4.6 Montel’s Theorem

Let $X$ be a topological space. We denote by $\mathcal{C}(X)$ the set of complex valued continuous functions on $X$.

**Definition 4.26.** A topological space is called *separable* iff it contains a countable dense subset.

**Definition 4.27.** Let $X$ be a topological space, $F \subseteq \mathcal{C}(X)$. $F$ is called *pointwise bounded* iff for each $a \in X$ there is a constant $M > 0$ such that $|f(a)| < M$ for all $f \in F$. $F$ is called *locally bounded* iff for each $a \in X$ there is a constant $M > 0$ and a neighborhood $U \subseteq X$ of $a$ such that $|f(x)| < M$ for all $x \in U$ and for all $f \in F$.

**Definition 4.28.** Let $X$ be a topological space. A subset $F \subseteq \mathcal{C}(X)$ is called *equicontinuous at $a \in X$* iff for every $\epsilon > 0$ there exists a neighborhood $U \subseteq X$ of $a$ such that

$$|f(x) - f(y)| < \epsilon \quad \text{for all } x, y \in U.$$ 

A subset $F \subseteq \mathcal{C}(X)$ is called *locally equicontinuous* iff $F$ is equicontinuous at $a$ for all $a \in X$.

**Definition 4.29.** Let $X$ be a topological space. A subset $F \subseteq \mathcal{C}(X)$ is called *normal* iff every sequence of elements of $F$ has a subsequence that converges uniformly on every compact subset of $X$.

**Theorem 4.30 (Arzela-Ascoli).** Let $X$ be a separable topological space and $F \subseteq \mathcal{C}(X)$. Suppose that $F$ is pointwise bounded and locally equicontinuous. Then, $F$ is normal.

**Proof.** Let $\{f_n\}_{n \in \mathbb{N}}$ be a sequence of elements of $F$. We have to show that there exists a subsequence that converges uniformly on any compact subset of $X$. We encode subsequences of a sequence through infinite subsets of $\mathbb{N}$ in the obvious way. Let $\{x_k\}_{k \in \mathbb{N}}$ be a sequence of points which is dense in $X$. Set $N_0 := \mathbb{N}$ and construct iteratively $N_k \subseteq N_{k-1}$ as follows. The sequence $\{f_n(x_k)\}_{n \in N_{k-1}}$ is bounded by the assumption of pointwise boundedness of $F$. Thus there exists a convergent subsequence given by an infinite subset $N_k \subseteq N_{k-1}$. Proceeding in this way we obtain a sequence of decreasing infinite subsets $N_0 \supset N_1 \supset N_2 \supset \ldots$. Now consider the sequence $\{n_l\}_{l \in \mathbb{N}}$ of strictly increasing natural numbers $n_l$ obtained as follows: $n_l$ is the $l$th element of the set $N_l$. It is then clear that the sequence $\{f_{n_l}(x_k)\}_{l \in \mathbb{N}}$ converges for every $k \in \mathbb{N}$. 
Now let $K \subseteq X$ be compact and choose $\epsilon > 0$. Since $F$ is locally equicontinuous, we find for each $a \in K$ an open neighborhood $U_a \subseteq X$ such that $|f(x) - f(y)| < \epsilon$ for all $f \in F$ if $x, y \in U_a$. Since $K$ is compact there are finitely many points $a_1, \ldots, a_m \in K$ such that $U_{a_1}, \ldots, U_{a_m}$ cover $K$. Since $\{x_k\}_{k \in \mathbb{N}}$ is dense in $X$ there exists for each $j \in 1, \ldots, m$ an index $k_j$ such that $x_{k_j} \in U_{a_j}$. Now, $\{f_{n_l}(x_{k_j})\}_{l \in \mathbb{N}}$ converges and is Cauchy for all $j \in 1, \ldots, m$. In particular, by taking a maximum if necessary we can find $l_0 \in \mathbb{N}$ such that $|f_{n_l}(x_{k_j}) - f_{n_l}(x_{k_j})| < \epsilon$ for all $i, l \geq l_0$ and for all $j \in 1, \ldots, m$.

Now fix $p \in K$. Then, there is $j \in 1, \ldots, m$ such that $p \in U_{a_j}$. For $i, l \geq l_0$ we thus obtain the estimate

$$|f_{n_i}(p) - f_{n_l}(p)| \leq |f_{n_i}(p) - f_{n_l}(x_{k_j})| + |f_{n_l}(x_{k_j}) - f_{n_l}(x_{k_j})| + |f_{n_l}(x_{k_j}) - f_{n_l}(p)| < 3\epsilon.$$

In particular, this implies that $\{f_{n_l}\}_{l \in \mathbb{N}}$ converges uniformly on $K$. \hfill \Box

**Theorem 4.31 (Montel).** Let $D \subseteq \mathbb{C}$ be a region and $F \subseteq \mathcal{O}(D)$. Suppose that $F$ is locally bounded. Then, $F$ is normal.

