6 The Riemann Sphere

6.1 Definition

Definition 6.1. A topological space is called *locally compact* iff every point has a compact neighborhood.

Proposition 6.2 (One-Point Compactification)**.** *Let X be a Hausdorff topological space that is locally compact. Consider the set $\hat{X} := X \cup \{\infty\}$ equipped with the following topology: A set $U \subseteq \hat{X}$ is open iff $U \subseteq X$ and U is open in X or if $U = V \cup \{\infty\}$ where $V \subseteq X$ such that $X \setminus V$ is compact in X. Then, $\hat{X}$ is a compact Hausdorff space.*

Proof. **Exercise.** □

Remark 6.3. The metric introduced above can be obtained from the *stereographic projection* of $\hat{\mathbb{C}}$ identified with the unit disk to the complex plane.

Proposition 6.4. *Consider the topological space $\hat{\mathbb{C}}$ with the subsets $U_0 := \hat{\mathbb{C}} \setminus \{\infty\}$ and $U_\infty := \hat{\mathbb{C}} \setminus \{0\}$. Consider the maps $\phi_0 : U_0 \to \mathbb{C}$ given by $\phi_0(z) := z$ for all $z \in U_0$ and $\phi_\infty : U_\infty \to \mathbb{C}$ given by $\phi_\infty(z) := 1/z$ for all $z \in U_\infty \setminus \{\infty\}$ and $\phi_\infty(\infty) := 0$. Then, ϕ_0 and ϕ_∞ are homeomorphisms. Moreover, $\phi_0 \circ \phi_\infty^{-1}|_{\mathbb{C}\setminus\{0\}}$ is the biholomorphism $\mathbb{C} \setminus \{0\} \to \mathbb{C} \setminus \{0\}$ given by $z \mapsto 1/z$.*

Remark 6.5. The topological space $\hat{\mathbb{C}}$ together with the structures introduced in the preceding Proposition is called the *Riemann sphere*. It is an example of a *complex manifold*. The maps ϕ_0, ϕ_∞ are called *charts*.

Exercise 62. Let $\{z_n\}_{n\in\mathbb{N}}$ be a sequence of complex numbers such that for each $M > 0$ there exists $n_0 \in \mathbb{N}$ such that $|z_n| > M$ for all $n \geq n_0$. Show that $\lim_{n\to\infty} z_n = \infty$ in $\hat{\mathbb{C}}$.

Exercise 63. Consider the symmetric function $d : \hat{\mathbb{C}} \times \hat{\mathbb{C}} \to \mathbb{R}_0^+$ given by

$$
\begin{aligned}
d(z, z') &:= \frac{2|z - z'|}{\sqrt{(1 + |z|^2)(1 + |z'|^2)}} \quad \forall z, z' \in \mathbb{C} \\
d(\infty, z) &:= \frac{2}{\sqrt{1 + |z|^2}} \quad \forall z \in \mathbb{C} \\
d(\infty, \infty) &:= 0.
\end{aligned}
$$

Show that d defines a metric on the Riemann sphere that is compatible with its topology.

6.2 Functions on $\hat{\mathbb{C}}$

<u>Exercise</u> 64. Let $D \subseteq \hat{\mathbb{C}}$ be a region and $f : D \to \mathbb{C}$ be continuous. Let $a \in D \setminus \{0, \infty\}$. Show that $f \circ \phi_0^{-1}$ is holomorphic/conformal at $\phi_0(a)$ iff $f \circ \phi_\infty^{-1}$ is holomorphic/conformal at $\phi_\infty(a)$.

Definition 6.6. Let $D \subseteq \hat{\mathbb{C}}$ be a region and $f : D \to \mathbb{C}$ be continuous. Let $a \in D$. If $a \neq \infty$, we say that f is holomorphic/conformal at a iff $f \circ \phi_0^{-1}$ is holomorphic/conformal at $\phi_0(a)$. If $a \neq 0$, we say that f is holomorphic/conformal at a iff $f \circ \phi_\infty^{-1}$ is holomorphic/conformal at $\phi_\infty(a)$. We say that f is holomorphic/conformal in D iff f is holomorphic/conformal at each point $a \in D$.

<u>Exercise</u> 65. Let $D \subseteq \hat{\mathbb{C}}$ be a region and $a \in D \setminus \{0, \infty\}$. Let $f \in \mathcal{O}(D \setminus \{a\})$. Show that the type and order of the singularity of $f \circ \phi_0^{-1}$ at $\phi_0(a)$ is the same as the type and order of the singularity of $f \circ \phi_\infty^{-1}$ at $\phi_\infty(a)$.

