Distributivity of the algebra of regular open subsets of $\beta\mathbb{R} \setminus \mathbb{R}$

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Abstract

We compare the structure of the algebras $\mathcal{P}(\omega)/\text{fin}$ and $\mathcal{A}_\omega/\text{Fin}$, where $\mathcal{A}_\omega$ denotes the algebra of clopen subsets of the Cantor set. We show that the distributivity number of the algebra $\mathcal{A}_\omega/\text{Fin}$ is bounded by the distributivity number of the algebra $\mathcal{P}(\omega)/\text{fin}$ and by the additivity of the meager ideal on the reals. As a corollary we obtain a result of A. Dow, who showed that in the iterated Mathias model the spaces $\beta\omega \setminus \omega$ and $\beta\mathbb{R} \setminus \mathbb{R}$ are not co-absolute. We also show that under the assumption $t = \mathfrak{b}$ the spaces $\beta\omega \setminus \omega$ and $\beta\mathbb{R} \setminus \mathbb{R}$ are co-absolute, improving on a result of E. van Douwen.

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0. Introduction

The object studied here is the algebra \(RO(\beta\mathbb{R} \setminus \mathbb{R})\) of regular open subsets of the Čech–Stone remainder of the real line. This algebra is naturally isomorphic to the completion of the Boolean algebra \(\mathcal{A}^\omega/\text{Fin}\), an object combinatorially easier to grasp. It is easily seen that \(\mathcal{A}^\omega/\text{Fin}\) is \(\sigma\)-closed (i.e., every countable decreasing chain of non-zero elements of \(\mathcal{A}^\omega/\text{Fin}\) has a non-zero lower bound), homogeneous Boolean algebra of cardinality \(\mathfrak{c}\) satisfying \(c^+\)-c.c. The algebra \(\mathcal{A}^\omega/\text{Fin}\) is closely related to the algebra \(\mathcal{P}(\omega)/\text{fin} = \text{Clop}(\beta\omega \setminus \omega)\). It is a consequence of the classical theorem of Parovičenko that, assuming the Continuum Hypothesis, the two algebras are, in fact, isomorphic. Recall that topological spaces \(X\) and \(Y\) are coabsolutes if the Boolean algebras \(\mathcal{RO}(X)\) and \(\mathcal{RO}(Y)\) are isomorphic.

Answering a longstanding question of E. van Douwen, Dow in [7] showed that the spaces \(\omega_1\) and \(\beta\mathbb{R} \setminus \mathbb{R}\) are not co-absolute in the iterated Mathias model. His method for proving this was to show that the distributivity number \(\mathfrak{d}\) of \(\mathcal{P}(\omega)/\text{fin}\) is equal to \(\omega_2\) while the distributivity number \(\mathfrak{d}(\mathcal{A}^\omega/\text{Fin})\) of \(\mathcal{A}^\omega/\text{Fin}\) equals \(\omega_1\) in this model. We present a ZFC theorem, from which Dow’s result easily follows. We also study the tower number \(t(\mathcal{A}^\omega/\text{Fin})\), a natural lower bound for \(\mathfrak{d}(\mathcal{A}^\omega/\text{Fin})\), and show that \(t(\mathcal{A}^\omega/\text{Fin}) = t\). This we use to show that under the assumption \(t = \mathfrak{d}\) the spaces \(\beta\omega \setminus \omega\) and \(\beta\mathbb{R} \setminus \mathbb{R}\) are co-absolutes, improving on a result of E. van Douwen.

1. Preliminaries

In this section we review basic notions concerning Boolean algebras and their cardinal invariants as well as standard cardinal invariants of the continuum. The terminology used here is standard and follows [12,13,10]. Throughout this paper \(A\) denotes the Boolean algebra of closed and open subsets of the Cantor set \(2^\omega\). It is well known that it is up to isomorphism the unique countable atomless Boolean algebra. Given a Boolean algebra \(B\) the product \(B^\omega\) is also a Boolean algebra with the operations defined coordinatewise. For \(f \in B^\omega\) the support of \(f\) is the set \(\{n \in \omega: f(n) \neq 0\}\). By Fin we denote the ideal \(\{f \in B^\omega: \text{support}(f) < \omega\}\). Algebras studied in the paper are the quotient algebras \(B^\omega/\text{Fin}\). It easily follows that \(\mathcal{P}(\omega)/\text{fin}\) can be regularly embedded into \(B^\omega/\text{Fin}\) for any Boolean algebra \(B\). It is therefore natural to compare cardinal invariants of the algebra \(B^\omega/\text{Fin}\) with those of \(\mathcal{P}(\omega)/\text{fin}\).