**Proof.** We show that $F$ is locally equicontinuous. The result follows then from the Arzela-Ascoli Theorem 4.30. Let $z_0 \in D$ and choose $\epsilon > 0$. Since $F$ is locally bounded, there exists a constant $M > 0$ and $r > 0$ with $B_{2r}(z_0) \subseteq D$ and such that $|f(z)| < M$ for all $z \in B_{2r}(z_0)$ and all $f \in F$. The Cauchy Integral Formula (Theorem 2.20) yields for all $f \in F$ and $z, w \in B_{2r}(z_0)$

$$f(z) - f(w) = \frac{1}{2\pi i} \int_{\partial B_{2r}(z_0)} \left( \frac{f(\xi)}{\xi - z} - \frac{f(\xi)}{\xi - w} \right) d\xi$$

$$= \frac{z - w}{2\pi i} \int_{\partial B_{2r}(z_0)} \frac{f(\xi)}{(\xi - z)(\xi - w)} d\xi.$$

If we restrict to $z, w \in B_r(z_0)$ we have the estimate $|(|\xi - z)(\xi - w)| > r^2$ for all $\xi \in \partial B_{2r}(z_0)$. Combining this with the standard integral estimate (Proposition 2.7) we obtain,

$$|f(z) - f(w)| \leq |z - w| \frac{2\|f\|_{\partial B_{2r}(z_0)}}{r} < |z - w| \frac{2M}{r}.$$

Choosing $\delta := \min \{r; \frac{r}{4M}\}$ yields the estimate

$$|f(z) - f(w)| < \epsilon \quad \forall z, w \in B_\delta(z_0),$$

showing local equicontinuity. This completes the proof. \hfill \Box
**Exercise 58.** Let $X$ be a metric space and $F \subseteq C(X)$. Suppose that $F$ is normal. Show that $F$ is locally bounded.

**Exercise 59 (Vitali’s Theorem).** Let $D \subseteq \mathbb{C}$ be a region and $\{f_n\}_{n \in \mathbb{N}}$ a locally bounded sequence of holomorphic functions on $D$. Let $f \in \mathcal{O}(D)$ and $A := \{z \in D : \lim_{n \to \infty} f_n(z) \text{ exists and } f(z) = \lim_{n \to \infty} f_n(z)\}$. Suppose that $A$ has a limit point in $D$. Show that $f_n \to f$ uniformly on compact subsets of $D$ for $n \to \infty$.

### 4.7 The Riemann Mapping Theorem

**Proposition 4.32.** Let $D \subseteq \mathbb{C}$ be a region and $\{f_n\}_{n \in \mathbb{N}}$ a sequence of holomorphic functions $f_n \in \mathcal{O}(D)$ that converges uniformly on any compact subset of $D$ to $f$. Then, $f \in \mathcal{O}(D)$ and the sequence $\{f_n^{(k)}\}_{n \in \mathbb{N}}$ converges uniformly on any compact subset of $D$ to $f^{(k)}$ for all $k \in \mathbb{N}$.

**Proof.** Let $z_0 \in D$ and set $r > 0$ such that $B_r(z_0) \subseteq D$. By Corollary 2.15 $f_n$ is integrable in $B_r(z_0)$. For any closed path $\gamma$ in $B_r(z_0)$ we thus have

$$\int_{\gamma} f = \int_{\gamma} \lim_{n \to \infty} f_n = \lim_{n \to \infty} \int_{\gamma} f_n = 0,$$

where we have used Proposition 2.8 to interchange the integral with the limit. Thus, $f$ is integrable in $B_r(z_0)$ and hence holomorphic there by Corollary 2.23. Since the choice of $z_0$ was arbitrary we find that $f$ is holomorphic in all of $D$.

Fix $k \in \mathbb{N}$ and consider $z_0 \in D$. Choose $r > 0$ such that $B_{2r}(z_0) \subseteq D$. Now for each $z \in B_r(z_0)$ we have the Cauchy estimate (Proposition 2.31),

$$|f_n^{(k)}(z) - f^{(k)}(z)| \leq k! \frac{1}{r^k} \|f_n - f\|_{\partial B_r(z)} \leq k! \frac{1}{r^k} \|f_n - f\|_{B_{2r}(z_0)}.$$

For $\epsilon > 0$ there is by uniform convergence of $\{f_n\}_{n \in \mathbb{N}}$ an $n_0 \in \mathbb{N}$ such that $|f_n(z) - f(z)| < \epsilon n^k/k!$ for all $n \geq n_0$ and all $z \in B_{2r}(z_0)$. Hence, $|f_n^{(k)}(z) - f^{(k)}(z)| < \epsilon$ for all $n \geq n_0$ and all $z \in B_r(z_0)$. That is, $\{f_n^{(k)}\}_{n \in \mathbb{N}}$ converges to $f^{(k)}$ uniformly on some neighborhood of every point of $D$. To obtain uniform convergence on a compact subset $K \subseteq D$ it is merely necessary to cover $K$ with finitely many such neighborhoods.

