Rend. Circ. Mat. Palermo (2011) 60:445–454
DOI 10.1007/s12215-011-0064-0

Adding ultrafilters by definable quotients

Michael Hrušák · Jonathan L. Verner

Received: 25 October 2010 / Accepted: 29 June 2011 / Published online: 31 August 2011

Abstract Forcing notions of the type $\mathcal{P}(\omega)/\mathcal{I}$ which do not add reals naturally add ultrafilters on ω. We investigate what classes of ultrafilters can be added in this way when $\mathcal{I}$ is a definable ideal. In particular, we show that if $\mathcal{I}$ is an F_σ P-ideal the generic ultrafilter will be a P-point without rapid RK-predecessors which is not a strong P-point. This provides an answer to long standing open questions of Canjar and Laflamme.

Keywords Ultrafilter · P-point · Q-point · Selective ultrafilter · Rapid ultrafilter · F_σ-ideal

Mathematics Subject Classification (2000) O3E05 · 03E17 · 03E35

1 Introduction

Various types of ultrafilters can be added by definable approximations. C. Laflamme [12] studied ultrafilters which can be constructed using forcing with definable ideals ordered by inclusion. This paper broaches a similar topic by studying the generic ultrafilters added by the quotient algebra $\mathcal{P}(\omega)/\mathcal{I}$, where $\mathcal{I}$ is some definable ideal (we only consider ideals such that $\mathcal{P}(\omega)/\mathcal{I}$ does not add reals, for example F_σ-ideals). This serves two purposes: (1) It is a simple method for consistently constructing various types of ultrafilters. In particular, forcing with quotient algebras over F_σ P-ideals adds a P-point with no rapid RK-predecessor

The research of the first author was partially supported by PAPIIT grant IN101608 and CONACyT grant 80355.

The second author would like to acknowledge the support of GAČR 401/09/H007 Logické základy sémantiky.

M. Hrušák
Instituto de Matemáticas, UNAM, Apartado postal 61-3, Xangari, 58089, Morelia, Michoacán, México
e-mail: michael@matmor.unam.mx

J.L. Verner (✉)
KTIML, Charles University, Malostranské náměstí 25, 118 00 Praha 1, Czech Republic
e-mail: jonathan.verner@matfyz.cz

which is not a strong P-point. (2) An attempt to classify definable ideals whose quotients do not add reals.

We now review standard definitions and theorems we will be using.

1.1 Ideals

In the paper we shall always assume that an ideal $\mathcal{I}$ is a nonprincipal proper ideal. The ideal of finite subsets of ω will be denoted by *fin*. An ideal $\mathcal{I}$ is *tall* if for each $A \in [\omega]^\omega$ there is $B \in [A]^\omega \cap \mathcal{I}$. It is called *nowhere tall* (or *Fréchet* or *locally fin*) if for each $A \in [\omega]^\omega$ there is $B \in [A]^\omega$ such that $\mathcal{I} \restriction B = [B]^{<\omega}$, where $\mathcal{I} \restriction B = \{I \cap B : I \in \mathcal{I}\}$. An ideal is F_σ (or analytic, etc.) if it is F_σ (or analytic, etc.) when considered as a subset of 2^ω viewed as a Polish space.

Definition 1.1 If φ is a property of ideals we shall say that $\mathcal{I}$ is *locally* φ if for each $\mathcal{I}$-positive A there is an $\mathcal{I}$-positive $B \subseteq A$ such that $\mathcal{I} \restriction B$ satisfies φ.

Definition 1.2 A function $\mu : \mathcal{P}(\omega) \to \mathbb{R}_0^+ \cup \{\infty\}$ is a *lower semicontinuous submeasure* (lscsm for short) on $\mathcal{P}(\omega)$ if the following holds:

(i) $\mu(\emptyset) = 0$,
(ii) If $A \subseteq B \in \mathcal{P}(\omega)$ then $\mu(A) \leq \mu(B)$ (monotonicity),
(iii) If $A, B \subseteq \omega$ then $\mu(A \cup B) \leq \mu(A) + \mu(B)$ (subadditivity),
(iv) If $A \subseteq \omega$ then $\mu(A) = \lim_{n\to\infty} \mu(A \cap n)$ (lower semicontinuity).

If μ moreover satisfies

(v) $\mu(A \cup B) = \mu(A) + \mu(B)$, for disjoint $A, B \in \mathcal{P}(\omega)$ (additivity),

we say it is a *measure*.

Definition 1.3 Given a lscsm μ define ideals $Fin(\mu) = \{A \subseteq \omega : \mu(A) < \infty\}$ and $Exh(\mu) = \{A \subseteq \omega : \lim_{n\to\infty} \mu(A \setminus n) = 0\}$.

It is not hard to see that $Fin(\mu)$ is an F_σ ideal. The following theorem of Mazur (see [14]) shows that each F_σ ideal is of this form.

