FUNCTIONAL ANALYSIS – Semester 2012-1

Contents

1 Topological and metric spaces

1.1 Basic Definitions

Definition 1.1 (Topology)**.** Let S be a set. A subset $\mathcal{T}$ of the set $\mathfrak{P}(S)$ of subsets of S is called a *topology* iff it has the following properties:

- $\emptyset \in \mathcal{T}$ and $S \in \mathcal{T}$.
- Let $\{U_i\}_{i\in I}$ be a family of elements in $\mathcal{T}$. Then $\bigcup_{i\in I} U_i \in \mathcal{T}$.
- Let $U, V \in \mathcal{T}$. Then $U \cap V \in \mathcal{T}$.

A set equipped with a topology is called a *topological space*. The elements of $\mathcal{T}$ are called the *open* sets in S. A complement of an open set in S is called a *closed* set.

Definition 1.2. Let S be a topological space and $x \in S$. Then a subset $U \subseteq S$ is called a *neighborhood* of x iff it contains an open set which in turn contains x.

Definition 1.3. Let S be a topological space and U a subset. The *closure* $\overline{U}$ of U is the smallest closed set containing U. The *interior* $\mathring{U}$ of U is the largest open set contained in U.

Definition 1.4 (base)**.** Let $\mathcal{T}$ be a topology. A subset $\mathcal{B}$ of $\mathcal{T}$ is called a *base* of $\mathcal{T}$ iff the elements of $\mathcal{T}$ are precisely the unions of elements of $\mathcal{B}$. It is called a *subbase* iff the elements of $\mathcal{T}$ are precisely the finite intersections of unions of elements of $\mathcal{B}$.

Proposition 1.5. *Let S be a set and $\mathcal{B}$ a subset of $\mathfrak{P}(S)$. $\mathcal{B}$ is the base of a topology on S iff it satisfies all of the following properties:*

- $\emptyset \in \mathcal{B}$.
- *For every $x \in S$ there is a set $U \in \mathcal{B}$ such that $x \in U$.*
- *Let $U, V \in \mathcal{B}$. Then there exits a family $\{W_\alpha\}_{\alpha\in A}$ of elements of $\mathcal{B}$ such that $U \cap V = \bigcup_{\alpha\in A} W_\alpha$.*

Proof. **<u>Exercise</u>.** □

Definition 1.6 (Filter)**.** Let S be a set. A subset $\mathcal{F}$ of the set $\mathfrak{P}(S)$ of subsets of S is called a *filter* iff it has the following properties:

- $\emptyset \notin \mathcal{F}$ and $S \in \mathcal{F}$.

- Let $U, V \in \mathcal{F}$. Then $U \cap V \in \mathcal{F}$.
- Let $U \in \mathcal{F}$ and $U \subseteq V \subseteq S$. Then $V \in \mathcal{F}$.

Definition 1.7. Let $\mathcal{F}$ be a filter. A subset $\mathcal{B}$ of $\mathcal{F}$ is called a *base* of $\mathcal{F}$ iff every element of $\mathcal{F}$ contains an element of $\mathcal{B}$.

Proposition 1.8. *Let S be a set and $\mathcal{B} \subseteq \mathfrak{P}(S)$. Then $\mathcal{B}$ is the base of a filter on S iff it satisfies the following properties:*

- *$\emptyset \notin \mathcal{B}$ and $\mathcal{B} \neq \emptyset$.*
- *Let $U, V \in \mathcal{B}$. Then there exists $W \in \mathcal{B}$ such that $W \subseteq U \cap V$.*

Proof. **Exercise**. □

Let S be a topological space and $x \in S$. It is easy to see that the set of neighborhoods of x forms a filter. It is called the *filter of neighborhoods* of x and denoted by $\mathcal{N}_x$. The family of filters of neighborhoods in turn encodes the topology:

Proposition 1.9. *Let S be a topological space and $\{\mathcal{N}_x\}_{x\in S}$ the family of filters of neighborhoods. Then a subset U of S is open iff for every $x \in U$, there is a set $W_x \in \mathcal{N}_x$ such that $W_x \subseteq U$.*

Proof. **Exercise**. □

Proposition 1.10. *Let S be a set and $\{\mathcal{F}_x\}_{x\in S}$ an assignment of a filter to every point in S. Then this family of filters are the filters of neighborhoods of a topology on S iff they satisfy the following properties:*

1. *For all $x \in S$, every element of $\mathcal{F}_x$ contains x.*
2. *For all $x \in S$ and $U \in \mathcal{F}_x$, there exists $W \in \mathcal{F}_x$ such that $U \in \mathcal{F}_y$ for all $y \in W$.*

Proof. If $\{\mathcal{F}_x\}_{x\in S}$ are the filters of neighborhoods of a topology it is clear that the properties are satisfied: 1. Every neighborhood of a point contains the point itself. 2. For a neighborhood U of x take W to be the interior of U. Then W is a neighborhood for each point in W.

Conversely, suppose $\{\mathcal{F}_x\}_{x\in S}$ satisfies Properties 1 and 2. Given x we define an open neighborhood of x to be an element $U \in \mathcal{F}_x$ such that $U \in \mathcal{F}_y$ for all $y \in U$. This definition is not empty since at least S itself is an open neighborhood of every point x in this way. Moreover, for any $y \in U$, by the same definition, U is an open neighborhood of y. Now take $y \notin U$. Then, by Property 1, U is not an open neighborhood of y. Thus, we obtain a good definition of open set: An open set is a set that is an open neighborhood for one (and thus any) of its points. We also declare the empty set to be open.

We proceed to verify the axioms of a topology. Property 1 of Definition 1.1 holds since S is open and we have declared the empty set to be open. Let $\{U_\alpha\}_{\alpha\in I}$ be a family of open sets and consider their union $U = \bigcup_{\alpha\in I} U_\alpha$. Assume U is not empty (otherwise it is trivially open) and pick $x \in U$. Thus, there is $\alpha \in I$ such that $x \in U_\alpha$. But then $U_\alpha \in \mathcal{F}_x$ and also $U \in \mathcal{F}_x$. This is true for any $x \in U$. Hence, U is open. Consider now open sets U and V. Assume the intersection $U \cap V$ to be non-empty (otherwise its openness is trivial) and pick a point x in it. Then $U \in \mathcal{F}_x$ and $V \in \mathcal{F}_x$ and therefore $U \cap V \in \mathcal{F}_x$. The same is true for any point in $U \cap V$, hence it is open.

It remains to show that $\{\mathcal{F}_x\}_{x\in S}$ are the filters of neighborhoods for the topology just defined. It is already clear that any open neighborhood of a point x is contained in $\mathcal{F}_x$. We need to show that every element of $\mathcal{F}_x$ contains an open neighborhood of x. Take $U \in \mathcal{F}_x$. We define W to be the set of points y such that $U \in \mathcal{F}_y$. This cannot be empty as $x \in W$. Moreover, Property 1 implies $W \subseteq U$. Let $y \in W$, then $U \in \mathcal{F}_y$ and we can apply Property 2 to obtain a subset $V \subseteq W$ with $V \in \mathcal{F}_y$. But this implies $W \in \mathcal{F}_y$. Since the same is true for any $y \in W$ we find that W is an open neighborhood of x. This completes the proof. □

Definition 1.11 (Continuity)**.** Let S, T be topological spaces. A map $f : S \to T$ is called *continuous at* $p \in S$ iff $f^{-1}(\mathcal{N}_{f(p)}) \subseteq \mathcal{N}_p$. f is called *continuous* iff it is continuous at every $p \in S$. We denote the space of continuous maps from S to T by $\mathrm{C}(S, T)$.

Proposition 1.12. *Let S, T be topological spaces and $f : S \to T$ a map. Then, f is continuous iff for every open set $U \in T$ the preimage $f^{-1}(U)$ in S is open.*

Proof. **Exercise.** □

Proposition 1.13. *Let S, T, U be topological spaces, $f \in \mathrm{C}(S, T)$ and $g \in \mathrm{C}(T, U)$. Then, the composition $g \circ f : S \to U$ is continuous.*

Proof. Immediate. □

Definition 1.14. Let S, T be topological spaces. A bijection $f : S \to T$ is called a *homeomorphism* iff f and f^{-1} are both continuous. If such a homeomorphism exists S and T are called *homeomorphic*.

Definition 1.15. Let $\mathcal{T}_1$, $\mathcal{T}_2$ be topologies on the set S. Then, $\mathcal{T}_1$ is called *finer* than $\mathcal{T}_2$ and $\mathcal{T}_2$ is called *coarser* than $\mathcal{T}_1$ iff all open sets of $\mathcal{T}_2$ are also open sets of $\mathcal{T}_1$.

Definition 1.16 (Induced Topology)**.** Let S be a topological space and U a subset. Consider the topology given on U by the intersection of each open set on S with U. This is called the *induced topology* on U.

Definition 1.17 (Product Topology)**.** Let S be the cartesian product $S = \prod_{\alpha\in I} S_\alpha$ of a family of topological spaces. Consider subsets of S of the form $\prod_{\alpha\in I} U_\alpha$ where finitely many U_α are open sets in S_α and the others coincide with the whole space $U_\alpha = S_\alpha$. These subsets form the base of a topology on S which is called the *product topology.*

Exercise 1. Let S be the cartesian product $S = \prod_{\alpha\in I} S_\alpha$ of a family of topological spaces. Show that the product topology is the coarsest topology on S that makes all projections $S \to S_\alpha$ continuous.

Proposition 1.18. *Let S, T, X be topological spaces and $f \in C(S \times T, X)$. Then the map $f_x : T \to X$ defined by $f_x(y) = f(x, y)$ is continuous for every $x \in S$.*

Proof. Fix $x \in S$. Let U be an open set in X. We want to show that $W := f_x^{-1}(U)$ is open. We do this by finding for any $y \in W$ an open neighborhood of y contained in W. If W is empty we are done, hence assume that this is not so. Pick $y \in W$. Then $(x, y) \in f^{-1}(U)$ with $f^{-1}(U)$ open by continuity of f. Since $S\times T$ carries the product topology there must be open sets $V_x \subseteq S$ and $V_y \subseteq T$ with $x \in V_x$, $y \in V_y$ and $V_x \times V_y \subseteq f^{-1}(U)$. But clearly $V_y \subseteq W$ and we are done. □

Definition 1.19 (Quotient Topology)**.** Let S be a topological space and $\sim$ an equivalence relation on S. Then, the *quotient topology* on $S/\sim$ is the finest topology such that the quotient map $S \twoheadrightarrow S/\sim$ is continuous.

Definition 1.20. Let S, T be topological spaces and $f : S \to T$. For $a \in S$ we say that f is *open at a* iff for every neighborhood U of a the image $f(U)$ is a neighborhood of $f(a)$. We say that f is *open* iff it is open at every $a \in S$.

Proposition 1.21. *Let S, T be topological spaces and $f : S \to T$. f is open iff it maps any open set to an open set.*

Proof. Straightforward. □

Definition 1.22 (Ultrafilter)**.** Let $\mathcal{F}$ be a filter. We call $\mathcal{F}$ an *ultrafilter* iff $\mathcal{F}$ cannot be enlarged as a filter. That is, given a filter $\mathcal{F}'$ such that $\mathcal{F} \subseteq \mathcal{F}'$ we have $\mathcal{F}' = \mathcal{F}$.

Lemma 1.23. *Let S be a set, $\mathcal{F}$ an ultrafilter on S and $U \subseteq S$ such that $U \cap V \neq \emptyset$ for all $V \in \mathcal{F}$. Then $U \in \mathcal{F}$.*

Proof. Let $\mathcal{F}$ be an ultrafilter on S and $U \subseteq S$ such that $U \cap V \neq \emptyset$ for all $V \in \mathcal{F}$. Then, $\mathcal{B} := \{U \cap V : V \in \mathcal{F}\}$ forms the base of a filter $\mathcal{F}'$ such that $\mathcal{F} \subseteq \mathcal{F}'$ and $U \in \mathcal{F}'$. But since $\mathcal{F}$ is ultrafilter we have $\mathcal{F} = \mathcal{F}'$ and hence $U \in \mathcal{F}$. □

Proposition 1.24 (Ultrafilter lemma)**.** *Let $\mathcal{F}$ be a filter. Then there exists an ultrafilter $\mathcal{F}'$ such that $\mathcal{F} \subseteq \mathcal{F}'$.*

Proof. **Exercise.**Use Zorn's Lemma. □

1.2 Some properties of topological spaces

In a topological space it is useful if two distinct points can be distinguished by the topology. A strong form of this distinguishability is the *Hausdorff property*.