**Theorem 4.33 (Hurwitz).** Let $D \subseteq \mathbb{C}$ be a region and $\{f_n\}_{n \in \mathbb{N}}$ a sequence of functions $f_n \in \mathcal{O}(D)$ converging uniformly in every compact subset of $D$ to $f$. Let $a \in D$ and $r > 0$ such that $B_r(a) \subseteq D$. Suppose that $f(z) \neq 0$ for all $z \in \partial B_r(a)$. Then, there exists $n_0 \in \mathbb{N}$ such that $f$ and $f_n$ have the same number of zeros in $B_r(a)$ for all $n \geq n_0$. 


Proof. Set \( \delta := \inf\{|f(z)| : z \in \partial B_r(a)\} \). By the assumptions \( \delta > 0 \) and \( \{f_n\}_{n \in \mathbb{N}} \) converges uniformly on \( \partial B_r(a) \). Thus, there exists \( n_0 \in \mathbb{N} \) such that \( |f_n(z) - f(z)| < \delta/2 \) for all \( n \geq n_0 \) and all \( z \in \partial B_r(a) \). But this implies,

\[
|f(z) - f_n(z)| < \frac{\delta}{2} < |f(z)| \leq |f(z)| + |f_n(z)| \quad \forall n \geq n_0, \forall z \in \partial B_r(a).
\]

Applying Rouché’s Theorem 3.21 yields the desired result. \( \Box \)

**Proposition 4.34.** Let \( D \subseteq \mathbb{C} \) be a region and \( \{f_n\}_{n \in \mathbb{N}} \) a sequence of functions \( f_n \in \mathcal{O}(D) \) converging uniformly in every compact subset of \( D \) to \( f \). Suppose that for all \( n \in \mathbb{N} \), \( f_n \) has no zeros. Then, either \( f = 0 \) or \( f \) has no zeros.

**Proof. Exercise.** \( \Box \)

**Proposition 4.35.** Let \( D \subseteq \mathbb{C} \) be a region and \( \{f_n\}_{n \in \mathbb{N}} \) a sequence of injective functions \( f_n \in \mathcal{O}(D) \) converging uniformly in every compact subset of \( D \) to \( f \). Then, either \( f \) is constant or \( f \) is injective.

**Proof.** Suppose that \( f \) is not constant. Let \( a \in D \) and set \( p := f(a) \) and \( p_n := f_n(a) \) for all \( n \in \mathbb{N} \). By injectivity \( f_n - p_n \) never vanishes on \( D \setminus \{a\} \). On the other hand, the sequence \( \{f_n - p_n\}_{n \in \mathbb{N}} \) converges uniformly in any compact subset of \( D \) to \( f - p \). Since \( f - p \neq 0 \), Proposition 4.34 implies that \( f - p \) has no zeros in \( D \setminus \{a\} \). In other words, \( f \) does not take the value \( p \) at any point of \( D \setminus \{a\} \). Since we chose \( a \) arbitrarily it follows that \( f \) is injective. \( \Box \)

**Theorem 4.36** (Riemann Mapping Theorem). Every homologically simply connected region which is different from \( \mathbb{C} \) is conformally equivalent to \( \mathbb{D} \).

**Proof.** Let \( D \) be the region in question. Fix \( z_0 \in D \) arbitrarily. Let \( F \subseteq \mathcal{O}(D) \) be the set of holomorphic functions \( f \in \mathcal{O}(D) \) which are injective, whose image is contained in \( \mathbb{D} \) and such that \( f(z_0) = 0 \). Our strategy is to find an element of \( F \) which is a biholomorphism \( D \to \mathbb{D} \).