Theorem 1.4 (Mazur) *An ideal I on ω is an F_σ ideal if and only if there is a lscsm μ such that $I = Fin(\mu)$.*

We will also need a related theorem of Solecki (see [18, 19]):

Definition 1.5 An ideal $\mathcal{I}$ is a P$^+$-ideal if any descending sequence of $\mathcal{I}$-positive sets has an $\mathcal{I}$-positive pseudointersection. It is a P-ideal if any descending sequence of sets from $\mathcal{I}^* = \{\omega \setminus I : I \in \mathcal{I}\}$ has a pseudointersection in $\mathcal{I}^*$.

Theorem 1.6 (Solecki) *An ideal I on ω is an analytic P-ideal if and only if there is a lscsm μ such that $I = Exh(\mu)$. Moreover an ideal is an F_σ P-ideal if and only if it is of the form $I = Fin(\mu)$ for some* measure μ.

For more on analytic P-ideals consult [5].

1.2 Ultrafilters

The following definition makes sense for any pair of ideals or filters.

Definition 1.7 Let $\mathcal{I}, \mathcal{J}$ be ideals (or filters) on ω. Recall that

(i) (Rudin-Keisler ordering, [11]) $\mathcal{I} \leq_{\mathrm{RK}} \mathcal{J}$ if there is a function $f : \omega \to \omega$ such that $\mathcal{I} = f_*(\mathcal{J}) = \{A \subseteq \omega : f^{-1}[A] \in \mathcal{J}\}$.
(ii) (Rudin-Blass ordering, [12]) $\mathcal{I} \leq_{\mathrm{RB}} \mathcal{J}$ if $\mathcal{I} \leq_{\mathrm{RK}} \mathcal{J}$ and the function witnessing this can be chosen to be finite-to-one.
(iii) (Katětov ordering, [11]) $\mathcal{I} \leq_{\mathrm{K}} \mathcal{J}$ if there is a function $f : \omega \to \omega$ such that preimages of $\mathcal{I}$-small sets are $\mathcal{J}$-small.
(iv) (Katětov-Blass ordering, [6]) $\mathcal{I} \leq_{\mathrm{KB}} \mathcal{J}$ if $\mathcal{I} \leq_{\mathrm{K}} \mathcal{J}$ and the witnessing function can be chosen to be finite-to-one.

We now review definitions of various types ultrafilters. Recall that these ultrafilters need not exist in ZFC.

Definition 1.8 [17] An ultrafilter $\mathcal{U}$ is a P-point if for any sequence $\langle X_n : n < \omega\rangle \subseteq \mathcal{U}$ there is an $X \in \mathcal{U}$ such that $(\forall n < \omega)(X \subseteq^* X_n)$ or, equivalently, for any partition $\langle X_n : n < \omega\rangle$ of ω consisting of $\mathcal{U}$-small sets, there is a $U \in \mathcal{U}$ such that $|U \cap X_n| < \omega$.

Fact Any RK-predecessor of a P-point is its RB-predecessor.

Definition 1.9 [3] An ultrafilter is an *Q-point* if for any interval partition $\langle I_n : n < \omega\rangle$ of ω there is an $U \in \mathcal{U}$ such that $|U \cap I_n| \leq 1$.

Definition 1.10 [16] An ultrafilter $\mathcal{U}$ is *rapid* if the family $\{e_X : X \in \mathcal{U}\}$ of enumerating functions of sets in $\mathcal{U}$ is a dominating family of functions in $(\omega^\omega, \leq^*)$ or, equivalently, if for any interval partition $\langle I_n : n < \omega\rangle$ of ω there is a $U \in \mathcal{U}$ such that $|U \cap I_n| \leq n$.

As far as we know, the following concept first appeared in [1] and/or in [3]:

Definition 1.11 An ultrafilter is *selective* (or *Ramsey*) if for any partition of ω there is a selector in the ultrafilter. An ultrafilter which is a P-point and rapid is called *semiselective*

Fact 1.12 An ultrafilter is selective if and only if it is a P-point and a Q-point. A selective ultrafilter is a minimal ultrafilter in the RK-ordering.

1.3 Forcing

Instead of dealing with the quotient algebra $\mathcal{P}(\omega)/\mathcal{I}$ we will implicitly use the equivalent forcing notion $(\mathcal{I}^+, \subseteq)$. When we say $\mathcal{I}$ (or $\mathcal{P}(\omega)/I$) adds an ultrafilter with some property we, in fact, mean that every generic ultrafilter on $(\mathcal{I}^+, \subseteq)$ will have this property. Beware that this is somewhat different from the usual meaning of "a forcing P adds an object O". We use "add" in this sense, since, e.g. under CH, the usual sense trivializes: under CH all forcings of this form are isomorphic and thus an ultrafilter added by one of the forcings is added by any other.

Definition 1.13 A forcing P is σ-closed if any countable descending sequence of conditions has a lower bound.

2 P-points

The following theorem may be found in [22]

Theorem 2.1 (Zapletal) *An ultrafilter $\mathcal{U}$ is a P-point if all analytic ideals disjoint from $\mathcal{U}$ can be separated from $\mathcal{U}$ by an F_σ ideal* (*i.e. are contained in an F_σ ideal disjoint from $\mathcal{U}$*).