Definition 1.25 (Hausdorff)**.** Let S be a topological space. Assume that given any two distinct points $x, y \in S$ we can find open sets $U, V \subset S$ such that $x \in U$ and $y \in V$ and $U \cap V = \emptyset$. Then, S is said to have the *Hausdorff property*. We also say that S is a *Hausdorff space*.

Definition 1.26. A topological space S is called *completely regular* iff given a closed subset $C \subseteq S$ and a point $p \in S \setminus C$ there exists a continuous function $f : S \to [0, 1]$ such that $f(C) = \{0\}$ and $f(p) = 1$.

Definition 1.27. A topological space is called *normal* iff it is Hausdorff and if given two disjoint closed sets A and B there exist disjoint open sets U, V such that $A \subseteq U$ and $B \subseteq V$.

Lemma 1.28. *Let S be a normal topological space, U an open subset and C a closed subset such that $C \subseteq U$. Then, there exists an open subset U' and a closed subset C' such that $C \subseteq U' \subseteq C' \subseteq U$.*

Proof. **Exercise.** □

Theorem 1.29 (Uryson's Lemma)**.** *Let S be a normal topological space and A, B disjoint closed subsets. Then, there exists a continuous function $f : S \to [0, 1]$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$.*

Proof. Let $C_0 := A$ and $U_1 := S \setminus B$. Applying Lemma 1.28 we find an open subset $U_{1/2}$ and a closed subset $C_{1/2}$ such that

$$C_0 \subseteq U_{1/2} \subseteq C_{1/2} \subseteq U_1.$$

Performing the same operation on the pairs $C_0 \subseteq U_{1/2}$ and $C_{1/2} \subseteq U_1$ we obtain

$$C_0 \subseteq U_{1/4} \subseteq C_{1/4} \subseteq U_{1/2} \subseteq C_{1/2} \subseteq U_{3/4} \subseteq C_{3/4} \subseteq U_1.$$

We iterate this process, at step n replacing the pairs $C_{(k-1)/2^n} \subseteq U_{k/2^n}$ by $C_{(k-1)/2^n} \subseteq U_{(2k-1)/2^{n+1}} \subseteq C_{(2k-1)/2^{n+1}} \subseteq U_{k/2^n}$ for all $k \in \{1, \ldots, n\}$.

Now define

$$f(p) := \begin{cases} 1 & \text{if}\, p \in B \\ \inf\{x \in (0,1] : p \in U_x\} & \text{if}\, p \notin B \end{cases}$$

Obviously $f(B) = \{1\}$ and also $f(A) = \{0\}$. To show that f is continuous it suffices to show that $f^{-1}([0,a))$ and $f^{-1}((b,1])$ are continuous for $0 < a \leq 1$ and $0 \leq b < 1$. But,

$$f^{-1}([0,a)) = \bigcup_{x<a} U_x, \quad f^{-1}((b,1]) = \bigcup_{x>b} (S \setminus C_x).$$

□

Corollary 1.30. *Every normal space is completely regular.*

Definition 1.31. Let S be a topological space. S is called *first-countable* iff for each point in S there exists a countable base of its filter of neighborhoods. S is called *second-countable* iff the topology of S admits a countable base.

Definition 1.32. Let S be a topological space and $U, V \subseteq S$ subsets. U is called *dense* in V iff $V \subseteq \overline{U}$.

Definition 1.33 (separable)**.** A topological space is called *separable* iff it contains a countable dense subset.

Proposition 1.34. *A topological space that is second-countable is separable.*

Proof. **Exercise.** □

Definition 1.35 (open cover)**.** Let S be a topological space and $U \subseteq S$ a subset. A family of open sets $\{U_\alpha\}_{\alpha \in A}$ is called an *open cover* of U iff $U \subseteq \bigcup_{\alpha \in A} U_\alpha$.

Proposition 1.36. *Let S be a second-countable topological space and $U \subseteq S$ a subset. Then, every open cover of U contains a countable subcover.*

Proof. **<u>Exercise</u>.** □

Definition 1.37 (compact)**.** Let S be a topological space and $U \subseteq S$ a subset. U is called *compact* iff every open cover of U contains a finite subcover.

Definition 1.38. Let S be a topological space and $U \subseteq S$ a subset. Then, U is called *relatively compact* in S iff the closure of U in S is compact.

Proposition 1.39. *A closed subset of a compact space is compact. A compact subset of a Hausdorff space is closed.*

Proof. **<u>Exercise</u>.** □

Proposition 1.40. *The image of a compact set under a continuous map is compact.*

Proof. **<u>Exercise</u>.** □

Lemma 1.41. *Let T_1 be a compact Hausdorff space, T_2 be a Hausdorff space and $f : T_1 \to T_2$ a continuous bijective map. Then, f is a homeomorphism.*

Proof. The image of a compact set under f is compact and hence closed in T_2. But every closed set in T_1 is compact, so f is open and hence a homeomorphism. □

Lemma 1.42. *Let T be a Hausdorff topological space and C_1, C_2 disjoint compact subsets of T. Then, there are disjoint open subsets U_1, U_2 of T such that $C_1 \subseteq U_1$ and $C_2 \subseteq U_2$. In particular, if T is compact, then it is normal.*

Proof. We first show a weaker statement: Let C be a compact subset of T and $p \notin C$. Then there exist disjoint open sets U and V such that $p \in U$ and $C \subseteq V$. Since T is Hausdorff, for each point $q \in C$ there exist disjoint open sets U_q and V_q such that $p \in U_q$ and $q \in V_q$. The family of sets $\{V_q\}_{q\in C}$ defines an open covering of C. Since C is compact there is a finite subset $S \subseteq C$ such that the family $\{V_q\}_{q\in S}$ already covers C. Define $U := \bigcap_{q\in S} U_q$ and $V := \bigcup_{q\in S} V_q$. These are open sets with the desired properties.

We proceed to the prove the first statement of the lemma. By the previous demonstration, for each point $p \in C_1$ there are disjoint open sets U_p and V_p such that $p \in U_p$ and $C_2 \subseteq V_p$. The family of sets $\{U_p\}_{p\in C_1}$ defines an open covering of C_1. Since C_1 is compact there is a finite subset $S \subseteq C_1$ such that the family $\{U_p\}_{p\in S}$ already covers C_1. Define $U_1 := \bigcup_{p\in S} U_p$ and $U_2 := \bigcap_{p\in S} V_p$.

For the second statement of the lemma observe that if T is compact, then every closed subset is compact. □

Definition 1.43. A topological space is called *locally compact* iff every point has a compact neighborhood.

Definition 1.44. A topological space is called *σ-compact* iff it is locally compact and admits a covering by countably many compact subsets.

Definition 1.45. Let T be a topological space. A *compact exhaustion* of T is a sequence $\{U_i\}_{i\in\mathbb{N}}$ of open and relatively compact subsets such that $\overline{U_i} \subseteq U_{i+1}$ for all $i \in \mathbb{N}$ and $\bigcup_{i\in\mathbb{N}} U_i = T$.

Proposition 1.46. *A topological space admits a compact exhaustion iff it is σ-compact.*

Proof. Suppose the topological space T is σ-compact. Then there exists a sequence $\{K_n\}_{n\in\mathbb{N}}$ of compact subsets such that $\bigcup_{n\in N} K_n = T$. Since T is locally compact, every point possesses an open and relatively compact neighborhood. (Take an open subneighborhood of a compact neighborhood.) We cover K_1 by such open and relatively compact neighborhoods around every point. By compactness a finite subset of those already covers K_1. Their union, which we call U_1, is open and relatively compact. We proceed inductively. Suppose we have constructed the open and relatively compact set U_n. Consider the compact set $\overline{U_n} \cup K_{n+1}$. Covering it with open and relatively compact neighborhoods and taking the union of a finite subcover we obtain the open and relatively compact set U_{n+1}. It is then clear that the sequence $\{U_n\}_{n\in\mathbb{N}}$ obtained in this way provides a compact exhaustion of T since $\overline{U_i} \subseteq U_{i+1}$ for all $i \in \mathbb{N}$ and $T = \bigcup_{n\in\mathbb{N}} K_n \subseteq \bigcup_{n\in\mathbb{N}} U_n$.

Conversely, suppose T is a topological space and $\{U_n\}_{n\in\mathbb{N}}$ is a compact exhaustion of T. Then, the sequence $\{\overline{U_n}\}_{n\in\mathbb{N}}$ provides a countable covering of T by compact sets. Also, given $p \in T$ there exists $n \in \mathbb{N}$ such that $p \in U_n$. Then, the compact set $\overline{U_n}$ is a neighborhood of p. That is, T is locally compact. □

Proposition 1.47. *Let T be a topological space, $K \subseteq T$ a compact subset and $\{U_n\}_{n\in\mathbb{N}}$ a compact exhaustion of T. Then, there exists $n \in \mathbb{N}$ such that $K \subseteq U_n$.*

Proof. **Exercise.** □

Exercise 2 (One-point compactification)**.** Let S be a locally compact Hausdorff space. Let $\tilde{S} := S \cup \{\infty\}$ to be the set S with an extra element ∞ ajoint. Define a subset U of $\tilde{S}$ to be open iff either U is an open subset of S or U is the complement of a compact subset of S. Show that this makes $\tilde{S}$ into a compact Hausdorff space.

1.3 Sequences and convergence

Definition 1.48 (convergence of sequences)**.** Let $x := \{x_n\}_{n\in\mathbb{N}}$ be a sequence of points in a topological space S. We say that x *converges* to $p \in S$ iff for any neighborhood U of p there is a number $n \in \mathbb{N}$ such that $x_k \in U$ for all $k \geq n$. Then, p is said to be a *limit* of x. We also say that $p \in S$ is *accumulation point* of x iff for every neighborhood U of p, $x_k \in U$ for infinitely many $k \in \mathbb{N}$.

Definition 1.49. Let S be a topological space and $U \subseteq S$ a subset. Consider the set B_U of sequences of elements of U. Then the set $\overline{U}^s$ consisting of the points to which some element of B_U converges is called the *sequential closure* of U.

Proposition 1.50. *Let S be a topological space and $U \subseteq S$ a subset. Then, $U \subseteq \overline{U}^s \subseteq \overline{U}$. If, moreover, S is first-countable, then $\overline{U}^s = \overline{U}$.*

Proof. **Exercise.** □

Proposition 1.51. *Let S, T be topological spaces and $f : S \to T$. If f is continuous, then for any $p \in S$ and sequence $\{x_n\}_{n\in\mathbb{N}}$ converging to p, the sequence $f\{(x_n)\}_{n\in\mathbb{N}}$ in T converges to $f(p)$. Conversely, if S is first-countable and for any $p \in S$ and sequence $\{x_n\}_{n\in\mathbb{N}}$ converging to p, the sequence $f\{(x_n)\}_{n\in\mathbb{N}}$ in T converges to $f(p)$, then f is continuous.*

Proof. **Exercise.** □

Proposition 1.52. *Let S be Hausdorff space and $\{x_n\}_{n\in\mathbb{N}}$ a sequence in S which converges to a point $p \in S$. Then, $\{x_n\}_{n\in\mathbb{N}}$ does not converge to any other point in S.*

Proof. **Exercise.** □

Definition 1.53. Let S be a topological space and $U \subseteq S$ a subset. U is called *limit point compact* iff every sequence in U has an accumulation point. U is called *sequentially compact* iff every sequence in U contains a converging subsequence.