We will often treat \(B^+\) as a set partially ordered by the canonical ordering on \(B\). The distributivity number of a Boolean algebra \(B\), denoted by \(\mathfrak{d}(B)\), is defined as the minimal size of a family of maximal antichains in \(B\) without common refinement. For homogeneous \(B\), \(\mathfrak{d}(B)\) is equal to the minimal size of a collection of dense open subsets of \(B\) whose intersection is empty. By a tower in a Boolean algebra \(B\) we mean a well ordered decreasing chain in \(B^+\) without a lower bound in \(B^+\). The tower number \(t(B)\) equals the minimal size of a tower in \(B\). For atomless Boolean algebras \(B\), \(t(B)\) is a regular infinite cardinal and \(t(B) \leq \mathfrak{d}(B)\). Furthermore, \(t(B)\) is uncountable if and only if \(B^+\) is \(\sigma\)-closed. All algebras of the type \(B^\omega/\text{Fin}\) have this property. A fundamental difference between \(\mathfrak{d}(B)\) and \(t(B)\) is that \(\mathfrak{d}(B) = \mathfrak{d}(\mathcal{RO}(B))\) whereas \(t(B) = \omega\) for every complete atomless Boolean algebra \(B\).
Next we recall the definitions and basic facts about the relevant cardinal invariants of the continuum (see, e.g., [4]). The symbol $b$ denotes the unbounding number of $(\omega^\omega, \leq^*)$ and $d$ denotes the dominating number of $(\omega^\omega, \leq^*)$, $\text{cov}(M)$ is the minimal size of a family of meager subsets of $2^\omega$ that cover $2^\omega$ and $\text{add}(M)$ stands for the additivity of the meager ideal, i.e., the minimal size of a family of meager subsets of $2^\omega$ whose union is not meager. Using standard notation the tower number of $\mathcal{P}(\omega)/\text{fin}$ is written simply as $t$ and the distributivity number of $\mathcal{P}(\omega)/\text{fin}$ as $h$. The following proposition sums up provable relationships between these cardinal invariants:

**Proposition 1.1.**

(i) $t \leq h \leq b \leq d$,
(ii) (Piotrowski–Szymański) $t \leq \text{add}(M)$,
(iii) (Bartoszyński–Miller) $\text{add}(M) = \min\{b, \text{cov}(M)\}$.

The invariants $h$ and $\text{add}(M)$ as well as $b$ and $\text{cov}(M)$ are in ZFC not provably comparable. For proofs and additional information consult [4]. In the proof of our main result we will use the following reformulation of a result of Keremedis [11], for an alternative proof see [1]. The symbol $\mathbb{Q}$ denotes the set of rational numbers equipped with its usual topology.

**Theorem 1.2** (Keremedis). $\text{cov}(M)$ is equal to the minimal size of a family $\mathcal{F}$ of nowhere dense subsets of $\mathbb{Q}$ such that for every infinite set $Y \subseteq \mathbb{Q}$ there is an $F \in \mathcal{F}$ intersecting $Y$ in an infinite set.

2. Main results

As mentioned above $\mathcal{P}(\omega)/\text{fin}$ can be regularly embedded into $\mathbb{B}^\omega/\text{Fin}$ for any Boolean algebra $\mathbb{B}$. It is well known, that the algebra $\mathcal{P}(\omega)/\text{fin}$ generically adds a selective ultrafilter $U$ on $\omega$. Next, we will show that $\mathbb{B}^\omega/\text{Fin}$ can be written as an iteration of $\mathcal{P}(\omega)/\text{fin}$ and an ultra-power of $\mathbb{B}$ modulo $U$. In particular, $\mathcal{P}(\omega)/\text{fin}$ is a regular subalgebra of $\mathcal{A}^\omega/\text{Fin}$. The relation $\simeq$ denotes forcing equivalence or, in other words, the fact that the completions of the algebras are isomorphic.

**Proposition 2.1.** $\mathbb{B}^\omega/\text{Fin} \simeq \mathcal{P}(\omega)/\text{fin} \ast \mathbb{B}^\omega/\dot{U}$, where $\dot{U}$ is the $\mathcal{P}(\omega)/\text{fin}$-name for the selective ultrafilter added by $\mathcal{P}(\omega)/\text{fin}$.