Using Mazur's theorem it is easy to prove the following which was first observed in [10]:

Observation 2.2 (Just-Krawczyk) *If $\mathcal{I}$ is an F_σ ideal then $\mathcal{P}(\omega)/\mathcal{I}$ is σ-closed*, *in fact*, *$\mathcal{I}$ is a P^+-ideal.*

Proof By Mazur's Theorem 1.4 we may find a lscsm μ such that $\mathcal{I} = Fin(\mu)$. Assume $\langle A_n : n < \omega\rangle \subseteq \mathcal{I}^+$ is a descending sequence of conditions. Without loss of generality we may assume that $A_{n+1} \subseteq A_n$ (since $A_{n+1} \setminus A_n \in \mathcal{I}$). Using lower semicontinuity and the fact that $\mu(A_n) = \infty$ we define by induction finite sets $a_n \in [A_n]^{<\omega}$ such that $\mu(a_n) \geq n$. Finally let $A = \bigcup_{n<\omega} a_n$. By monotonicity $\mu(A) = \infty$ so $A \in I^+$ is a condition. Since $\mu(A \setminus A_n) \leq \mu(\bigcup_{i<n} a_i) \leq \sum_{i<n} \mu(a_i) < \infty$ so $A \setminus A_n \in \mathcal{I}$ so A is stronger than each A_n. □

Question 2.3 *Suppose I is Borel and $\mathcal{P}(\omega)/\mathcal{I}$ does not add new reals. Is $\mathcal{P}(\omega)/\mathcal{I}$ σ-closed?*

Observation 2.4 (Folklore) *If $\mathcal{I}$ is F_σ then $\mathcal{P}(\omega)/\mathcal{I}$ adds a P-point.*

Proof To prove that the generic ultrafilter G is a P-point suppose $A \in \mathcal{I}^+$ forces $\langle \dot{A}_n : n < \omega\rangle \subseteq \dot{G}$. Since $\mathcal{P}(\omega)/\mathcal{I}$ is σ-closed we may assume $\langle A_n : n < \omega\rangle \in V$. As above in the proof of Observation 2.2 find $B \subseteq A$, $\mu(B) = \infty$ with $\mu(B \setminus A_n) < \infty$. Then $B \in \mathcal{I}^+$ is stronger than A and forces that $B \in \dot{G}$ is a pseudointersection of $\langle A_n : n < \omega\rangle$. □

We shall show that this is essentially the only case when a definable $\mathcal{I}$ not adding reals adds a P-point:

Theorem 2.5 *Suppose $\mathcal{I}$ is analytic and $\mathcal{P}(\omega)/\mathcal{I}$ adds no new reals. Then $\mathcal{P}(\omega)/\mathcal{I}$ adds a P-point if and only if $\mathcal{I}$ is locally F_σ.*

Proof If $\mathcal{I}$ is locally F_σ then $\mathcal{P}(\omega)/\mathcal{I}$ adds a P-point by Observation 2.4.

Suppose on the other hand that $\mathcal{P}(\omega)/\mathcal{I}$ adds a P-point and that $A \in \mathcal{I}^+$. Work in the extension by some generic filter G containing A. Clearly $G \cap \mathcal{I} = \emptyset$ so, by Zapletal's Theorem 2.1 there is an F_σ ideal J extending $\mathcal{I}$. Since $\mathcal{J}$ is given by a lscsm, which is essentially given by a real, and $\mathcal{P}(\omega)/\mathcal{I}$ adds no new reals, this J is already in the ground model and we may assume that $A \Vdash$ "$\dot{G} \cap J = \emptyset$". Since $\mathcal{J} \restriction A$ is an F_σ ideal it is sufficient to show $\mathcal{I} \restriction A = \mathcal{J} \restriction A$. The inclusion from left to right is clear. So suppose some there was some $C \subseteq A$, $C \in \mathcal{J} \setminus \mathcal{I}$. Then $C \Vdash$ "$C \in \dot{G} \cap J$" which would be a contradiction. □

Note that being locally F_σ is not the same as being F_σ even in the class of Borel tall ideals:

Example 2.6 There are tall Borel ideals of arbitrarily high complexity which are locally F_σ.

Proof Given a set $A \subseteq \omega^\omega$ let $\mathcal{I}_A$ be the ideal generated by sets of the form (1) $\{f \restriction n : n < \omega\}$ for $f \in A$, (2) $\{f \restriction n : n \in X\}$ for $f \notin A$, $X \in \mathcal{I}_{1/n}$ and (3) antichains in $\omega^{<\omega}$. This is clearly a tall ideal.

The complexity of $\mathcal{I}_A$ is at least the complexity of A: Consider $\Phi : \omega^\omega \to \mathcal{P}(\omega^{<\omega})$ defined as follows $\Phi(f) = \{f \restriction n : n < \omega\}$. This is a continuous function and $\Phi^{-1}[\mathcal{I}_A] = A$.