Proposition 1.54. *Let S be a first-countable topological space and $x = \{x_n\}_{n\in\mathbb{N}}$ a sequence in S with accumulation point p. Then, x has a subsequence that converges to p.*

Proof. By first-countability choose a countable base $\{U_n\}_{n\in\mathbb{N}}$ of the filter of neighborhoods at p. Now consider the family $\{W_n\}_{n\in\mathbb{N}}$ of open neighborhoods $W_n := \bigcap_{k=1}^n U_k$ at p. It is easy to see that this is again a countable neighborhood base at p. Moreover, it has the property that $W_n \subseteq W_m$ if $n \geq m$. Now, Choose $n_1 \in \mathbb{N}$ such that $x_{n_1} \in W_1$. Recursively, choose $n_{k+1} > n_k$ such that $x_{n_{k+1}} \in W_{k+1}$. This is possible since W_{k+1} contains infinitely many points of x. Let V be a neighborhood of p. There exists some $k \in \mathbb{N}$ such that $U_k \subseteq V$. By construction, then $W_m \subseteq W_k \subseteq U_k$ for all $m \geq k$ and hence $x_{n_m} \in V$ for all $m \geq k$. Thus, the subsequence $\{x_{n_m}\}_{m\in\mathbb{N}}$ converges to p. □

Proposition 1.55. *Sequential compactness implies limit point compactness. In a first-countable space the converse is also true.*

Proof. **Exercise.** □

Proposition 1.56. *A compact space is limit point compact.*

Proof. Consider a sequence x in a compact space S. Suppose x does not have an accumulation point. Then, for each point $p \in S$ we can choose an open neighborhood U_p which contains only finitely many points of x. However, by compactness, S is covered by finitely many of the sets U_p. But their union can only contain a finite number of points of x, a contradiction. □

1.4 Filters and convergence

Definition 1.57 (convergence of filters)**.** Let S be a topological space and $\mathcal{F}$ a filter on a subset A of S. $\mathcal{F}$ is said to *converge* to $p \in S$ iff every neighborhood of p is contained in $\mathcal{F}$, i.e., $\mathcal{N}_p \subseteq \mathcal{F}$. Then, x is said to be a *limit* of x. Also, $p \in S$ is called *accumulation point* of $\mathcal{F}$ iff $p \in \bigcap_{U\in\mathcal{F}} \overline{U}$.

Proposition 1.58. *Let S be a topological space and $\mathcal{F}$ a filter on a subset A of S converging to $p \in S$. Then, p is accumulation point of $\mathcal{F}$.*

Proof. **Exercise.** □

Proposition 1.59. *Set S be a topological space and $\mathcal{F}, \mathcal{F}'$ filters on a subset A of S such that $\mathcal{F} \subseteq \mathcal{F}'$. If $p \in S$ is accumulation point of $\mathcal{F}'$, then it is also accumulation point of $\mathcal{F}$. If $\mathcal{F}$ converges to $p \in S$, then so does $\mathcal{F}'$.*

Proof. Immediate. □

Let $x = \{x_n\}_{n\in\mathbb{N}}$ be a sequence of points in a topological space S. We define the filter $\mathcal{F}_x$ associated with this sequence as follows: $\mathcal{F}_x$ contains all the subsets U of S such that U contains all x_n, except possibly finitely many.

Proposition 1.60. *Let $x := \{x_n\}_{n\in\mathbb{N}}$ be a sequence of points in a topological space S. Then x converges to a point $p \in S$ iff the associated filter $\mathcal{F}_x$ converges to p. Also, $p \in S$ is accumulation point of x iff it is accumulation point of $\mathcal{F}_x$.*

Proof. **Exercise.** □

Proposition 1.61. *Let S be a topological space and $U \subseteq S$ a subset. Consider the set A_U of filters on U. Then, the closure $\overline{U}$ of U coincides with the set of points to which some element in A_U converges.*

Proof. If $U = \emptyset$, then A_U is empty and the proof is trivial. Assume the contrary. If $x \in \overline{U}$, then the intersection of U with the filter $\mathcal{N}_x$ of neighborhoods of x is a filter on U that converges to x as desired. If $x \notin \overline{U}$, then there exists a neighborhood V of x such that $U \cap V = \emptyset$. So no filter in U can contain V. □

Proposition 1.62. *Let S, T be topological spaces and $f : S \to T$. If f is continuous, then for any $p \in S$ and filter $\mathcal{F}$ converging to p, the filter generated by $f(\mathcal{F})$ in T converges to $f(p)$. Conversely, if for any $p \in S$ and filter $\mathcal{F}$ converging to p, the filter generated by $f(\mathcal{F})$ in T converges to $f(p)$, then f is continuous.*

Proof. **Exercise.** □

Proposition 1.63. *Let S be a Hausdorff topological space, $\mathcal{F}$ a filter on a subset A of S converging to a point $p \in S$. Then $\mathcal{F}$ does not converge to any other point in S.*

Proof. **Exercise.** □

Proposition 1.64. *Let S be a topological space and $K \subseteq S$ a subset. Then, K is compact iff every filter in K has at least one accumulation point in K.*

Proof. Let $K \subseteq S$ be compact. We suppose that there is a filter $\mathcal{F}$ on K that has no accumulation point. For each $U \in \mathcal{F}$ consider the open set $O_U := S \setminus \overline{U}$. By assumption, these open sets cover K. Since K is compact, there must be a finite subset $\{U_1, \dots, U_n\}$ of elements of $\mathcal{F}$ such that $\{O_{U_1}, \dots, O_{U_n}\}$ covers K. But this implies $\bigcap_{i=1}^n \overline{U_i} = \emptyset$ and thus, in particular, also $\bigcap_{i=1}^n U_i = \emptyset$, contradicting the fact that $\mathcal{F}$ is a filter. Thus, any filter on K must have an accumulation point.

Now suppose that $K \subseteq S$ is not compact. Then, there exists a cover of K by open sets $\{U_\alpha\}_{\alpha \in A}$ which does not admit any finite subcover. Now consider finite intersections of the sets $C_\alpha := K \setminus U_\alpha$. These are non-empty and form the base of a filter in K. But this filter clearly has no accumulation point. Thus, if every filter in K is to posses an accumulation point, K must be compact. □

1.5 Metric and pseudometric spaces

Definition 1.65. Let S be a set and $d : S \times S \to \mathbb{R}_0^+$ a map with the following properties:

- $d(x, y) = d(y, x) \quad \forall x, y \in S$. (symmetry)
- $d(x, z) \leq d(x, y) + d(y, z) \quad \forall x, y, z \in S$. (triangle inequality)
- $d(x, x) = 0 \quad \forall x \in S$.

Then d is called a *pseudometric* on S. S is also called a *pseudometric space*. Suppose d also satisfies

- $d(x, y) = 0 \implies x = y \quad \forall x, y \in S$. (definiteness)

Then d is called a *metric* on S and S is called a *metric space*.

Definition 1.66. Let S be a pseudometric space, $x \in S$ and $r > 0$. Then the set $B_r(x) := \{y \in S : d(x, y) < r\}$ is called the *open ball* of radius r centered around x in S. The set $\overline{B}_r(x) := \{y \in S : d(x, y) \leq r\}$ is called the *closed ball* of radius r centered around x in S.

Proposition 1.67. *Let S be a pseudometric space. Then, the open balls in S together with the empty set form the basis of a topology on S. This topology is first-countable and such that closed balls are closed. Moreover, the topology is Hausdorff iff S is metric.*

Proof. **Exercise.** □

Definition 1.68. A topological space is called *(pseudo)metrizable* iff there exists a (pseudo)metric such that the open balls given by the (pseudo)metric are a basis of its topology.

Proposition 1.69. *In a pseudometric space any open ball can be obtained as the countable union of closed balls. Similarly, any closed ball can be obtained as the countable intersection of open balls.*

Proof. **Exercise**. □

Proposition 1.70. *Let S be a set equipped with two pseudometrics d^1 and d^2. Then, the topology generated by d^2 is finer than the topology generated by d^1 iff for all $x \in S$ and $r_1 > 0$ there exists $r_2 > 0$ such that $B^2_{r_2}(x) \subseteq B^1_{r_1}(x)$. In particular, d^1 and d^2 generate the same topology iff the condition holds both ways.*

Proof. **Exercise**. □

Proposition 1.71 (epsilon-delta criterion)**.** *Let S, T be pseudometric spaces and $f : S \to T$ a map. Then, f is continuous at $x \in S$ iff for every $\epsilon > 0$ there exists $\delta > 0$ such that $f(B_\delta(x)) \subseteq B_\epsilon(f(x))$.*

Proof. **Exercise**. □

1.6 Elementary properties of pseudometric spaces

Proposition 1.72. *Every metric space is normal.*

Proof. Let A, B be disjoint closed sets in the metric space S. For each $x \in A$ choose $\epsilon_x > 0$ such that $B_{\epsilon_x}(x) \cap B = \emptyset$ and for each $y \in B$ choose $\epsilon_y > 0$ such that $B_{\epsilon_y}(y) \cap A = \emptyset$. Then, for any pair (x, y) with $x \in A$ and $y \in B$ we have $B_{\epsilon_x/2}(x) \cap B_{\epsilon_y/2}(y) = \emptyset$. Consider the open sets $U := \bigcup_{x\in A} B_{\epsilon_x/2}(x)$ and $V := \bigcup_{y\in B} B_{\epsilon_y/2}(y)$. Then, $U \cap V = \emptyset$, but $A \subseteq U$ and $B \subseteq V$. So S is normal. □

Proposition 1.73. *Let S be a pseudometric space and $x := \{x_n\}_{n\in\mathbb{N}}$ a sequence in S. Then x converges to $p \in S$ iff for any $\epsilon > 0$ there exists an $n_0 \in \mathbb{N}$ such that $d(x_n, p) < \epsilon$ for all $n \geq n_0$.*

Proof. Immediate. □

Definition 1.74. Let S be a pseudometric space and $x := \{x_n\}_{n\in\mathbb{N}}$ a sequence in S. Then x is called a *Cauchy sequence* iff for all $\epsilon > 0$ there exists an $n_0 \in \mathbb{N}$ such that for all $n, m \geq n_0 : d(x_n, x_m) < \epsilon$.

Proposition 1.75. *Any converging sequence in a pseudometric space is a Cauchy sequence.*

Proof. **Exercise**. □

Proposition 1.76. *Suppose x is a Cauchy sequence in a pseudometric space. If p is accumulation point of x then x converges to p.*

Proof. **Exercise**. □

Definition 1.77. Let S be a pseudometric space and $U \subseteq S$ a subset. If every Cauchy sequence in U converges to a point in U, then U is called *complete*.

Proposition 1.78. *A complete subset of a metric space is closed. A closed subset of a complete pseudometric space is complete.*

Proof. **Exercise**. □

Definition 1.79 (Totally boundedness)**.** Let S be a pseudometric space. A subset $U \subseteq S$ is called *totally bounded* iff for any $r > 0$ the set U admits a cover by finitely many open balls of radius r.

Proposition 1.80. *A subset of a pseudometric space is compact iff it is complete and totally bounded.*

Proof. We first show that compactness implies totally boundedness and completeness. Let U be a compact subset. Then, for $r > 0$ cover U by open balls of radius r centered at every point of U. Since U is compact, finitely many balls will cover it. Hence, U is totally bounded. Now, consider a Cauchy sequence x in U. Since U is compact x must have an accumulation point $p \in U$ (Proposition 1.56) and hence (Proposition 1.76) converge to p. Thus, U is complete.

