every nowhere dense subset of $\beta\mathbb{N} \setminus \mathbb{N}$ a $\mathfrak{c}$-set?*

The topological language allows to formulate a seemingly easier problem, also open till now:

Problem 6. *Is every nowhere dense subset of $\beta\mathbb{N} \setminus \mathbb{N}$ a 2-set?* 409?

Still, this is not the end of the story. A.I. Veksler introduced the following order on the family of all nowhere dense subsets of a topological space: If C, D are nowhere dense in X, let $C < D$ mean that $C \subseteq D$ and C is nowhere dense in D. He studied this order in a series of paper; we quote here [**25**] as a sample. A theorem from [**22**] says that the next problem is again Problem 3 in disguise:

Problem 7. *Is it true that a family of all nowhere dense subsets of $\beta\mathbb{N} \setminus \mathbb{N}$, when ordered by $<$, has no maximal elements?*

While studying the sequential order of compact spaces A. Dow [6] introduced the notion of a totally MAD family:

Definition. Given $\mathcal{A}$ and $\mathcal{B}$ infinite families of subsets of ω, say that $\mathcal{A}$ is *totally bounded* with respect to $\mathcal{B}$, if for each infinite $\mathcal{B}' \subseteq \mathcal{B}$ and each $h \in \mathcal{B}'^{\omega}$, there is an $A \in \mathcal{A}$ such that $A \cap \bigcup\{B \setminus h[B] : B \in \mathcal{B}'\}$ is infinite. A MAD family $\mathcal{A}$ is *totally MAD* if for each infinite $\mathcal{B} \subseteq \mathcal{A}$ no subset of cardinality less than $\mathfrak{c}$ is totally bounded with respect to $\mathcal{B}$.

Dow showed that a totally MAD family exists assuming $\mathfrak{b} = \mathfrak{c}$, noted that every totally MAD family has a refinement which is a completely separable MAD family and asked:

410? **Problem 8.** *Is there a totally MAD family in* ZFC*? Does $\mathfrak{b} = \omega_1$ imply there is a totally MAD family?*

A positive answer to Dow's second problem, implies a positive answer to the following weak form of Problem 1.

411? **Problem 9.** *Is there a completely separable MAD family assuming $\mathfrak{c} \leq \omega_2$?*

3. MAD families in forcing extensions

While (as mentioned in the introduction) MAD families with strong combinatorial properties are hard to come by in ZFC, there is also a definite lack of negative (i.e., consistency) results. In this section we present some of the open test problems for understanding the behavior of MAD families in forcing extensions. The first of these problems is due to J. Steprāns [24]:

412? **Problem 10.** *Is there a Cohen-indestructible MAD family in* ZFC*?*

K. Kunen [16] showed that the answer is positive under CH. J. Steprāns showed that the answer is also positive in any model obtained by adding $\aleph_1$-many Cohen reals. Each of $\mathfrak{b} = \mathfrak{c}$, $\mathfrak{a} < \text{cov(meagre)}$ and $\diamondsuit(\mathfrak{d})$ ([13, 14, 17]) is also sufficient for the positive answer. The problem has the following combinatorial translation:

Theorem ([13, 17]). *The following statements are equivalent for a MAD family $\mathcal{A}$.*

(1) *$\mathcal{A}$ is Cohen-indestructible.*
(2) *For every $f : \mathbb{Q} \to \omega$ there is an $A \in \mathcal{A}$ such that $f^{-1}[A]$ is somewhere dense.*

Surprisingly, it is not even known whether there is (in ZFC) a MAD family which survives some forcing extension adding new reals (equivalently, a single Sacks real extension):

413? **Problem 11.** *Is there a Sacks-indestructible MAD family in* ZFC*?*

A flawed proof of this appeared in [13]. This and other flaws of the paper were rectified in [5]. The following old problem (sometimes attributed to J. Roitman) can be formulated as a problem about cardinal invariants of the continuum:

Problem 12. *Does* $\mathfrak{d} = \omega_1$ *imply* $\mathfrak{a} = \omega_1$*?* 414?

Consult [**12**, 18] for some partial positive results. Recently S. Shelah [**20**] using a novel technique of iteration along templates showed that $\mathfrak{d} < \mathfrak{a}$ is relatively consistent with ZFC. J. Brendle [4] presented an axiomatic treatment of Shelah's technique and showed that it can not be used to solve Problem 12.

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