Note that $\mathcal{I}_{\omega^\omega}$ is F_σ, as is $\mathcal{I}_A \restriction \{f \restriction n : n < \omega\}$ for $f \notin A$. Suppose $X \in \mathcal{I}_A^+$. Then either $\mathcal{I}_A \restriction X = \mathcal{I}_{\omega^\omega} \restriction X$ and then $\mathcal{I}_A \restriction X$ is F_σ or not, and then there is $f \notin A$ such that $Y = X \cap \{f \restriction n : n < \omega\} \in \mathcal{I}_A^+$. Then $\mathcal{I}_A \restriction Y$ is F_σ. □

3 Selectivity

Recall the following classical theorem of A.R.D. Mathias [13]

Theorem 3.1 (Mathias) *An ultrafilter $\mathcal{U}$ is selective if and only if $\mathcal{U}$ meets all tall analytic ideals.*

The following fact is folklore:

Fact 3.2 $\mathcal{P}(\omega)/\mathit{fin}$ adds a selective ultrafilter.

We shall show that, in the class of analytic ideals not adding reals, *fin* is in a sense the only ideal adding a selective ultrafilter:

Theorem 3.3 *Suppose $\mathcal{I}$ is analytic and $\mathcal{P}(\omega)/\mathcal{I}$ does not add reals. Then $\mathcal{P}(\omega)/\mathcal{I}$ adds a selective ultrafilter if and only if $\mathcal{I}$ is Fréchet.*

Proof Suppose first that $\mathcal{I}$ is Fréchet. Given $A \in \mathcal{I}^+$ there is an $\mathcal{I}$-positive $B \subseteq A$ such that $\mathcal{I} \restriction B \simeq \mathcal{P}(\omega)/\mathit{fin}$. Now use Fact 3.2.

The other direction is a direct corollary of Theorem 3.1: Suppose $A \in \mathcal{I}^+$. By assumption $A \Vdash$ "$\dot{G}$ is selective". We need to find a $B \in [A]^\omega$ such that $\mathcal{I} \restriction B = \mathit{fin}$. Since $\mathcal{I} \cap \dot{G} = \emptyset$ and $\mathcal{I}$ is analytic, we may apply Theorem 3.1 (taking A instead of ω) to see that $\mathcal{I} \restriction A$ is not tall. So there is an infinite $B \subseteq A$ such that $\mathcal{I} \restriction B = \mathit{fin}$. □

Remark 3.4 This, of course, fails badly in the non-definable case, e.g. $\mathcal{P}(\omega)/\mathcal{I}(\mathcal{A})$ adds a selective ultrafilter for every MAD family $\mathcal{A}$ (see [13]).

4 Q-points and rapid ultrafilters

Now we turn our attention to other properties of ultrafilters and prove two more characterizations.

Definition 4.1 Let $\Delta = \{(x, y) : x \leq y\}$. The ideal $\mathcal{ED}_{\mathit{fin}}$ on Δ consists of those sets which can be covered by finitely many functions.

Proposition 4.2 *Suppose $\mathcal{P}(\omega)/\mathcal{I}$ does not add new reals. Then the forcing $\mathcal{I}$ adds a Q-point if and only if it is locally not KB-above $\mathcal{ED}_{\mathit{fin}}$.*

Proof Suppose that $\mathcal{I}$ adds a Q-point. We must show that $\mathcal{I}$ is locally not KB-above $\mathcal{ED}_{fin}$. Pick some $\mathcal{I}$-positive set A and a finite-to-one function $f : A \to \Delta$. Aiming towards a contradiction suppose this function witnesses $\mathcal{I} \restriction A \geq_{\mathrm{KB}} \mathcal{ED}_{fin}$. Let $A_n = f^{-1}[\{(n, y) : y \leq n\}]$. Then A_n is a partition of A into finite sets with no positive selector so $A \Vdash$ "$\dot{G}$ is not a Q-point" a contradiction.

On the other hand suppose $A \in \mathcal{I}^+$ and $A \Vdash$ "$\dot{G}$ is not a Q-point". Since $\mathcal{P}(\omega)/\mathcal{I}$ does not add new reals, we can assume that there is an interval partition $\langle I_n : n < \omega\rangle$ such that A forces each selector to be outside of $\dot{G}$. Fix an increasing sequence k_n of natural numbers such that $|I_n| \leq k_n$ and also fix bijections $\varphi_n : I_n \to k_n$. Finally define $f : A \to \Delta$ as follows. For $x \in A$ find $n < \omega$ such that $x \in I_n$ and let $f(x) = \varphi_n(x)$. It is easy to see that this function witnesses $\mathcal{I} \restriction A$ is KB-above $\mathcal{ED}_{fin}$. □

For dealing with rapid ultrafilters we use the following theorem of P. Vojtáš [21].

Definition 4.3 Given a function $f : \omega \to \mathbb{R}_0^+$ tending to zero such that $\sum_{n<\omega} f(n) = \infty$, we define $\mathcal{I}_f = \{A \subseteq \omega : \sum_{n\in A} f(n) < \infty\}$ and we call an ideal $\mathcal{I}$ *summable* if $\mathcal{I} = \mathcal{I}_f$ for some such function.