We proceed to show that completeness together with totally boundedness imply compactness. Let U be a complete and totally bounded subset. Assume U is not compact and choose a covering $\{U_\alpha\}_{\alpha \in A}$ of U that does not admit a finite subcovering. On the other hand, U is totally bounded and admits a covering by finitely many open balls of radius $1/2$. Hence, there must be at least one such ball B_1 such that $C_1 := B_1 \cap U$ is not covered by finitely many U_α. Choose a point x_1 in C_1. Observe that C_1 itself is totally bounded. Inductively, cover C_n by finitely many open balls of radius $2^{-(n+1)}$. For at least one of those, call it B_{n+1}, $C_{n+1} := B_{n+1} \cap C_n$ is not covered by finitely many U_α. Choose a point x_{n+1} in C_{n+1}. This process yields a Cauchy sequence $x := \{x_k\}_{k\in\mathbb{N}}$. Since U is complete the sequence converges to a point $p \in U$. There must be $\alpha \in A$ such that $p \in U_\alpha$. Since U_α is open there exists $r > 0$ such that $B(p,r) \subseteq U_\alpha$. This implies, $C_n \subseteq U_\alpha$ for all $n \in \mathbb{N}$ such that $2^{-n+1} < r$. However, this is a contradiction to the C_n not being finitely covered. Hence, U must be compact. □

Proposition 1.81. *The notions of compactness, limit point compactness and sequential compactness are equivalent in a pseudometric space.*

Proof. **Exercise.** □

Proposition 1.82. *A totally bounded pseudometric space is second-countable.*

Proof. **Exercise.** □

Proposition 1.83. *The notions of separability and second-countability are equivalent in a pseudometric space.*

Proof. **Exercise.** □

Theorem 1.84 (Baire's Theorem)**.** *Let S be a complete metric space and $\{U_n\}_{n\in\mathbb{N}}$ a sequence of open and dense subsets of S. Then, the intersection $\bigcap_{n\in\mathbb{N}} U_n$ is dense in S.*

Proof. Set $U := \bigcap_{n\in\mathbb{N}} U_n$. Let V be an arbitrary open set in S. It suffices to show that $V \cap U \neq \emptyset$. To this end we construct a sequence $\{x_n\}_{n\in\mathbb{N}}$ of elements of S and a sequence $\{\epsilon_n\}_{n\in\mathbb{N}}$ of positive numbers. Choose $x_1 \in U_1 \cap V$ and then $0 < \epsilon_1 \leq 1$ such that $\overline{B_{\epsilon_1}(x_1)} \subseteq U_1 \cap V$. Now, consecutively choose $x_{n+1} \in U_{n+1} \cap B_{\epsilon_n/2}(x_n)$ and $0 < \epsilon_{n+1} < 2^{-n}$ such that $\overline{B_{\epsilon_{n+1}}(x_{n+1})} \subseteq U_{n+1} \cap B_{\epsilon_n}(x_n)$. The sequence $\{x_n\}_{n\in\mathbb{N}}$ is Cauchy since by construction $d(x_n, x_{n+1}) < 2^{-n}$ for all $n \in \mathbb{N}$. So by completeness it converges to some point $x \in S$. Indeed, $x \in \overline{B_{\epsilon_1}(x_1)} \subseteq V$. On the other hand, $x \in \overline{B_{\epsilon_n}(x_n)} \subseteq U_n$ for all $n \in \mathbb{N}$ and hence $x \in U$. This completes the proof. □

Proposition 1.85. *Let S be equipped with a pseudometric d. Then $p \sim q \iff d(p,q) = 0$ for $p, q \in S$ defines an equivalence relation on S. The prescription $\tilde{d}([p],[q]) := d(p,q)$ for $p, q \in S$ is well defined and yields a metric $\tilde{d}$ on the quotient space $S/\!\sim$. The topology induced by this metric on $S/\!\sim$ is the quotient topology with respect to that induced by d on S. Moreover, $S/\!\sim$ is complete iff S is complete.*

Proof. **Exercise.** □

1.7 Completion of metric spaces

Often it is desirable to work with a complete metric space when one is only given a non-complete metric space. To this end one can construct the *completion* of a metric space. This is detailed in the following exercise.

<u>Exercise</u> 3. Let S be a metric space.

- Let $x := \{x_n\}_{n\in\mathbb{N}}$ and $y := \{y_n\}_{n\in\mathbb{N}}$ be Cauchy sequences in S. Show that the limit $\lim_{n\to\infty} d(x_n, y_n)$ exists.
- Let T be the set of Cauchy sequences in S. Define the function $\tilde{d} : T \times T \to \mathbb{R}_0^+$ by $\tilde{d}(x, y) := \lim_{n\to\infty} d(x_n, y_n)$. Show that $\tilde{d}$ defines a pseudometric on T.
- Show that T is complete.
- Define $\overline{S}$ as the metric quotient $T/\sim$ as in Proposition 1.85. Then, $\overline{S}$ is complete.
- Show that there is a natural isometric embedding (i.e., a map that preserves the metric) $i_S : S \to \overline{S}$. Furthermore, show that this is a bijection iff S is complete.

Definition 1.86. The metric space $\overline{S}$ constructed above is called the *completion* of the metric space S.

Proposition 1.87 (Universal property of completion)**.** *Let S be a metric space, T a complete metric space and $f : S \to T$ an isometric map. Then, there is a unique isometric map $\overline{f} : \overline{S} \to T$ such that $f = \overline{f} \circ i_S$. Furthermore, the closure of $f(S)$ in T is equal to $\overline{f}(\overline{S})$.*

Proof. **<u>Exercise</u>.** □

2 Vector spaces with additional structure

In the following $\mathbb{K}$ denotes a field which might be either $\mathbb{R}$ or $\mathbb{C}$.

Definition 2.1. Let V be a vector space over $\mathbb{K}$. A subset A of V is called *balanced* iff for all $v \in A$ and all $\lambda \in \mathbb{K}$ with $|\lambda| \leq 1$ the vector λv is contained in A. A subset A of V is called *convex* iff for all $x, y \in V$ and $t \in [0, 1]$ the vector $(1-t)x + ty$ is in A. Let A be a subset of V. Consider the smallest subset of V which is convex and which contains A. This is called the *convex hull* of A, denoted $\mathrm{conv}(A)$.

Proposition 2.2. *(a) Intersections of balanced sets are balanced. (b) The sum of two balanced sets is balanced. (c) A scalar multiple of a balanced set is balanced.*

Proof. **Exercise.** $\square$

Proposition 2.3. *Let V be vector space and A a subset. Then*

$$\mathrm{conv}(A) = \left\{ \sum_{i=1}^{n} \lambda_i x_i : \lambda_i \in [0, 1], x_i \in A, \sum_{i=1}^{n} \lambda_i = 1 \right\}.$$

Proof. **Exercise.** $\square$

We denote the space of linear maps between a vector space V and a vector space W by $\mathrm{L}(V, W)$.

2.1 Topological vector spaces

Definition 2.4. A set V that is equipped both with a vector space structure over $\mathbb{K}$ and a topology is called a *topological vector space (tvs)* iff the vector addition $+ : V \times V \to V$ and the scalar multiplication $\cdot : \mathbb{K} \times V \to V$ are both continuous. (Here the topology on $\mathbb{K}$ is the standard one.)

Proposition 2.5. *Let V be a tvs, $\lambda \in \mathbb{K} \setminus 0$, $w \in V$. The maps $V \to V : v \mapsto \lambda v$ and $V \to V : v \mapsto v + w$ are automorphisms of V as a tvs. In particular, the topology $\mathcal{T}$ of V is invariant under rescalings and translations: $\lambda \mathcal{T} = \mathcal{T}$ and $\mathcal{T} + w = \mathcal{T}$. In terms of filters of neighborhoods, $\lambda \mathcal{N}_v = \mathcal{N}_{\lambda v}$ and $\mathcal{N}_v + w = \mathcal{N}_{v+w}$ for all $v \in V$.*

Proof. It is clear that non-zero scalar multiplication and translation are vector space automorphisms. To see that they are also continuous use Proposition 1.18. The inverse maps are of the same type hence also continuous. Thus we have homeomorphisms. The scale- and translation invariance of the topology follows. $\square$

Note that this implies that the topology of a tvs is completely determined by the filter of neighborhoods of one of its points, say 0.

Definition 2.6. Let V be a tvs and U a subset. U is called *bounded* iff for every neighborhood W of 0 there exists $\lambda \in \mathbb{R}^+$ such that $U \subseteq \lambda W$.

Remark: Changing the allowed range of λ in the definition of boundedness from $\mathbb{R}^+$ to $\mathbb{K}$ leads to an equivalent definition, i.e., is not weaker. However, the choice of $\mathbb{R}^+$ over $\mathbb{K}$ is more convenient in certain applications.

Proposition 2.7. *Let V be a tvs. Then:*

1. *Every point set is bounded.*
2. *Every neighborhood of 0 contains a balanced subneighborhood of 0.*
3. *Let U be a neighborhood of 0. Then there exists a subneighborhood W of 0 such that $W + W \subseteq U$.*

Proof. We start by demonstrating Property 1. Let $x \in V$ and U some open neighborhood of 0. Then $Z := \{(\lambda, y) \in \mathbb{K} \times V : \lambda y \in U\}$ is open by continuity of multiplication. Also $(0, x) \in Z$ so that by the product topology there exists an $\epsilon > 0$ and an open neighborhood W of x in V such that $B_\epsilon(0) \times W \subseteq Z$. In particular, there exists $\mu > 0$ such that $\mu x \in U$, i.e., $\{x\} \subseteq \mu^{-1} U$ as desired.

We proceed to Property 2. Let U be an open neighborhood of 0. By continuity $Z := \{(\lambda, x) \in \mathbb{K} \times V : \lambda x \in U\}$ is open. By the product topology, there are open neighborhoods X of $0 \in \mathbb{K}$ and W of $0 \in V$ such that $X \times W \subseteq Z$. Thus, $X \cdot W \subseteq U$. Now X contains an open ball of some radius $\epsilon > 0$ around 0 in $\mathbb{K}$. Set $Y := B_\epsilon(0) \cdot W$. This is an (open) neighborhood of 0 in V, it is contained in U and it is balanced.

We end with Property 3. Let U be an open neighborhood of 0. By continuity $Z := \{(x, y) \in V \times V : x + y \in U\}$ is open. By the product topology, there are open neighborhoods W_1 and W_2 of 0 such that $W_1 \times W_2 \subseteq Z$. This means $W_1 + W_2 \subseteq U$. Now define $W := W_1 \cap W_2$. $\square$

Proposition 2.8. *Let V be a vector space and $\mathcal{F}$ a filter on V. Then $\mathcal{F}$ is the filter of neighborhoods of 0 for a compatible topology on V iff 0 is contained in every element of $\mathcal{F}$ and $\lambda \mathcal{F} = \mathcal{F}$ for all $\lambda \in \mathbb{K} \setminus \{0\}$ and $\mathcal{F}$ satisfies the properties of Proposition 2.7.*

Proof. It is already clear that the properties in question are necessary for $\mathcal{F}$ to be the filter of neighborhoods of 0 of V. It remains to show that they are sufficient. If $\mathcal{F}$ is to be the filter of neighborhoods of 0 then, by translation invariance, $\mathcal{F}_x := \mathcal{F} + x$ must be the filter of neighborhoods of the point x. We show that the family of filters $\{\mathcal{F}_x\}_{x \in V}$ does indeed define a topology on V. To this end we will use Proposition 1.10. Property 1 is satisfied by assumption. It remains to show Property 2. By translation invariance it will be enough to consider $x = 0$. Suppose $U \in \mathcal{F}$. Using Property 3 of Proposition 2.7 there is $W \in \mathcal{F}$ such that $W + W \subseteq U$. We claim that Property 2 of Proposition 1.10 is now satisfied with this choice of W. Indeed, let $y \in W$ then $y + W \in \mathcal{F}_y$ and $y + W \subseteq U$ so $U \in \mathcal{F}_y$ as required.

We proceed to show that the topology defined in this way is compatible with the vector space structure. Take an open set $U \subseteq V$ and consider its preimage $Z = \{(x, y) \in V \times V : x + y \in U\}$ under vector addition. Take some point $(x, y) \in Z$. $U - x - y$ is an open neighborhood of 0. By Property 3 of Proposition 2.7 there is an open neighborhood W of 0 such that $W + W \subseteq U - x - y$, i.e., $(x + W) + (y + W) \subseteq U$. But $x + W$ is an open neighborhood of x and $y + W$ is an open neighborhood of y so $(x + W) \times (y + W)$ is an open neighborhood of (x, y) in $V \times V$ contained in Z. Hence vector addition is continuous.