**Proof.** Define a function $\Phi : \mathbb{B}^\omega/\text{Fin} \rightarrow \mathcal{P}(\omega)/\text{fin} \ast \mathbb{B}^\omega/\dot{U}$ by putting $\Phi(f) = (\text{support}(f), [\hat{f}]_U)$, where $[\hat{f}]_U$ is a $\mathcal{P}(\omega)/\text{fin}$-name for $\{g \in \mathbb{B}^\omega : \{n \in \omega : f(n) = g(n)\} \in U\}$. It is easy to verify that $\Phi$ is a dense embedding. □

We use the proposition to show that $h(\mathbb{A}^\omega/\text{Fin}) \leq h$, a fact known already to van Douwen (see [6]).
Note that the Boolean algebra $\mathcal{A}/\text{Fin}$ is naturally isomorphic to the algebra $\text{Clop}(\beta X \setminus X)$, where the space $X = 2^\omega \setminus 0$ or, equivalently, $X$ is a disjoint sum of countably many copies of $2^\omega$. In particular, the spaces $\beta X \setminus X$ and $\beta \mathbb{R} \setminus \mathbb{R}$ are co-absolute.

Now we are ready to state and prove our main result:

**Theorem 2.2.** $h(\mathcal{A}/\text{Fin}) \leq \min\{h, \text{add}(\mathcal{M})\}$.

**Proof.** In this proof we identify the rationals $\mathbb{Q}$ with $\{q \in 2^\omega: (\forall \omega)n \in \omega) q(n) = 0\}$. Recall that if $f \in \mathcal{A}$ and $n \in \text{support}(f)$ then $f(n)$ is a nonempty clopen subset of $2^\omega$.

By Proposition 2.1, $\mathcal{P}(\omega)/\text{fin}$ can be regularly embedded into $\mathcal{A}/\text{Fin}$. Consequently, $h(\mathcal{A}/\text{Fin}) \leq \text{h}$. Since $h \leq b$ and $\text{add}(\mathcal{M}) = \min\{b, \text{cov}(\mathcal{M})\}$ all that needs to be shown is that $h(\mathcal{A}/\text{Fin}) \leq \text{cov}(\mathcal{M})$.

To that end fix a family $\{K_\alpha: \alpha < \text{cov}(\mathcal{M})\}$ of nowhere dense subsets of $\mathbb{Q}$ such that for every infinite set $Y \subseteq \mathbb{Q}$ there is an $\alpha < \text{cov}(\mathcal{M})$ such that $K_\alpha$ intersects $Y$ in an infinite set. For $\alpha < \text{cov}(\mathcal{M})$ put

$$\mathcal{H}_\alpha = \{f \in \mathcal{A}: (\forall \omega)n \in \omega) f(n) \cap K_\alpha = \emptyset\}.$$  

Note that:

(a) $\mathcal{H}_\alpha$ is closed under finite changes, i.e., if $f \in \mathcal{H}_\alpha$ and $g = \text{Fin} f$ then $g \in \mathcal{H}_\alpha$.

(b) $\mathcal{H}_\alpha$ is dense in the algebra $\mathcal{A}/\text{Fin}$.

To see this, let $f \in \mathcal{A}$ have infinite support. As $K_\alpha$ is nowhere dense, it is easy to find $g \in \mathcal{A}$ with the same support as $f$ such that $g(n) \subseteq f(n)$ and $g(n) \cap K_\alpha = \emptyset$, for all $n \in \omega$.

(c) $\mathcal{H}_\alpha$ is open (downward closed).

To finish the proof it suffices to check that:

(d) $\bigcap\{\mathcal{H}_\alpha: \alpha < \text{cov}(\mathcal{M})\} = \text{Fin}$.

Obviously, $\text{Fin} \subseteq \mathcal{H}_\alpha$ for all $\alpha < \text{cov}(\mathcal{M})$. Now, take $f \in \mathcal{A}$ with infinite support. For every $n$ in the support of $f$ recursively pick $q_n \in \mathbb{Q} \cap f(n) \setminus \{q_m: m < n\}$ and set $Y = \{q_m: n \in \text{support}(f)\}$. There is an $\alpha < \text{cov}(\mathcal{M})$ such that $|Y \cap K_\alpha| = \omega$. Consequently, there are infinitely many $n \in \omega$ such that $f(n) \cap K_\alpha \neq \emptyset$ and hence $f \notin \mathcal{H}_\alpha$. \qed

As a corollary we now get Dow’s result.