Theorem 4.4 (Vojtáš) *An ultrafilter is rapid if and only if it meets every tall summable ideal.*

The following proposition, which can be found in [6], shows that we can replace tall summable ideal with tall analytic P ideal in the above theorem.

Proposition 4.5 (Hrušák-Hernandez) *Suppose $\mathcal{I}$ is a tall analytic P-ideal. Then there is a tall summable ideal contained in $\mathcal{I}$.*

Proof By Theorem 1.6 there is a lscsm μ such that $\mathcal{I} = Exh(\mu)$. We shall show that $\mu(\{n\}) \to 0$: Suppose otherwise. Then there is an $\varepsilon > 0$ and an infinite $A \subseteq \omega$ such that $\mu(\{a\}) \geq \varepsilon$ for each $a \in A$. Then any infinite subset of A has submeasure $\geq \varepsilon$ so is not in $\mathcal{I}$ contradicting the tallness of $\mathcal{I}$. Now let $g(n) = \mu(\{n\})$. By the preceding $\mathcal{I}_g$ is a tall summable ideal. We claim that $\mathcal{I}_g \subseteq \mathcal{I}$. To see this, let $A \subseteq \omega$ with $\sum_{a\in A} g(a) < \infty$. Since the sum converges, necessarily $\sum_{a\in A\setminus n} g(a) \to 0$. Moreover $\mu(A \setminus n) \leq \sum_{a\in A\setminus n} g(a)$ so also $\mu(A \setminus n) \to 0$ so $A \in \mathcal{I}$. □

Proposition 4.6 *Suppose $\mathcal{P}(\omega)/\mathcal{I}$ does not add new reals. Then forcing with $\mathcal{I}$ adds a rapid ultrafilter if and only if it is locally not KB-above a tall summable ideal.*

Proof Suppose $\mathcal{I}$ adds a rapid ultrafilter and $A \in \mathcal{I}^+$. We shall show that $\mathcal{I} \restriction A$ is not a tall summable ideal. Aiming towards a contradiction suppose that $\mathcal{I} \restriction A \geq_{\mathrm{KB}} \mathcal{I}_g$ as witnessed by some $f : A \to \omega$. and define $\mu(n) = g(f(n))$. Then $Fin(\mu) \subseteq \mathcal{I} \restriction A$ so, in particular, $A \Vdash$ "$\dot{G} \cap Fin(\mu) = \emptyset$" contradicting Vojtáš's characterization 4.4 ($Fin(\mu)$ is tall since f was finite-to-one, so $\mu \to 0$).

Suppose on the other hand that $\mathcal{I}$ does not add a rapid ultrafilter. Using Vojtáš's characterization again, since $\mathcal{P}(\omega)/\mathcal{I}$ does not add any new reals, there must be a condition $A \in \mathcal{I}^+$ and a tall summable ideal $\mathcal{J}$ such that $A \Vdash$ "$\dot{G} \cap \mathcal{J} = \emptyset$". Then necessarily $\mathcal{J} \restriction A \subseteq \mathcal{I} \restriction A$ and the identity map shows that $\mathcal{I} \restriction A$ is KB-above a summable ideal. Same argument shows that each $\mathcal{I}$-positive B below A is also Katětov-above a summable ideal. □

5 Canjar ultrafilters

On the other hand forcing with tall a analytic P-ideal gives a special kind of ultrafilter:

Definition 5.1 An ultrafilter $\mathcal{U}$ is *Canjar* if the forcing $\mathbb{M}_{\mathcal{U}}$ does not add a dominating real.

M. Canjar (in [2]) showed:

Proposition 5.2 (Canjar) *If $\mathcal{U}$ is Canjar, then it is a P-point and has no rapid RK-predecessors.*

Conjecture 5.3 (Canjar, Laflamme) *$\mathcal{U}$ is Canjar if and only if $\mathcal{U}$ is a P-point with no rapid RK-predecessor.*

Canjar filters have a combinatorial characterization due to M. Hrušák and H. Minami [7]:

Definition 5.4 Given a filter $\mathcal{F}$ on ω we let $\mathcal{F}^{<\omega}$ be the filter generated by $\{[F]^{<\omega} : F \in \mathcal{F}\}$.

Note that $[\mathcal{F}]^{<\omega}$ is a filter on $[\omega]^{<\omega}$ and it is *not* an ultrafilter even if $\mathcal{F}$ is.

Theorem 5.5 (Hrušák-Minami) *$\mathcal{U}$ is Canjar if and only if $\mathcal{U}^{<\omega}$ is a P^+-filter.*

In [8] we have shown, assuming $\mathrm{cov}(\mathcal{M}) = \mathfrak{c}$, that there are P-points with no rapid RK-predecessors which are, nevertheless, not Canjar. The following theorem shows, that such ultrafilters can be added by forcing with $\mathcal{P}(\omega)/\mathcal{I}$ for a tall F_σ P-ideal $\mathcal{I}$.