We proceed to show continuity of scalar multiplication. Consider an open set $U \subseteq V$ and consider its preimage $Z = \{(\lambda, x) \in \mathbb{K} \times V : \lambda x \in U\}$ under scalar multiplication. Take some point $(\lambda, x) \in Z$. $U - \lambda x$ is an open neighborhood of 0 in V. By Property 3 of Proposition 2.7 there is an open neighborhood W of 0 such that $W + W = U - \lambda x$. By Property 2 of Proposition 2.7 there exists a balanced subneighborhood X of W. By Property 1 of Proposition 2.7 (boundedness of points) there exists $\epsilon > 0$ such that $\epsilon x \in X$. Now define $Y := (\epsilon + |\lambda|)^{-1} X$. Note that scalar multiples of (open) neighborhoods of 0 are (open) neighborhoods of 0 by assumption. Hence Y is open since X is. Thus $B_\epsilon(\lambda) \times (x + Y)$ an open neighborhood of (λ, x) in $\mathbb{K} \times V$. We claim that it is contained in Z. First observe that since X is balanced, $B_\epsilon(0) \cdot x \subseteq X$. Similarly, we have $B_\epsilon(\lambda) \cdot Y \subseteq B_{\epsilon+|\lambda|}(0) \cdot Y = B_1(0) \cdot X \subseteq X$. Thus we have $B_\epsilon(0) \cdot x + B_\epsilon(\lambda) \cdot Y \subseteq X + X \subseteq W + W \subseteq U - \lambda x$. But this implies $B_\epsilon(\lambda) \cdot (x + Y) \subseteq U$ as required. $\square$

Proposition 2.9. *(a) The interior of a balanced set is balanced. (b) The closure of a balanced set is balanced.*

Proof. Let U be balanced and let $\lambda \in \mathbb{K}$ with $0 < |\lambda| \leq 1$. It is then enough to observe that for (a) $\lambda \mathring{U} = \mathring{\overline{\lambda U}}$ $\subseteq \mathring{U}$ and for (b) $\lambda \overline{U} = \overline{\lambda U} \subseteq \overline{U}$. $\square$

Proposition 2.10. *In a tvs every neighborhood of* 0 *contains a closed and balanced subneighborhood.*

Proof. Let U be a neighborhood of 0. By Proposition 2.7.3 there exists a subneighborhood $W \subseteq U$ such that $W + W \subset U$. By Proposition 2.7.2 there exists a balanced subneighborhood $X \subseteq W$. Let $Y := \overline{X}$. Then, Y is obviously a closed neighborhood of 0. Also Y is balanced by Proposition 2.9. Finally, let $y \in Y = \overline{X}$. Any neighborhood of y must intersect X. In particular, $y + X$ is such a neighborhood. Thus, there exist $x \in X$, $z \in X$ such that $x = y + z$, i.e., $y = x - z \in X - X = X + X \subseteq U$. So, $Y \subseteq U$. □

Proposition 2.11. *(a) Subsets of bounded sets are bounded. (b) Finite unions of bounded sets are bounded. (c) The closure of a bounded set is bounded. (d) The sum of two bounded sets is bounded. (e) A scalar multiple of a bounded set is bounded.*

Proof. **Exercise**. □

Definition 2.12. Let V be a tvs and $C \subseteq V$ a subset. Then, C is called *totally bounded* iff for each neighborhood U of 0 in V there exists a finite subset $F \subseteq C$ such that $C \subseteq F + U$.

Proposition 2.13. *(a) Subsets of totally bounded sets are totally bounded. (b) Finite unions of totally bounded sets are totally bounded. (c) The closure of a totally bounded set is totally bounded. (d) The sum of two totally bounded sets is totally bounded. (e) A scalar multiple of a totally bounded set is totally bounded.*

Proof. **Exercise**. □

Proposition 2.14. *Compact sets are totally bounded. Totally bounded sets are bounded.*

Proof. **Exercise**. □

Let A, B be topological vector spaces. We denote the space of maps from A to B that are linear and continuous by $\mathrm{CL}(A, B)$.

Definition 2.15. Let A, B be tvs. A linear map $f : A \to B$ is called *bounded* iff there exists a neighborhood U of 0 in A such that $f(U)$ is bounded. A linear map $f : A \to B$ is called *compact* iff there exists a neighborhood U of 0 in A such that $\overline{f(U)}$ is compact.

Let A, B be tvs. We denote the space of maps from A to B that are linear and bounded by $\mathrm{BL}(A, B)$. We denote the space of maps from A to B that are linear and compact by $\mathrm{KL}(A, B)$.

Proposition 2.16. *Let A, B be tvs and $f \in \mathrm{L}(A, B)$. (a) f is continuous iff the preimage of any neighborhood of 0 in B is a neighborhood of 0 in A. (b) If f is continuous it maps bounded sets to bounded sets. (c) If f is bounded then f is continuous, i.e., $\mathrm{BL}(A, B) \subseteq \mathrm{CL}(A, B)$. (d) If f is compact then f is bounded.*

Proof. **Exercise.** □

A useful property for a topological space is the Hausdorff property, i.e., the possibility to separate points by open sets. It is not the case that a tvs is automatically Hausdorff. However, the way in which a tvs may be non-Hausdorff is severely restricted. Indeed, we shall see int the following that a tvs may be split into a part that is Hausdorff and another one that is maximally non-Hausdorff in the sense of carrying the trivial topology.

Proposition 2.17. *Let V be a tvs and $C \subseteq V$ a vector subspace. Then, the closure $\overline{C}$ of C is also a vector subspace of V.*

Proof. **Exercise.**[Hint: Use Proposition 1.61.] □

Proposition 2.18. *Let V be a tvs. The closure of $\{0\}$ in V coincides with the intersection of all neighborhoods of 0. Moreover, V is Hausdorff iff $\overline{\{0\}} = \{0\}$.*

Proof. **Exercise.** □

Proposition 2.19. *Let V be a tvs and $C \subseteq V$ a vector subspace.*

1. *The quotient space V/C is a tvs.*
2. *V/C is Hausdorff iff C is closed in V.*
3. *The quotient map $q : V \to V/C$ is linear, continuous and open. Moreover, the quotient topology on V/C is the only topology such that q is continuous and open.*
4. *The image of a base of the filter of neighborhoods of 0 in V is a base of the filter of neighborhoods of 0 in V/C.*

Proof. **Exercise.** □

Thus, for a tvs V the exact sequence

$$0 \to \overline{\{0\}} \to V \to V/\overline{\{0\}} \to 0$$

describes how V is composed of a Hausdorff piece $V/\overline{\{0\}}$ and a piece $\overline{\{0\}}$ with trivial topology. We can express this decomposition also in terms of a direct sum, as we shall see in the following.

A (vector) subspace of a tvs is a tvs with the subset topology. Let A and B be tvs. Then the direct sum $A \oplus B$ is a tvs with the product topology. Note that as subsets of $A \oplus B$, both A and B are closed.

Definition 2.20. Let V be a tvs and A a subspace. Then another subspace B of A in V is called a *topological complement* iff $V = A \oplus B$ as tvs (i.e., as vector spaces and as topological spaces). A is called *topologically complemented* if such a topological complement B exists.

Note that algebraic complements (i.e., complements merely with respect to the vector space structure) always exist (using the Axiom of Choice). However, an algebraic complement is not necessarily a topological one. Indeed, there are examples of subspaces of tvs that have no topological complement.

Proposition 2.21 (Structure Theorem for tvs)**.** *Let V be a tvs and B an algebraic complement of $\overline{\{0\}}$ in V. Then B is also a topological complement of $\overline{\{0\}}$ in V. Moreover, B is canonically isomorphic to $V/\overline{\{0\}}$ as a tvs.*

Proof. **Exercise.** □

We conclude that every tvs is a direct sum of a Hausdorff tvs and a tvs with the trivial topology.

2.2 Metrizable and pseudometrizable vector spaces

In this section we consider *(pseudo)metrizable vector spaces* (mvs), i.e., tvs that admit a (pseudo)metric compatible with the topology.

Definition 2.22. A pseudometric on a vector space V is called *translation-invariant* iff $d(x + a, y + a) = d(x, y)$ for all $x, y, a \in V$. A translation-invariant pseudometric on a vector space V is called *balanced* iff its open balls around the origin are balanced.

As we shall see it will be possible to limit ourselves to balanced translation-invariant pseudometrics on mvs. Moreover, these can be conveniently described by pseudo-seminorms.

Definition 2.23. Let V be a vector space over $\mathbb{K}$. Then a map $V \to \mathbb{R}^+_0 : x \mapsto \|x\|$ is called a *pseudo-seminorm* iff it satisfies the following properties:

1. For all $\lambda \in \mathbb{K}$, $|\lambda| \leq 1$ implies $\|\lambda x\| \leq \|x\|$ for all $x \in V$.

2. For all $x, y \in V : \|x + y\| \leq \|x\| + \|y\|$.

$\|\cdot\|$ is called a *pseudo-norm* iff it satisfies in addition the following property.

3. $\|x\| = 0$ implies $x = 0$.

Proposition 2.24. *There is a one-to-one correspondence between pseudo-seminorms and balanced translation invariant pseudometrics on a vector space via $d(x, y) := \|x - y\|$. This specializes to a correspondence between pseudo-norms and balanced translation invariant metrics.*

Proof. **Exercise.** □

Proposition 2.25. *Let V be a vector space. The topology generated by a pseudo-seminorm on V is compatible with the vector space structure iff for every $x \in V$ and $\epsilon > 0$ there exists $\lambda \in \mathbb{R}^+$ such that $x \in \lambda B_\epsilon(0)$.*

Proof. Assume we are given a pseudo-seminorm on V that induces a compatible topology. It is easy to see that the stated property of the pseudo-seminorm then follows from Property 1 in Proposition 2.7 (boundedness of points).

Conversely, suppose we are given a pseudo-seminorm on V with the stated property. We show that the filter $\mathcal{N}_0$ of neighborhoods of 0 defined by the pseudo-seminorm has the properties required by Proposition 2.8 and hence defines a compatible topology on V. Firstly, it is already clear that every $U \in \mathcal{N}_0$ contains 0. We proceed to show that $\mathcal{N}_0$ is scale invariant. It is enough to show that for $\epsilon > 0$ and $\lambda \in \mathbb{K} \setminus \{0\}$ the scaled ball $\lambda B_\epsilon(0)$ is open. Choose a point $\lambda x \in \lambda B_\epsilon(0)$. Take $\delta > 0$ such that $\|x\| < \epsilon - \delta$. Then $B_\delta(0) + x \subseteq B_\epsilon(0)$. Choose $n \in \mathbb{N}$ such that $2^{-n} \leq |\lambda|$. Observe that the triangle inequality implies $B_{2^{-n}\delta}(0) \subseteq 2^{-n} B_\delta(0)$ (for arbitrary δ and n in fact). Hence $B_{2^{-n}\delta}(\lambda x) = B_{2^{-n}\delta}(0) + \lambda x \subseteq \lambda B_\delta(0) + \lambda x \subseteq \lambda B_\epsilon(0)$ showing that $\lambda B_\epsilon(0)$ is open.

