Integrability and complex structures adapted to smooth vector fields on the plane

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Abstract. We describe the relations between two integrability notions for C^∞ vector fields X on the plane. The first integrability notion is the existence of non trivial first integrals. The second is related to Cauchy–Riemann equations under suitable complex structures; it means that a vector field X is integrable when is J–complex analytic, under a suitable complex structure J (a priori not from complex multiplication by $\sqrt{-1}$). Geometrically, this last condition means that X admits a global flow box map outside of their singularities. Topological obstructions to both integrability notions are given.

1. Introduction

Any paracompact, Hausdorff, orientable, C^1, two–dimensional manifold $\mathcal{M}$ admits a complex structure J, *i.e.* $(\mathcal{M}, J)$ is a Riemann surface. We study the analogous problem for C^∞ vector fields X on the $\mathcal{M} = \mathbb{R}^2$, requiring that X becomes the real part, $\Re\mathbb{X}$, of a complex analytic vector field on a Riemann surface $(\mathcal{M}, J)$.

Let $X \in \mathfrak{X}^\infty(\mathbb{R}^2)$ be a vector field on $\mathbb{R}^2$. We consider two notions of integrability. X is *integrable* if there exists an integrating factor μ such that

$$\mu X = X_f, \qquad (I)$$

here X_f is the Hamiltonian vector field of a suitable C^∞ function f. The second notion seems more recent, X *admits a global flow box* if there exists a scaling factor ρ and a local diffeomorphism map (both of C^∞ class) $(g, f) : \mathbb{R}^2 \backslash \mathcal{Z}(X) \to \mathbb{R}^2$ such that

$$(g, f)_*(\rho X) = \frac{\partial}{\partial t} \qquad (GFB).$$

Geometrically it means that outside of the zeros $\mathcal{Z}(X)$, the associated foliation $\mathcal{F}(X)$ is a lift of the trivial foliation on $\mathbb{R}^2$. Liftable vector fields appear in many problems, in singularity theory Arnol'd [1] pp. 561 or du Plessis and