Definition 6.7. Let $D \subseteq \hat{\mathbb{C}}$ be a region, $a \in D$ and $f \in \mathcal{O}(D \setminus \{a\})$. If $a \neq \infty$, we say that f has a removable singularity/a pole of order n/an essential singularity at a iff $f \circ \phi_0^{-1}$ has a removable singularity/a pole of order n/an essential singularity at $\phi_0(a)$. If $a \neq 0$, we say that f has a removable singularity/a pole of order n/an essential singularity at a iff $f \circ \phi_\infty^{-1}$ has a removable singularity/a pole of order n/an essential singularity at $\phi_\infty(a)$.

Proposition 6.8. *Let $f \in \mathcal{O}(\hat{\mathbb{C}})$. Then, f is constant.*

Proof. **<u>Exercise</u>.** □

Definition 6.9. Let $D \subseteq \hat{\mathbb{C}}$ be a region and $A \subset D$ be a discrete and relatively closed subset. A function $f \in \mathcal{O}(D \setminus A)$ is called *meromorphic* iff each point $a \in A$ is either a removable singularity or a pole of f.

Proposition 6.10. *Let $f \in \mathcal{M}(\hat{\mathbb{C}})$. Then, f is a rational function.*

Proof. **<u>Exercise</u>.**[Hint: First assume that f has a pole only at ∞ and show that $|f(z)| < M|z|^n$ for some constants $M > 0$ and $n \in \mathbb{N}$. Conclude that f must be a polynomial. In the general case show and use the fact that f can only have finitely many poles.] □

6.3 Functions onto $\hat{\mathbb{C}}$ and $\mathrm{Aut}(\hat{\mathbb{C}})$

<u>Exercise</u> 66. Let $D \subseteq \hat{\mathbb{C}}$ be a region and $f \in \mathcal{M}(D)$. Let $P \subset D$ be the set of poles of f and $Z \subseteq D$ the set of zeros of f. Define $\hat{f} : D \to \hat{\mathbb{C}}$ by $\hat{f}(z) := \phi_0^{-1}(f(z))$ if $z \in D \setminus P$ and $\hat{f}(z) := \infty$ if $z \in P$. Show that $\hat{f}$ is continuous and that $\phi_0 \circ \hat{f}|_{D \setminus P}$ as well as $\phi_\infty \circ \hat{f}|_{D \setminus Z}$ are holomorphic.

<u>**Exercise**</u> **67.** Let $D \subseteq \hat{\mathbb{C}}$ be a region and $\hat{f} : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be continuous. Let $Z := \{z \in \hat{\mathbb{C}} : \hat{f}(z) = 0\}$ and $P := \{z \in \hat{\mathbb{C}} : \hat{f}(z) = \infty\}$. Suppose that $\phi_0 \circ \hat{f}|_{D\setminus P}$ as well as $\phi_\infty \circ \hat{f}|_{D\setminus Z}$ are holomorphic. Define $f : D \setminus P \to \mathbb{C}$ by $f := \phi_0 \circ \hat{f}|_{D\setminus P}$. If $P \neq D$, then $f \in \mathcal{M}(D)$.

Definition 6.11. Let $D \subseteq \hat{\mathbb{C}}$ be a region and $f : D \to \hat{\mathbb{C}}$ be continuous. Let $a \in D$. If $f(a) \neq \infty$, we say that f is conformal at a iff $\phi_0 \circ f$ is conformal at a. If $f(a) \neq 0$, we say that f is conformal at a iff $\phi_\infty \circ f$ is conformal at a. We say that f is conformal in D iff f is conformal at each point $a \in D$.

Definition 6.12. A conformal mapping $\hat{\mathbb{C}} \to \hat{\mathbb{C}}$ that has a conformal inverse is called a *conformal automorphism* of $\hat{\mathbb{C}}$.

Proposition 6.13. *Möbius transformations are conformal automorphisms of* $\hat{\mathbb{C}}$.

Proof. <u>**Exercise**</u>. □

Theorem 6.14. *Suppose that* $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ *is conformal and injective. Then,* f *is a Möbius transformation.*

Proof. (Sketch.) As in Exercise 67 we can think of f as a meromorphic function on $\hat{\mathbb{C}}$. Thus, by Proposition 6.10, f is rational, i.e., a quotient p/q of polynomials. Without loss of generality we may assume p and q not to have common divisors. Since f is injective, p can only have one zero which must be simple. Similarly, q can only have one pole which must be simple. Thus, f is a Möbius transformation. □

Corollary 6.15. $\mathrm{Aut}(\hat{\mathbb{C}}) = \boldsymbol{Möb}$.

Theorem 6.16. *Let* (a, b, c) *and* (a', b', c') *be triples of distinct points in* $\hat{\mathbb{C}}$. *Then, there exists exactly one Möbius transformation* f *such that* $f(a) = a'$, $f(b) = b'$, $f(c) = c'$.

Proof. <u>**Exercise**</u>. □