It now remains to show the properties of $\mathcal{N}_0$ listed in Proposition 2.7. As for Property 3, we may take U to be an open ball of radius ϵ around 0 for some $\epsilon > 0$. Define $W := B_{\epsilon/2}(0)$ Then $W + W \subseteq U$ follows from the triangle inequality. Concerning Property 2 we simple notice that open balls are balanced by construction. The only property that is not automatic for a pseudo-seminorm and does require the stated condition is Property 1 (boundedness of points). The equivalence of the two is easy to see. □

Theorem 2.26. *A tvs V is pseudometrizable iff it is first-countable, i.e., iff there exists a countable base for the filter of neighborhoods of* 0*. Moreover, if V is pseudometrizable it admits a compatible pseudo-seminorm.*

Proof. It is clear that pseudometrizability implies the existence of a countable base of $\mathcal{N}_0$. For example, the sequence of balls $\{B_{1/n}(0)\}_{n\in\mathbb{N}}$ provides such a base. Conversely, suppose that $\{U_n\}_{n\in\mathbb{N}}$ is a base of the filter of neighborhoods of 0 such that all U_n are balanced and $U_{n+1}+U_{n+1}\subseteq U_n$. (Given an arbitrary countable base of $\mathcal{N}_0$ we can always produce another one with the desired properties.) Now for each finite subset H of $\mathbb{N}$ define $U_H := \sum_{n\in H} U_n$ and $\lambda_H := \sum_{n\in H} 2^{-n}$. Note that each U_H is a balanced neighborhood of 0. Define now the function $V\to\mathbb{R}^+_0 : x\mapsto \|x\|$ by

$$\|x\| := \inf_H \{\lambda_H | x\in U_H\}$$

if $x\in U_H$ for some H and $\|x\|=1$ otherwise. We proceed to show that $\|\cdot\|$ defines a pseudo-seminorm and generates the topology of V.

Fix $x\in V$ and $\lambda\in\mathbb{K}$ with $|\lambda|\le 1$. Since U_H is balanced for each H, λx is contained at least in the same sets U_H as x. Because the definition of $\|\cdot\|$ uses an infimum, $\|\lambda x\|\le\|x\|$. This confirms Property 1 of Definition 2.23.

To show the triangle inequality (Property 2 of Definition 2.23) we first note that for finite subsets H,K of $\mathbb{N}$ with the property $\lambda_H+\lambda_K<1$ there is another unique finite subset L of $\mathbb{N}$ such that $\lambda_L=\lambda_H+\lambda_K$. Furthermore, $U_H+U_K\subseteq U_L$ in this situation. Now, fix $x,y\in V$. If $\|x\|+\|y\|\ge 1$ the triangle inequality is trivial. Otherwise, we can find $\epsilon>0$ such that $\|x\|+\|y\|+2\epsilon<1$. We now fix finite subsets H,K of $\mathbb{N}$ such that $x\in U_H$, $y\in U_K$ while $\lambda_H<\|x\|+\epsilon$ and $\lambda_K<\|y\|+\epsilon$. Let L be the finite subset of $\mathbb{N}$ such that $\lambda_L=\lambda_H+\lambda_K$. Then $x+y\in U_L$ and hence $\|x+y\|\le\lambda_L=\lambda_H+\lambda_K<\|x\|+\|y\|+2\epsilon$. Since the resulting inequality holds for any $\epsilon>0$ we must have $\|x+y\|\le\|x\|+\|y\|$ as desired.

It remains to show that the pseudo-seminorm generates the topology of the tvs. Since the topology generated by the pseudo-seminorm as well as that of the tvs are translation invariant, it is enough to show that the open balls around 0 of the pseudo-seminorm form a base of the filter of neighborhoods of 0 in the topology of the tvs. Let $n\in\mathbb{N}$. Clearly $B_{2^{-n}}(0)\subseteq U_n\subseteq B_{2^{-(n-1)}}(0)$. But this shows that $\{B_{2^{-n}}(0)\}_{n\in\mathbb{N}}$ generates the same filter as $\{U_n\}_{n\in\mathbb{N}}$. This completes the proof. □

Exercise 4. Show that for a tvs with a balanced translation-invariant pseudometric the concepts of totally boundedness of Definitions 1.79 and 2.12 coincide.

Proposition 2.27. *Let V be a mvs with pseudo-seminorm. Let $r > 0$ and $0 < \mu \le 1$. Then, $B_{\mu r}(0) \subseteq \mu B_r(0)$.*

Proof. **Exercise.** $\square$

Proposition 2.28. *Let V, W be mvs with compatible metrics and $f \in \mathrm{L}(V, W)$. (a) f is continuous iff for all $\epsilon > 0$ there exists $\delta > 0$ such that $f(B^V_\delta(0)) \subseteq B^W_\epsilon(0)$. (b) f is bounded iff there exists $\delta > 0$ such that for all $\epsilon > 0$ there is $\mu > 0$ such that $f(\mu B^V_\delta(0)) \subseteq B^W_\epsilon(0)$.*

Proof. **Exercise.** $\square$

Proposition 2.29. *Let V be a mvs and C a subspace. Then, the quotient space V/C is a mvs.*

Proof. **Exercise.** $\square$

2.3 Locally convex tvs

Definition 2.30. A tvs is called *locally convex* iff every neighborhood of 0 contains a convex neighborhood of 0.

Definition 2.31. Let V be a vector space over $\mathbb{K}$. Then a map $V \to \mathbb{R}^+_0 : x \mapsto \|x\|$ is called a *seminorm* iff it satisfies the following properties:

1. $\|\lambda x\| = |\lambda| \|x\|$ for all $\lambda \in \mathbb{K}, x \in V$.
2. For all $x, y \in V : \|x + y\| \le \|x\| + \|y\|$. (triangle inequality)

A seminorm is called a *norm* iff it satisfies in addition the following property:

3. $\|x\| = 0 \implies x = 0$.

Proposition 2.32. *A seminorm induces a balanced translation-invariant pseudometric via $d(x, y) := \|x - y\|$. Moreover, the open balls of this metric are convex.*

Proof. **Exercise.** $\square$

Proposition 2.33. *Let V be a vector space and $\{\|\cdot\|_\alpha\}_{\alpha \in A}$ a set of seminorms on V. For any finite subset $I \subseteq A$ and any $\epsilon > 0$ define*

$$U_{I,\epsilon} := \{x \in V : \|x\|_\alpha < \epsilon \; \forall \alpha \in I\}.$$

Then, the sets $U_{I,\epsilon}$ form the base of the filter of neighborhoods of 0 in a topology on V that makes it into a locally convex tvs. If A is countable, then V is pseudometrizable. Moreover, the topology is Hausdorff iff for any $x \in V \setminus \{0\}$ there exists $\alpha \in A$ such that $\|x\|_\alpha > 0$.

Proof. Let $I, I' \subseteq A$ be finite and $\epsilon, \epsilon' > 0$. Set $I'' := I \cup I'$ and $\epsilon'' := \min(\epsilon, \epsilon')$. Then, $U_{I'',\epsilon''} \subseteq U_{I,\epsilon} \cap U_{I',\epsilon'}$. So the $U_{I,\epsilon}$ really form the basis of a filter $\mathcal{F}$. We proceed to verify that $\mathcal{F}$ satisfies the properties required by Proposition 2.8. Clearly, $0 \in U$ for all $U \in \mathcal{F}$ since $\|0\|_\alpha = 0$ and so $0 \in U_{I,\epsilon}$ for all $I \subseteq A$ finite and $\epsilon > 0$. Also $\lambda\mathcal{F} = \mathcal{F}$ since $\lambda U_{I,\epsilon} = U_{I,|\lambda|\epsilon}$ for all $I \subseteq A$ finite and $\epsilon > 0$ by linearity of seminorms. As for property 1 of Proposition 2.7 consider $x \in V$, $I \subseteq A$ finite and $\epsilon > 0$ arbitrary. Set $\mu := \max_{\alpha\in I}\{\|x\|_\alpha\}$. Then, $x \in \frac{\mu+1}{\epsilon} U_{I,\epsilon}$. Property 2 of Proposition 2.7 is satisfied since open balls of a seminorm are balanced and the sets $U_{I,\epsilon}$ are finite intersections of such open balls and hence also balanced. Property 3 of Proposition 2.7 is sufficient to satisfy for a base. Observe then, $U_{I,\epsilon/2} + U_{I,\epsilon/2} \subseteq U_{I,\epsilon}$ for all $I \subseteq A$ finite and $\epsilon > 0$ due to the triangle inequality. Thus, the so defined topology makes V into a tvs.

Observe that the sets $U_{I,\epsilon}$ are convex, being finite intersections of open balls which are convex by Proposition 2.32. Thus, V is locally convex. If A is countable, then there is an enumeration $I_1, I_2, \dots$ of the finite subsets of A. It is easy to see that $U_{I_j,1/n}$ with $j \in \{1, \dots\}$ and $n \in \mathbb{N}$ provides then a countable basis of the filter of neighborhoods of 0. That is, V is pseudometrizable. Concerning the Hausdorff property suppose that for any $x \in V \setminus \{0\}$ there exists $\alpha \in A$ such that $\|x\|_\alpha > 0$. Then, for this x we have $x \notin U_{\{\alpha\},\|x\|_\alpha}$. So V is Hausdorff. Conversely, suppose V is Hausdorff. Given $x \in V \setminus \{0\}$ there exist thus $I \subseteq A$ finite and $\epsilon > 0$ such that $x \notin U_{I,\epsilon}$. In particular, there exists $\alpha \in I$ such that $\|x\|_\alpha \geq \epsilon > 0$. □

Exercise 5. In the context of Proposition 2.33 show that the topology is the coarsest such that all seminorms $\|\cdot\|_\alpha$ are continuous.

Definition 2.34. Let V be a tvs and $W \subseteq V$ a neighborhood of 0. The map $\|\cdot\|_W : V \to \mathbb{R}_0^+$ defined as

$$\|x\|_W := \inf\{\lambda \in \mathbb{R}_0^+ : x \in \lambda W\}$$

is called the *Minkowski functional* associated to W.

Proposition 2.35. *Let V be a tvs and $W \subseteq V$ a neighborhood of 0.*

1. $\|\mu x\|_W = \mu\|x\|_W$ *for all* $\mu \in \mathbb{R}_0^+$ *and* $x \in V$.
2. *If W is balanced, then* $\|cx\|_W = |c|\|x\|_W$ *for all* $c \in \mathbb{K}$ *and* $x \in V$.
3. *If W is convex, then* $\|x+y\|_W \leq \|x\|_W + \|y\|_W$ *for all* $x, y \in V$.
4. *If V is Hausdorff and W is bounded, then* $\|x\|_W = 0$ *implies* $x = 0$.

Proof. **<u>Exercise</u>.** □

Theorem 2.36. *Let V be a tvs. Then, V is locally convex iff there exists a set of seminorms inducing its topology as in Proposition 2.33. Also, V is locally convex and pseudometrizable iff there exists a countable such set.*

Proof. Given a locally convex tvs V, let $\{U_\alpha\}_{\alpha\in A}$ be a base of the filter of neighborhoods such that U_α is balanced and convex for all $\alpha \in A$. (**<u>Exercise</u>.**How can this be achieved?) In case that V is pseudometrizable we choose the base such that A is countable. Let $\|\cdot\|_\alpha$ be the Minkowski functional associated to U_α. Then, by Proposition 2.35, $\|\cdot\|_\alpha$ is a seminorm for each $\alpha \in A$. We claim that the topology generated by the seminorms is precisely the topology of V. **<u>Exercise</u>.**Complete the proof. □

<u>Exercise</u> 6. Let V be a locally convex tvs and W a balanced and convex neighborhood of 0. Show that the Minkowski functional associated to W is continuous on V.

<u>Exercise</u> 7. Let V be a vector space and $\{\|\cdot\|_n\}_{n\in\mathbb{N}}$ a sequence of seminorms on V. Define the function $q : V \to \mathbb{R}^+_0$ via

$$q(x) := \sum_{n=1}^{\infty} 2^{-n} \frac{\|x\|_n}{\|x\|_n + 1}.$$

(a) Show that q is a pseudo-seminorm on V. (b) Show that the topology generated on V by q is the same as that generated by the sequence $\{\|\cdot\|_n\}_{n\in\mathbb{N}}$.

2.4 Normed and seminormed vector spaces

Definition 2.37. A tvs is called *locally bounded* iff it contains a bounded neighborhood of 0.