Theorem 5.6 *If $\mathcal{I}$ is a tall F_σ P-ideal, then $\mathcal{P}(\omega)/\mathcal{I}$ adds a P-point with no rapid RK-predecessors which is not Canjar.*

Proof By Observation 2.4 $\mathcal{P}(\omega)/\mathcal{I}$ adds a P-point and by Observation 2.2 it is σ-closed. We first show that the generic has no RB-predecessors. By Proposition 4.5 and the characterization Theorem 4.4 of rapid ultrafilters, it will be sufficient to show that for each $f : \omega \to \omega$ finite-to-one, $f_*(\mathcal{I})$ is a tall analytic P-ideal. Let $\mathcal{I} = Fin(\mu) = Exh(\mu)$ by Theorem 1.6. Define $\mu_*(A) = \mu(f^{-1}[A])$. Then μ_* is a submeasure on ω and since f is finite-to-one it is lower semicontinuous. It is easy to see that $f_*(\mathcal{I}) = Fin(\mu_*)$. It remains to verify that $Exh(\mu_*) = Fin(\mu_*)$. The $\subseteq$ inclusion is clear and for the other one we use the fact that, since f is finite-to-one, for each n there is $k \geq n$ such that $f^{-1}[A \setminus k] \subseteq f^{-1}[A] \setminus n$. Since RK-predecessors of P-points are its RB-predecessors we are finished.

We next show that the generic is not Canjar. To do this, we show that it fails the combinatorial condition of Theorem 5.5. Let $X_n = \{a \in [\omega]^{<\omega} : \mu(a) \geq (n+1)\}$. Clearly $\mathcal{P}(\omega)/\mathcal{I} \Vdash X_n \in (\dot{G}^{<\omega})^+$. Pick $A \in \mathcal{I}^+$ and let X be a pseudointersection of the X_n's. We shall find a stronger condition $B \subseteq A$, $B \in \mathcal{I}^+$ which will force X to be in $(\dot{G}^{<\omega})^*$. Let $g(n) = \min\{k : a \in X \setminus X_n \to a \subseteq k\}$. By increasing g we may assume $1 \leq \mu([g(n), g(n+1)) \cap A)$ and $\mu(\{x\}) \leq 1/8$ for each $x \in [g(n), g(n+1)) \cap A$. For n let $b_n \subseteq [g(n), g(n+1)) \cap A$ be minimal such that $1/4 \leq \mu(b_n)$. Then, by the minimality of b_n, $\mu(b_n) < 1/2$. Let $B = \bigcup_{n<\omega} b_n$. Then $1/4 \leq \mu(B \setminus n)$ for each n so $B \in \mathcal{I}^+$ by exhaustivity and clearly $B \subseteq A$. We will show that $[B]^{<\omega} \cap X = \emptyset$. Let $b \in [B]^{<\omega}$ and let $k = \max\{n : b \setminus g(n) \neq \emptyset\}$. If $b \in X$ then $b \in X_k$ by the definition of g. Then $\mu(b) \geq (k+1)$. However $b \subseteq \bigcup_{i\leq k} b_i$ so $\mu(b) \leq \sum_{i\leq k} \mu(b_i) < (k+1)/2 \leq k+1$ which is absurd. This finishes the proof. $\square$

One way to construct Canjar ultrafilters is to force with F_σ ideals (ordered by reverse inclusion). This suggests the following question:

Question 5.7 *Is there a Borel ideal $\mathcal{I}$ on ω such that $\mathcal{P}(\omega)/\mathcal{I}$ adds a Canjar ultrafilter?*

By the previous results such an ideal would have to be locally F_σ, locally KB-above a tall analytic P-ideal and locally not P.

6 Examples

We conclude by presenting a few illustrative examples.

Example 6.1 The ideal $\mathcal{ED}_{fin}$ adds a semiselective ultrafilter with a selective ultrafilter RB-below.

Proof First notice that $\mathcal{ED}_{fin} = Fin(\mu)$ where $\mu(A) = \min\{|K| : K \subseteq \omega^\omega \ \& \ A \subseteq \bigcup K\}$. This shows that $\mathcal{ED}_{fin}$ is F_σ so it is σ-closed and adds a P-point.

To show it adds a rapid ultrafilter, pick $\langle a_n : n < \omega\rangle$ a partition of Δ into finite sets and a condition $A \in \mathcal{ED}_{fin}{}^+$. We must find $B \subseteq A$, $B \in \mathcal{ED}_{fin}{}^+$ such that $|B \cap a_n| \leq n$. We shall actually find a B such that $|B \cap a_n| \leq n^2$ which is clearly sufficient. For each $n < \omega$ let $k_n = \min\{k : (\forall i < n)(a_i \subseteq k \times k \cap \Delta)\}$. By recursion pick an increasing sequence $\langle l_n : n < \omega\rangle$ such that $k_n < l_n$ and $|A \cap \{l_n\} \times l_n| \geq n$. Then choose $b_n \in [A \cap \{l_n\} \times l_n]^n$ and let $B = \bigcup_{n<\omega} b_n$. Clearly $B \in \mathcal{ED}_{fin}{}^+$ and, moreover, $|B \cap a_n| \leq n^2$ by the definition of k_n. This finishes the proof that $\mathcal{ED}_{fin}$ adds a rapid ultrafilter.