First we show that \( F \) is not empty. By assumption \( D \neq \mathbb{C} \), so we can choose \( a \in \mathbb{C} \setminus D \). The function \( f(z) := z - a \) is holomorphic and zero-free in \( D \), so according to Theorem 4.25 there is a holomorphic square root \( g \in \mathcal{O}(D) \) with \( g^2 = f \). If \( g(z_1) = g(z_2) \) then \( (g(z_1))^2 = (g(z_2))^2 \) and so \( z_1 = z_2 \) since \( f \) is injective. Therefore also \( g \) is injective. Moreover, if \( g(z_1) = -g(z_2) \) we can draw the same conclusion \( z_1 = z_2 \), but this time we get a contradiction, since \( g \) is zero-free. Thus, if \( z \in \mathbb{C} \) is in the image of \( g \), then \(-z\) cannot be in the image of \( g \). Now since \( g \) is not constant the
Open Mapping Theorem 2.40 ensures that \( g(D) \) is open. In particular there exists \( w \in \mathbb{C} \) and \( r > 0 \) such that \( \overline{B_r(w)} \subseteq g(D) \). But applying the previous statement to all elements of \( B_r(w) \) we obtain \( \overline{B_r(-w)} \cap g(D) = \emptyset \). It is now easy to see that the function \( h \in \mathcal{O}(D) \) defined by \( h(z) := r/(g(z) + w) \) is also injective and satisfies \( h(D) \subseteq \mathbb{D} \). Setting \( v := h(z_0) \), we have \( D_v \circ h \in F \) since \( D_v \in \text{Aut}(\mathbb{D}) \) and \( D_v(v) = 0 \).

Since \( D \) is open, there exists \( r > 0 \) such that \( \overline{B_r(z_0)} \subset D \). Using the Cauchy estimate (Proposition 2.31) we find the bound \( |f'(z_0)| < 1/r \) for all \( f \in F \). This implies that

\[
M := \sup\{|f'(z_0)| : f \in F\}
\]

is well defined. On the other hand we will show that if \( f(D) \neq \mathbb{D} \) for some \( f \in F \), then there exists \( g \in F \) such that \( |g'(z_0)| > |f'(z_0)| \). This implies that \( h \in F \) is a biholomorphism \( D \rightarrow \mathbb{D} \) iff \( |h'(z_0)| = M \). We will then show that such an \( h \) exists.

Consider some \( f \in F \) such that \( f(D) \neq \mathbb{D} \). Choose \( p \in \mathbb{D} \setminus f(D) \). Since \( D_p \in \text{Aut}(\mathbb{D}) \), the composition \( D_p \circ f \) is injective and \( D_p \circ f(D) \subset \mathbb{D} \). Furthermore, \( D_p \circ f \) is zero-free since \( D_p^{-1}(0) = \{p\} \). Since \( D \) is homologically simply connected we can find a holomorphic square root \( g \in \mathcal{O}(D) \) with \( g^2 = D_p \circ f \) according to Theorem 4.25. In fact, it is clear that \( g \) is injective and \( g(D) \subseteq \mathbb{D} \). Set \( w := g(z_0) \). Then \( h := D_w \circ g \in F \). Consider now the holomorphic map \( k : \mathbb{D} \rightarrow \mathbb{D} \) given by \( k(z) = D_p((D_w(z))^2) \). Then, \( f = k \circ h \) and applying the chain rule for derivatives we obtain

\[
f'(z_0) = k'(h(z_0))h'(z_0) = k'(0)h'(z_0).
\]

Noting that \( k(0) = 0 \) we can apply the Schwarz Lemma 4.11. Since \( k \) is not a rotation, this implies \( |k'(0)| < 1 \). Hence, \( |f'(z_0)| < |h'(z_0)| \) since \( h'(z_0) \neq 0 \) by injectivity of \( h \).

The image of all functions in \( F \) is contained in the bounded set \( \mathbb{D} \), so in particular \( F \) is locally bounded. According to Montel’s Theorem 4.31 this implies that \( F \) is normal. Consider now a sequence \( \{f_n\}_{n \in \mathbb{N}} \) of elements of \( F \) such that \( |f_n'(z_0)| \rightarrow M \) as \( n \rightarrow \infty \). Since \( F \) is normal, there is a subsequence \( \{f_{n_k}\}_{k \in \mathbb{N}} \) which converges uniformly on any compact subset of \( D \) to a function \( f \in \mathcal{O}(D) \) by Proposition 4.32. By the same Proposition we have convergence of the derivative and thus \( |f'(z_0)| = M \) as desired. It remains to show that \( f \in F \). From the limit process it is clear that \( f(z_0) = 0 \) and \( f(D) \subseteq \overline{\mathbb{D}} \). Since \( f \) is not constant (in particular, \( f'(z_0) \neq 0 \)) the Open Mapping Theorem 2.40 implies that \( f(D) \) must be open and so we must have \( f(D) \subseteq \mathbb{D} \). The injectivity of \( f \) follows from Proposition 4.35. Hence \( f \in F \). This completes the proof. \( \Box \)
Proposition 4.37. Let \( D \subseteq \mathbb{C} \) be a homologically simply connected region, \( a \in D \). Then, there exists exactly one biholomorphism \( f : D \to \mathbb{D} \) such that \( f(a) = 0 \) and \( f'(a) > 0 \).

Proof. Exercise.

Exercise 60. Show that a homologically simply connected region cannot be conformally equivalent to a region that is not homologically simply connected.