Proposition 2.38. *A locally bounded tvs is pseudometrizable.*

Proof. Let V be a locally bounded tvs and U a bounded neighborhood of 0 in V. The sequence $\{U_n\}_{n\in N}$ with $U_n := \frac{1}{n}U$ is the base of a filter $\mathcal{F}$ on V. Take a neighborhood W of 0. By boundedness of U there exists $\lambda \in \mathbb{R}^+$ such that $U \subseteq \lambda W$. Choosing $n \in \mathbb{N}$ with $n \geq \lambda$ we find $U_n \subseteq W$, i.e., $W \in \mathcal{F}$. Hence $\mathcal{F}$ is the filter of neighborhoods of 0 and we have presented a countable base for it. By Theorem 2.26, V is pseudometrizable. □

Proposition 2.39. *Let A, B be a tvs and $f \in \mathrm{CL}(A, B)$. If A or B is locally bounded then f is bounded. Hence, $\mathrm{CL}(A, B) = \mathrm{BL}(A, B)$ in this case.*

Proof. **Exercise**. □

Definition 2.40. A tvs V is called *(semi)normable* iff the topology of V is induced by a (semi)norm.

Theorem 2.41. *A tvs V is seminormable iff V is locally bounded and locally convex.*

Proof. Suppose V is a seminormed vector space. Then, every ball is bounded and also convex, so in particular, V is locally bounded and locally convex.

Conversely, suppose V is a tvs that is locally bounded and locally convex. Take a bounded neighborhood U_1 of 0 and a convex subneighborhood U_2 of U_1. Now take a balanced subneighborhood U_3 of U_2 and its convex hull $W = \mathrm{conv}(U_3)$. Then W is a balanced, convex and bounded (since $W \subseteq U_2 \subseteq U_1$) neighborhood of 0 in V. Thus, by Proposition 2.35 the Minkowski functional $\|\cdot\|_W$ defines a seminorm on V. It remains to show that the topology generated by this seminorm coincides with the topology of V. Let U be an open set in the topology of V and $x \in U$. The ball $B_1(0)$ defined by the seminorm is bounded since $B_1(0) \subseteq W$ and W is bounded. Hence there exists $\lambda \in \mathbb{R}^+$ such that $B_1(0) \subseteq \lambda(U - x)$, i.e., $\lambda^{-1}B_1(0) \subseteq U - x$. But $\lambda^{-1}B_1(0) = B_{\lambda^{-1}}(0)$ by linearity and thus $B_{\lambda^{-1}}(x) \subseteq U$. Hence, U is open in the seminorm topology as well. Conversely, consider a ball $B_\epsilon(0)$ defined by the seminorm for some $\epsilon > 0$ and take $x \in B_\epsilon(0)$. Choose $\delta > 0$ such that $\|x\|_W < \epsilon - \delta$. Observe that $\frac{1}{2}W \subseteq B_1(0)$ and thus by linearity $\frac{\delta}{2}W \subseteq B_\delta(0)$. It follows that $\frac{\delta}{2}W + x \subseteq B_\epsilon(0)$. But $\frac{\delta}{2}W + x$ is a neighborhood of x so it follows that $B_\epsilon(0)$ is open. This completes the proof. □

Exercise 8. Let V be locally convex tvs with its topology generated by a finite family of seminorms. Show that V is seminormable.

Proposition 2.42. *Let V be a seminormed vector space and $U \subseteq V$ a subset. Then, U is bounded iff there exists $c \in \mathbb{R}^+$ such that $\|x\| \le c$ for all $x \in U$.*

Proof. **Exercise**. □

Proposition 2.43. *Let A, B be seminormed vector spaces and $f \in \mathrm{L}(A, B)$. f is bounded iff there exists $c \in \mathbb{R}^+$ such that $\|f(x)\| \le c\,\|x\|$ for all $x \in A$.*

Proof. **Exercise**. □

Proposition 2.44. *Let V be a tvs and C a vector subspace. If V is locally convex, then so is V/C. If V is locally bounded, then so is V/C.*

Proof. **Exercise**. □

2.5 Inner product spaces

As before $\mathbb{K}$ stands for a field that is either $\mathbb{R}$ or $\mathbb{C}$.

Definition 2.45. Let V be a vector space over $\mathbb{K}$ and $\langle\cdot,\cdot\rangle : V \times V \to \mathbb{K}$ a map. $\langle\cdot,\cdot\rangle$ is called a *bilinear* (if $\mathbb{K} = \mathbb{R}$) or *sesquilinear* (if $\mathbb{K} = \mathbb{C}$) form iff it satisfies the following properties:

- $\langle u + v, w\rangle = \langle u, w\rangle + \langle v, w\rangle$ and
  $\langle u, v + w\rangle = \langle u, v\rangle + \langle u, w\rangle$ for all $u, v, w \in V$.
- $\langle \lambda u, v\rangle = \lambda\langle u, v\rangle$ and $\langle u, \lambda v\rangle = \overline{\lambda}\langle u, v\rangle$ for all $\lambda \in \mathbb{K}$ and $v \in V$.

$\langle\cdot,\cdot\rangle$ is called *symmetric* (if $\mathbb{K} = \mathbb{R}$) or *hermitian* (if $\mathbb{K} = \mathbb{C}$) iff it satisfies in addition the following property:

- $\langle u, v\rangle = \overline{\langle v, u\rangle}$ for all $u, v \in V$.

$\langle\cdot,\cdot\rangle$ is called *positive* iff it satisfies in addition the following property:

- $\langle v, v\rangle \geq 0$ for all $v \in V$.

$\langle\cdot,\cdot\rangle$ is called *definite* iff it satisfies in addition the following property:

- If $\langle v, v\rangle = 0$ then $v = 0$ for all $v \in V$.

A map with all these properties is also called a *scalar product* or an *inner product*. V equipped with such a structure is called an *inner product space* or a *pre-Hilbert space*.

Theorem 2.46 (Schwarz Inequality)**.** *Let V be a vector space over $\mathbb{K}$ with a scalar product $\langle\cdot,\cdot\rangle : V \times V \to \mathbb{K}$. Then, the following inequality is satisfied:*

$$|\langle v, w\rangle|^2 \leq \langle v, v\rangle\langle w, w\rangle \quad \forall v, w \in V.$$

Proof. By definiteness $\alpha := \langle v, v\rangle \neq 0$ and we set $\beta := -\langle w, v\rangle$. By positivity we have,

$$0 \leq \langle \beta v + \alpha w, \beta v + \alpha w\rangle.$$

Using bilinearity and symmetry (if $\mathbb{K} = \mathbb{R}$) or sesquilinearity and hermiticity (if $\mathbb{K} = \mathbb{C}$) on the right hand side this yields,

$$0 \leq |\langle v, v\rangle|^2\langle w, w\rangle - \langle v, v\rangle|\langle v, w\rangle|^2.$$

(**Exercise**.Show this.) Since $\langle v, v\rangle \neq 0$ we can divide by it and arrive at the required inequality. □

Proposition 2.47. *Let V be a vector space over $\mathbb{K}$ with a scalar product $\langle\cdot,\cdot\rangle : V \times V \to \mathbb{K}$. Then, V is a normed vector space with norm given by $\|v\| := \sqrt{\langle v, v\rangle}$.*

Proof. **Exercise**.Hint: To prove the triangle inequality, show that $\|v + w\|^2 \leq (\|v\| + \|w\|)^2$ can be derived from the Schwarz inequality (Theorem 2.46). □

Proposition 2.48. *Let V be an inner product space. Then, $\forall v, w \in V$,*

$$\langle v, w\rangle = \frac{1}{4}\left(\|v + w\|^2 - \|v - w\|^2\right) \quad \textit{if} \quad \mathbb{K} = \mathbb{R},$$
$$\langle v, w\rangle = \frac{1}{4}\left(\|v + w\|^2 - \|v - w\|^2 + \mathrm{i}\|v + \mathrm{i}w\|^2 - \mathrm{i}\|v - \mathrm{i}w\|^2\right) \quad \textit{if} \quad \mathbb{K} = \mathbb{C}$$

Proof. **Exercise**. □

Proposition 2.49. *Let V be an inner product space. Then, its scalar product $V \times V \to \mathbb{K}$ is continuous.*

Proof. **Exercise**. □

Theorem 2.50. *Let V be a normed vector space. Then, there exists a scalar product on V inducing the norm iff the parallelogram equality holds,*

$$\|v + w\|^2 + \|v - w\|^2 = 2\|v\|^2 + 2\|w\|^2 \quad \forall v, w \in V.$$

Proof. **Exercise**. □

Example 2.51. The spaces $\mathbb{R}^n$ and $\mathbb{C}^n$ are inner product spaces via

$$\langle v, w\rangle := \sum_{i=1}^{n} v_i \overline{w_i},$$

where v_i, w_i are the coefficients with respect to the standard basis.

3 First examples and properties

3.1 Elementary topologies on function spaces

If V is a vector space over $\mathbb{K}$ and S is some set, then the set of maps $S \to V$ naturally forms a vector space over $\mathbb{K}$. This is probably the most important source of topological vector spaces in functional analysis. Usually, the spaces S and V carry additional structure (e.g. topologies) and the maps in question may be restricted, e.g. to be continuous etc. The topology given to this vector space of maps usually depends on these additional structures.

Example 3.1. Let S be a set and $F(S,\mathbb{K})$ be the set of functions on S with values in $\mathbb{K}$. Consider the set of seminorms $\{p_x\}_{x\in S}$ on $F(S,\mathbb{K})$ defined by $p_x(f) := |f(x)|$. This gives $F(S,\mathbb{K})$ the structure of a locally convex tvs. The topology defined in this way is also called the *topology of pointwise convergence*.

Exercise 9. Show that this topology is the coarsest topology making all evaluation maps, i.e., maps of the type $f \mapsto f(x)$, continuous. Show also that a sequence in $F(S,\mathbb{K})$ converges with respect to this topology iff it converges pointwise.

Example 3.2. Let S be a set and $B(S,\mathbb{K})$ be the set of bounded functions on S with values in $\mathbb{K}$. Then, $B(S,\mathbb{K})$ is a normed vector space with the supremum norm:

$$\|f\| := \sup_{x\in B(S,\mathbb{K})} |f(x)| \quad \forall f \in B(S,\mathbb{K}).$$

The topology defined in this way is also called the *topology of uniform convergence*.

Exercise 10. Show that a sequence in $B(S,\mathbb{K})$ converges with respect to this topology iff it converges uniformly on all of S.

Exercise 11. (a) Show that on $B(S,\mathbb{K})$ the topology of uniform convergence is finer than the topology of pointwise convergence. (b) Under which circumstances are both topologies equal?

Example 3.3. Let S be a topological space and $\mathfrak{K}$ the set of compact subsets of S. For $K \in \mathfrak{K}$ define on $\mathrm{C}(S,\mathbb{K})$ the seminorm

$$\|f\|_K := \sup_{x\in K} |f(x)| \quad \forall f \in \mathrm{C}(S,\mathbb{K}).$$

The topology defined in this way on $\mathrm{C}(S,\mathbb{K})$ is called the *topology of compact convergence*.

Exercise 12. Show that a sequence in $\mathrm{C}(S,\mathbb{K})$ converges with respect to this topology iff it converges compactly, i.e., uniformly in any compact subset.

Exercise 13. (a) Show that on $\mathrm{C}(S,\mathbb{K})$ the topology of compact convergence is finer than the topology of pointwise convergence. (b) Show that on the space $\mathrm{C_b}(S,\mathbb{K})$ of bounded continuous maps the topology of uniform convergence is finer than the topology of compact convergence. (c) Give a sufficient condition for them to be equal.

Definition 3.4. Let S be a set, V a tvs. Let $\mathfrak{S}$ a non-empty set of non-empty subsets of S with the property that for X, Y in $\mathfrak{S}$ there exists $Z \in \mathfrak{S}$ such that $X \cup Y \subseteq Z$. Let $\mathcal{B}$ be a base of the filter of neighborhoods of 0 in V. Then, for $X \in \mathfrak{S}$ and $U \in \mathcal{B}$ the sets

$$M(X,U) := \{f \in F(S,V) : f(X) \subseteq U\}$$

define a base of the filter of neighborhoods of 0 for a translation invariant topology on $F(S,V)$. This is called the $\mathfrak{S}$-topology on $F(S,V)$.