To see that the generic filter always has a selective ultrafilter below, let $\pi : \Delta \to \omega$ be the projection on the first coordinate. Given $A \in \mathcal{ED}_{fin}$ and $\langle I_n : n < \omega\rangle$ an interval partition of ω, choose an increasing sequence $\langle k_n : n < \omega\rangle$ such that $|\{k_n\} \times k_n \cap A| \geq n$. Then pick some infinite $N \subseteq \omega$ such that $(\forall j < \omega)(|\{k_n : n \in N\} \cap I_j| \leq 1)$, and let $B = \pi^{-1}\{k_n : n \in N\} \cap A$. Then clearly $B \in (\mathcal{ED}_{fin})^+$ and B forces that $\pi_*(G)$ contains a selector for the partition $\langle I_n : n < \omega\rangle$. $\square$

Recall that $Fin \times Fin = \{X \subseteq \omega \times \omega : (\forall^\infty k)(|X \cap \{k\} \times \omega| < \omega)\}$. Even though $Fin \times Fin$ is not an F_σ-ideal, $\mathcal{P}(\omega \times \omega)/Fin \times Fin$ is σ-closed (see [4] or [20]).

Example 6.2 The ideal $Fin \times Fin$ adds a Q-point which is not a P-point.

Proof To see that it does not add a P-point, notice that if we let $A_n = [n, \infty) \times \omega$ then $A_n \in (Fin \times Fin)^*$ so they will be in any generic. However any pseudointersection of the A_n's is in $Fin \times Fin$.

To see that the generic is a Q-point, fix $A \in \omega \times \omega$ positive and some partition $\langle a_n : n < \omega\rangle$ of $\omega \times \omega$ into finite sets. Enumerate $\{n : |\{n\} \times \omega \cap A| = \omega\}$ as $\langle n_k : k < \omega\rangle$ so that each number appears infinitely often. By induction choose $x_l \in A \cap \{n_l\} \times [l, \omega) \setminus \bigcup_{i<s(l)} a_i$ where $s(l) = \max\{i : (\exists j < l)(x_j \cap a_i \neq \emptyset)\}$. Then $B = \{x_l : l < \omega\}$ is a $Fin \times Fin$-positive subset of A which is a selector for the partition. This shows that the generic is a Q-point. $\square$

Example 6.3 Let $\mathcal{G}_C = \{X \subseteq [\omega]^2 : (\forall A \in [\omega]^\omega)([A]^2 \nsubseteq X)\}$ (see [15]). Then $\mathcal{G}_C$ adds a rapid ultrafilter which is neither a P-point nor a Q-point.

Notice that $\mathcal{P}([\omega]^2)/\mathcal{G}_C$ is not σ-closed, but does not add new reals, since it has a dense subset isomorphic to $\mathcal{P}(\omega)/fin$ (take the embedding $A \mapsto [A]^2$). This shows that the Question 2.3 can be answered in the negative for co-analytic ideals.

Proof of Example 6.3 Notice that $\mathcal{ED}_{fin} \leq_{\mathrm{KB}} \mathcal{G}_C$: let $F_n = \{\{m,n\} : m < n\}$ then $[\omega]^2 = \bigcup_{n<\omega} F_n$ and all selectors are in $\mathcal{G}_C$. Next $Fin \times Fin \leq_{\mathrm{KB}} \mathcal{G}_C$ let $I_n = \{\{m,n\} : n < m\}$ then $[\omega]^2 = \bigcup_{n<\omega} I_n$ and each infinite subset A of ω has $|[A]^2 \cap I_n| = \omega$ for infinitely many n (every $n \in A$). The first implies that the generic is not a Q-point while the second shows that it cannot be a P-point via the same argument as for $Fin \times Fin$.

To show that it adds a rapid ultrafilter, by Proposition 4.6 and homogeneity it suffices to show that $\mathcal{G}_C$ is not KB-above any tall summable ideal $\mathcal{I}_g$. So suppose $f : [\omega]^2 \to \omega$ is finite-to-one. Now construct a sequence $\langle n_i : i < \omega\rangle$ such that for each $i < j$, $g(f(\{n_i, n_j\})) < \frac{1}{j \cdot 2^j}$. This is easy to do and then $f''[\{n_i : i < \omega\}]^2 \in \mathcal{I}_g$, so $\mathcal{G}_C$ is not KB-above $\mathcal{I}_g$ via f. □

Definition 6.4 [9] If μ is a lscsm and there is a partition $\langle a_n : n < \omega\rangle$ of ω into finite sets and a sequence of submeasures $\langle \mu_n : n < \omega\rangle$, such that $\mu(A) = \sup\{\mu_n(A \cap a_n) : n < \omega\}$, then we say the submeasure μ is *fragmented*. The ideal $Fin(\mu)$ is then called a *fragmented ideal*.