**Theorem 2.3** (Dow). $h(\mathcal{A}/\text{Fin}) < h$ in the iterated Mathias model.

**Proof.** It is a folklore fact, first observed in [5], that $h = \epsilon = \omega_2$ in the Mathias model. Equally standard is the fact that $\text{cov}(\mathcal{M}) = \omega_1$ in the Mathias model (see, e.g., [4]). By Theorem 2.2 the result follows. \qed

It actually follows from Theorem 2.2, that $h(\mathbb{B}/\text{Fin}) \leq \min\{h, \text{add}(\mathcal{M})\}$ for any $\mathbb{B}$ which contains $\mathcal{A}$ as a regular subalgebra, i.e., for any $\mathbb{B}$ adding a Cohen real.
Theorem 2.4. \( t = t(\mathcal{A}^\omega/\text{Fin}) \).

**Proof.** Let \( \{ T_\alpha : \alpha < t \} \) be a tower in \( \mathcal{P}(\omega)/\text{fin} \). Let \( \chi_\alpha \in \mathcal{A}^\omega \) be the characteristic function of \( T_\alpha \), i.e., \( \chi_\alpha(n) = 1 \) if \( n \in T_\alpha \) and \( \chi_\alpha(n) = 0 \) if \( n \notin T_\alpha \). Then \( \{ \chi_\alpha : \alpha < t \} \) forms a tower in \( \mathcal{A}^\omega/\text{Fin} \), for if \( f \in \mathcal{A}^\omega \) were its lower bound then the support of \( f \) would be a lower bound for \( \{ T_\alpha : \alpha < t \} \). Hence \( t \geq t(\mathcal{A}^\omega/\text{Fin}) \).

In order to prove \( t \leq t(\mathcal{A}^\omega/\text{Fin}) \) let \( \{ f_\alpha : \alpha < \kappa \} \) be a decreasing chain in \( \mathcal{A}^\omega/\text{Fin} \), where \( \kappa < t \). We will show that \( \{ f_\alpha : \alpha < \kappa \} \) has a lower bound. To that end we first prove the following claim.

**Claim 1.** There is a one-to-one sequence \( \langle c_n : n \in \omega \rangle \in (2^\omega)^\omega \) such that for every \( \alpha < \kappa \) the set \( A_\alpha = \{ n \in \omega : c_n \in f_\alpha(n) \} \) is infinite.

To prove the claim, note that the set
\[
X = \bigcup_{\alpha < \kappa} \bigcup_{m \in \omega} \left\{ (c_n : n \in \omega) \in (2^\omega)^\omega : (\forall n \geq m) c_n \notin f_\alpha(n) \right\}
\]
\[
\cup \bigcup_{m \neq n \in \omega} \left\{ (c_n : n \in \omega) \in (2^\omega)^\omega : c_n = c_m \right\}
\]
is a union of \( \kappa \) many meager subsets of \( (2^\omega)^\omega \). As \( \kappa < t \leq \text{cov}(\mathcal{M}) \), \( (2^\omega)^\omega \setminus X \) is not empty and any \( \langle c_n : n \in \omega \rangle \in (2^\omega)^\omega \setminus X \) has the required property.

**Claim 2.** If \( \alpha < \beta < \kappa \) then \( A_\beta \subseteq^* A_\alpha \).

As \( f_\beta \leq_{\text{Fin}} f_\alpha \), \( f_\beta(n) \subseteq f_\alpha(n) \) for all but finitely many \( n \in \omega \). Hence, \( A_\beta \subseteq^* A_\alpha \).

So, \( \{ A_\alpha : \alpha < \kappa \} \) form a decreasing chain in \( \mathcal{P}(\omega)/\text{fin} \) and as \( \kappa < t \) the chain has a lower bound \( A \in [\omega]^{\omega} \). For \( \alpha < \kappa \) and \( n \in A \) set
\[
g_\alpha(n) = \begin{cases} \min\{k \in \omega : \langle c_n \mid k \rangle \subseteq f_\alpha(k)\} & \text{if } n \in A_\alpha \cap A, \\ 0 & \text{otherwise}, \end{cases}
\]
where \( \langle c_n \mid k \rangle \) denotes the clopen set \( \{ h \in 2^\omega : h \mid k = c_n \mid k \} \). As \( \kappa < t \leq b \) there is a function \( g : A \to \omega \) which \( \leq^* \)-dominates all \( g_\alpha, \alpha < \kappa \). Set
\[
f(n) = \begin{cases} \langle c_n \mid g(n) \rangle & \text{if } n \in A, \\ 0 & \text{otherwise}. \end{cases}
\]
Note that \( \text{support}(f) = A \) is an infinite set. To finish the proof it suffices to check that \( f \leq_{\text{Fin}} f_\alpha \) for every \( \alpha < \kappa \). This follows as for all but finitely many \( n \in A \), \( g(n) \geq g_\alpha(n) \) and hence \( f_\alpha(n) \geq \langle c_n \mid g_\alpha(n) \rangle \geq \langle c_n \mid g(n) \rangle = f(n) \). \( \square \)