Example 6.5 If $\mathcal{I}$ is a fragmented ideal, then $\mathcal{P}(\omega)/\mathcal{I}$ adds a P-point RB-above a selective ultrafilter.

Proof The forcing adds a P-point by Theorem 2.5. We show that the generic is RB-above a selective ultrafilter. Let $\langle a_n : n < \omega\rangle$ be the partition of ω witnessing the fragmentation of $\mathcal{I} = Fin(\mu)$. Fix some finite-to-one function such that $\mu(f^{-1}(n)) \geq n$ and each a_n is contained in some $f^{-1}(k)$. Suppose $A \in \mathcal{I}^+$ and $\langle X_n : n < \omega\rangle$ is a partition of ω. For $k < \omega$ choose n_k such that $\mu(f^{-1}(n_k) \cap A) \geq k$ (this is possible since $\mu(A) \leq \sup\{\mu(A \cap f^{-1}(n) : n < \omega\}$). Now either there is an infinite $X \subseteq \{n_k : k < \omega\}$ which is almost contained in some X_n or we can pick an infinite $X \subseteq \{n_k : k < \omega\}$ such that $|X \cap X_n| \leq 1$ for each $n < \omega$. Then $B = f^{-1}[X] \cap A \in \mathcal{I}^+$ and B forces that either some X_n is in the generic or there is a selector in the generic. □

Acknowledgements The research for this paper was initiated during the first author's sabbatical stay at the Mathematical Institute of the Czech Academy of Sciences and was concluded while the second author was visiting Instituto de Matemáticas Universidad Nacional Autónoma de México. The authors would like to thank both institutions for their support and hospitality.

References

1. Booth, D.: Ultrafilters on a countable set. Ann. Math. Log. **2**(1), 1–24 (1970/1971)
2. Canjar, R.M.: Mathias forcing which does not add dominating reals. Proc. Am. Math. Soc. **104**(4), 1239–1248 (1988)
3. Choquet, G.: Deux classes remarquables d'ultrafiltres sur N. Bull. Sci. Math. **92**, 143–153 (1968)
4. Dow, A.: Tree π-bases for $\beta\mathbb{N} \setminus \mathbb{N}$ in various models. Topol. Appl. **33**(1), 3–19 (1989)
5. Farah, I.: Analytic quotients: theory of liftings for quotients over analytic ideals on the integers. Mem. Am. Math. Soc. **148**(702), xvi+177 (2000)
6. Hernández-Hernández, F., Hrušák, M.: Cardinal invariants of analytic P-ideals. Can. J. Math. **59**(3), 575–595 (2007)
7. Hrušák, M., Minami, H.: Mathias-Prikry and Laver-Prikry type forcing (2010), preprint
8. Hrušák, M., Verner, J., Blass, A.: On strong P-points (2010), preprint
9. Hrušák, M., Zapletal, J., Rojas, D.: Cofinalities of Borel ideals (2010), preprint

10. Just, W., Krawczyk, A.: On certain Boolean algebras $\mathcal{P}(\omega)/I$. Trans. Am. Math. Soc. **285**(1), 411–429 (1984)
11. Katětov, M.: Products of filters. Comment. Math. Univ. Carol. **9**, 173–189 (1968)
12. Laflamme, C.: Forcing with filters and complete combinatorics. Ann. Pure Appl. Log. **42**(2), 125–163 (1989)
13. Mathias, A.R.D.: Happy families. Ann. Math. Log. **12**(1), 59–111 (1977)
14. Mazur, K.: F_σ-ideals and $\omega_1\omega_1^*$-gaps in the Boolean algebras $\mathcal{P}(\omega)/I$. Fundam. Math. **138**(2), 103–111 (1991)
15. Meza-Alcántara, D.: Ph.D. thesis, UNAM (2009)
16. Mokobodzki, G.: Ultrafiltres rapides sur N. Construction d'une densité relative de deux potentiels comparables, Séminaire de Théorie du Potentiel, dirigé par M. Brelot, G. Choquet et J. Deny: 1967/68, Exp. 12, Secrétariat mathématique, Paris (1969), p. 22
17. Rudin, W.: Homogeneity problems in the theory of Čech compactifications. Duke Math. J. **23**, 409–419 (1956)
18. Solecki, S.: Analytic ideals. Bull. Symb. Log. **2**(3), 339–348 (1996)
19. Solecki, S.: Analytic ideals and their applications. Ann. Pure Appl. Log. **99**(1–3), 51–72 (1999)
20. Szymański, A., Xua, Z.H.: The behaviour of ω^{2^*} under some consequences of Martin's axiom. In: General Topology and Its Relations to Modern Analysis and Algebra, V (Prague, 1981). Sigma Ser. Pure Math., vol. 3, pp. 577–584. Heldermann, Berlin (1983)
21. Vojtáš, P.: On ω^* and absolutely divergent series. Topol. Proc. **19**, 335–348 (1994)
22. Zapletal, J.: Preserving P-points in definable forcing. Fundam. Math. **204**(2), 145–154 (2009)