The following corollary is a strengthening of a theorem of van Douwen [6] who proved that, assuming \( p = \mathfrak{c}, \beta \omega \setminus \omega \) and \( \beta \mathbb{R} \setminus \mathbb{R} \) are coabsolute.

**Corollary 2.5.** \( (t = \mathfrak{h}) \beta \omega \setminus \omega \) and \( \beta \mathbb{R} \setminus \mathbb{R} \) are coabsolute.
Proof. If \( t = h = \kappa \) then by Theorems 2.2 and 2.4 also \( t(A^{\omega}/\text{Fin}) = h(A^{\omega}/\text{Fin}) = \kappa \). Hence by the base tree theorem of [2] (see also [3,8]) both algebras \( P(\omega)/\text{fin} \) and \( A^{\omega}/\text{Fin} \) have dense subsets isomorphic to \( c \)-branching tree of height \( \kappa \). As \( t = t(A^{\omega}/\text{Fin}) = \kappa \) these trees have no short branches and hence are isomorphic to the tree \( c^{<\kappa} \). So the algebras \( RO(P(\omega)/\text{fin}) \) and \( RO(A^{\omega}/\text{Fin}) \) are isomorphic and the result follows. □

We have shown that \( t \leq h(A^{\omega}/\text{Fin}) \leq \min\{h, \text{add}(\mathcal{M})\} \). It is a natural question whether one of the inequalities is, in fact, an equality.

Proposition 2.6. It is relatively consistent with ZFC that \( t < h(A^{\omega}/\text{Fin}) \).

Proof. Let \( V \) be a model of \( p = c > \omega_1 \) in which there is a Suslin tree \( T \). To obtain such model it suffices to add a single Cohen real to a model of Martin’s Axiom (see, e.g., [4]). Treat \( T \) as a forcing notion ordered by reversing the tree order. \( T \) is then a c.c.c. forcing of size \( \omega_1 \) which does not add any new reals. Let \( G \) be a \( T \)-generic filter over \( V \). In \( V[G] \), \( t = \omega_1 \) as \( T \) can be isomorphically embedded into \( P(\omega)/\text{fin} \) and the generic branch produces a tower. This fact was probably first explicitly stated in [9].

To see that \( V[G] \models h(A^{\omega}/\text{Fin}) = c \) fix some \( \kappa < c \) and assume that, \( \dot{A}_a, \alpha < \kappa \) are \( T \)-names for maximal antichains in \( A^{\omega}/\text{Fin} \). For \( \alpha < \kappa \) and \( t \in T \) let

\[
A^t_a = \{ f \in A^{\omega}/\text{Fin}: t \Vdash f \in A_a \}
\]

and extend each \( A^t_a \) to a maximal antichain \( B^t_a \) in \( A^{\omega}/\text{Fin} \). As \( V \) is a model of \( p = h(A^{\omega}/\text{Fin}) = c \) there is a maximal antichain \( A \) in \( A^{\omega}/\text{Fin} \) which refines \( B^t_a \) for every \( \alpha < \kappa \) and \( t \in T \). Then, in \( V[G] \), \( A \) refines \( A_a \) for all \( \alpha < \kappa \). □

Another model for \( t < h(A^{\omega}/\text{Fin}) \) can be found in [2]: Take the product forcing \( P \times Q \) where \( P \) is the Solovay–Tennenbaum c.c.c. poset forcing MA plus \( c = \omega_2 \), and \( Q \) is the forcing for adding more than \( \omega_2 \) subsets of \( \omega_1 \) with countable conditions. A completely analogous reasoning as in [2] shows that \( t < h(A^{\omega}/\text{Fin}) \) holds in this model.

We conjecture that Dow’s method for proving the consistency of \( h(A^{\omega}/\text{Fin}) < h \) can be used to prove the following consistency result.

Question 2.7. Is it consistent with ZFC that \( h(A^{\omega}/\text{Fin}) < \min\{h, \text{add}(\mathcal{M})\} \)?

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