Proposition 3.5. *Let S be a set, V a tvs and $\mathfrak{S} \subseteq \mathfrak{P}(S)$ as in Definition 3.4. Let $A \subseteq F(S,V)$ be a vector subspace. Then, A is a tvs with the the $\mathfrak{S}$-topology iff $f(X)$ is bounded for all $f \in A$ and $X \in \mathfrak{S}$.*

Proof. **Exercise.** □

Exercise 14. (a) Let S be a set and $\mathfrak{S}$ be the set of finite subsets of S. Show that the $\mathfrak{S}$-topology on $F(S,\mathbb{K})$ is the topology of pointwise convergence. (b) Let S be a topological space and $\mathfrak{K}$ the set of compact subsets of S. Show that the $\mathfrak{K}$-topology on $\mathrm{C}(S,\mathbb{K})$ is the topology of compact convergence. (c) Let S be a set and $\mathfrak{S}$ a set of subsets of S such that $S \in \mathfrak{S}$. Show that the $\mathfrak{S}$-topology on $B(S,\mathbb{K})$ is the topology of uniform convergence.

3.2 Completeness

In the absence of a pseudometric we can use the vector space structure of a tvs to complement the information contained in the topology in order to define a Cauchy property which in turn will be used to define an associated notion of completeness.

Definition 3.6. A sequence $\{x_n\}_{n\in\mathbb{N}}$ in a tvs V is called a *Cauchy sequence* iff for every neighborhood U of 0 in V there is a number $N > 0$ such that $x_n - x_m \in U$ for all $n, m \geq N$.

Proposition 3.7. *Let V be a mvs with translation-invariant pseudometric. Then, the Cauchy property for sequences in tvs coincide with the previuosly defined one in pseudometric spaces. That is, Definition 3.6 coincides then with Definition 1.74.*

Proof. Straightforward. □

This Proposition implies that there is no conflict with our previous definition of a Cauchy sequence in pseudometric spaces if we restrict ourselves to translation-invariant pseudometrics. Moreover, it implies that for this purpose it does not matter which pseudometric we use, as long as it is translation-invariant. This latter condition is indeed essential.

Exercise 15. Give an example of an mvs with two compatible metrics d^1, d^2 and a sequence x, such that x is Cauchy with respect to d^1, but not with respect to d^2.

In the following, whenever we talk about a Cauchy sequence in a tvs (possibly with additional) structure, we mean a Cauchy sequence according to Definition 3.6.

For a topologically sensible notion of completeness, we need something more general than Cauchy sequences: Cauchy filters.

Definition 3.8. A filter $\mathcal{F}$ on a subset A of a tvs V is called a *Cauchy filter* iff for every neighborhood U of 0 in V there is an element $W \in \mathcal{F}$ such that $W - W \subseteq U$.

Proposition 3.9. *A sequence is Cauchy iff the associated filter is Cauchy.*

Proof. **Exercise.** □

Proposition 3.10. *Let V be a tvs, $\mathcal{F}$ a Cauchy filter on a subset A of V. If $p \in V$ is accumulation point of $\mathcal{F}$, then $\mathcal{F}$ converges to p.*

Proof. Let U be a neighborhood of 0 in V. Then, there exists a neighborhood W of 0 in U such that $W + W \subseteq U$. Since $\mathcal{F}$ is a Cauchy filter there exists $F \in \mathcal{F}$ such that $F - F \subseteq W$. On the other hand, p is accumulation point of $\mathcal{F}$ so there exists $q \in F \cap (p + W)$. Then, we have $F - q \subseteq W$ and thus $F \subseteq q + W \subseteq p + W + W \subseteq p + U$. This shows that every neighborhood of p is contained in $\mathcal{F}$, i.e., $\mathcal{F}$ converges to p. □

Proposition 3.11. *A converging filter is Cauchy.*

Proof. **Exercise.** □

Definition 3.12. A subset U of a tvs is called *complete* iff every Cauchy filter on U converges to a point in U. It is called *sequentially complete* iff every Cauchy sequence in U converges to a point in U.

Since completeness is an important and convenient concept in functional analisis, the complete versions of Hausdorff tvs have special names. In particular, a complete metrizable locally convex tvs is called a *Fréchet space*, a complete normable tvs is called a *Banach space*, and a complete inner product space is called a *Hilbert space*.

Obviously, completeness implies sequential completeness, but not necessarily the other way round. Note that for an mvs with translation-invariant pseudometric, completeness in the sense of metric spaces (Definition 1.77) is now called sequential completeness. However, we will see that in this context it is equivalent to completeness in the sense of the above definition.

Proposition 3.13. *Let V be a mvs. Then, V is complete (in the sense of tvs) iff it is sequentially complete.*

Proof. We have to show that sequential completeness implies completeness. (The opposite direction is obvious.) We use a translation-invariant pseudometric on V. Suppose $\mathcal{F}$ is a Cauchy filter on V. That is, for any $\epsilon > 0$ there exists $U \in \mathcal{F}$ such that $U - U \subseteq B_\epsilon(0)$. Now, for each $n \in \mathbb{N}$ choose consecutively $U_n \in \mathcal{F}$ such that $U_n - U_n \subseteq B_{1/n}(0)$ and $U_n \subseteq U_{n-1}$ if $n > 1$ (possibly by using the intersection property). Thus, for every $N \in \mathbb{N}$ we have that for all $n, m \geq N : U_n - U_m \subseteq B_{1/N}(0)$. Now for each $n \in \mathbb{N}$ choose an element $x_n \in U_n$. These form a Cauchy sequence and by sequential completeness converge to a point $x \in V$. Given n observe that for all $y \in U_n : d(y, x) \leq d(y, x_n) + d(x_n, x) < \frac{1}{n} + \frac{1}{n}$, hence $U_n \subseteq B_{2/n}(x)$ and thus $B_{2/n}(x) \in \mathcal{F}$. Since this is true for all $n \in \mathbb{N}$, $\mathcal{F}$ contains arbitrarily small neighborhoods of x and hence all of them, i.e., converges to x. □

Proposition 3.14. *(a) Let V be a Hausdorff tvs and A be a complete subset. Then A is closed. (b) Let V be a tvs and A be a closed subset of a complete subset B. Then A is complete.*

Proof. **Exercise**. □

We proceed to show the analogue of Proposition 1.80.

Lemma 3.15. *Let V be a tvs, $C \subseteq V$ totally bounded and $\mathcal{F}$ an ultrafilter on C. Then $\mathcal{F}$ is Cauchy.*

Proof. Let U be a neighborhood of 0 in V. Choose another neighborhood W of 0 such that W is balanced and $W + W \subseteq U$. Since C is totally bounded there is a finite subset $F = \{x_1, \dots, x_n\}$ of V such that $C \subseteq F + W$. This implies in turn that there is $k \in \{1, \dots, \}$ such that $(x_k + W) \cap X \neq \emptyset$ for all $X \in \mathcal{F}$. To see that this is true suppose the contrary. Then for each $i \in \{1, \dots, n\}$ there is $X_i \in \mathcal{F}$ such that $(x_i + W) \cap X_i = \emptyset$. But, then $\emptyset = \bigcap_{i=1}^n X_i \in \mathcal{F}$, a contradiction. Thus, since $\mathcal{F}$ is ultrafilter we must have $x_k + W \in \mathcal{F}$ by Lemma 1.23. But $(x_k + W) - (x_k + W) = W - W = W + W \subseteq U$ by construction. So $\mathcal{F}$ is a Cauchy filter. □

Proposition 3.16. *Let V be a tvs and $C \subseteq V$ a compact subset. Then, C is complete and totally bounded.*

Proof. **Exercise.** □

Proposition 3.17. *Let V be a tvs and $C \subseteq V$ a subset. If C is totally bounded and complete then it is compact.*

Proof. Let $\mathcal{F}$ be a filter on C. By Proposition 1.24 there exists an ultrafilter $\mathcal{F}'$ in C such that $\mathcal{F} \subseteq \mathcal{F}'$. Since C is totally bounded, Lemma 3.15 implies that $\mathcal{F}'$ is Cauchy. Since C is complete, $\mathcal{F}'$ must converge to some point $p \in C$. By Proposition 1.58, this means that p is accumulation point of $\mathcal{F}'$. By Proposition 1.59 this implies that p is accumulation point of $\mathcal{F}$. Since $\mathcal{F}$ was arbitrary, Proposition 1.64 implies that C is compact. □

Proposition 3.18. *Let V be a complete mvs and C a vector subspace. Then V/C is complete.*

Proof. **Exercise.** □

Exercise 16. Which of the topologies defined above are complete? Which become complete under additional assumptions on the space S?

3.3 Finite dimensional tvs

Theorem 3.19. *Let V be a Hausdorff tvs of dimension $n \in \mathbb{N}$. Then, any isomorphism of vector spaces from $\mathbb{K}^n$ to V is also an isomorphism of tvs. Moreover, any linear map from V to any tvs is continuous.*

Proof. We first show that any linear map from $\mathbb{K}^n$ to any tvs W is continuous. Define the map $g : \mathbb{K}^n \times W^n \to W$ given by

$$g((\lambda_1, \dots, \lambda_n), (v_1, \dots, v_n)) := \lambda_1 v_1 + \cdots + \lambda_n v_n.$$

This map can be obtained by taking products and compositions of vector addition and scalar multiplication, which are continuous. Hence it is continuous. On the other hand, any linear map $f : \mathbb{K}^n \to W$ takes the form $f(\lambda_1, \ldots, \lambda_n) = g((\lambda_1, \ldots, \lambda_n), (v_1, \ldots, v_n))$ for some fixed set of vectors $\{v_1, \ldots, v_n\}$ in W and is thus continuous by Proposition 1.18.

We proceed to show that any linear map $V \to \mathbb{K}^n$ is continuous. We do this by induction in n starting with $n = 1$. For $n = 1$ any such non-zero map takes the form $g : \lambda e_1 \to \lambda$ for some $e_1 \in V \setminus \{0\}$. (If $g = 0$ continuity is trivial.) For $r > 0$ consider the element $re_1 \in V$. Since V is Hausdorff there exists an open neighborhood U of 0 in V that does not contain re_1. Moreover, we can choose U to be balanced. But then it is clear that $U \subseteq g^{-1}(B_r(0))$. That is, $g^{-1}(B_r(0))$ is a neighborhood of 0 in V. Since open balls centered at 0 form a base of neighborhoods of 0 in $\mathbb{K}$ this implies that the preimage of any neighborhood of 0 in $\mathbb{K}$ is a neighborhood of 0 in V. By Proposition 2.16.a this implies that g is continuous.

We now assume that we have proofed the statement in dimension $n-1$. Let V be a Hausdorff tvs of dimension n. Consider now some non-zero linear map $h : V \to \mathbb{K}$. We factorize h as $h = \tilde{h} \circ p$ into the projection $p : V \to V/\ker h$ and the linear map $\tilde{h} : V/\ker h \to \mathbb{K}$. $\ker h$ is a vector subspace of V of dimension $n-1$. In particular, it is a Hausdorff tvs and hence by assumption of the induction isomorphic as a tvs to $\mathbb{K}^{n-1}$. Thus, it is complete and by Proposition 3.14.a closed as a subspace of V. Therefore by Proposition 2.19 the quotient tvs $V/\ker h$ is Hausdorff. Since $V/\ker h$ is also one-dimensional it is isomorphic as a tvs to $\mathbb{K}$ as we have shown above. Thus, $\tilde{h}$ is continuous. Since the projection p is continuous by definition, the composition $h = \tilde{h} \circ p$ must be continuous. Hence, any linear map $V \to \mathbb{K}$ is continuous. But a linear map $V \to \mathbb{K}^n$ can be written as a composition of the continuous map $V \to V^n$ given by $v \mapsto (v, \ldots, v)$ with the product of n linear (and hence continuous) maps $V \to \mathbb{K}$. Thus, it must be continuous.

We have thus shown that for any n a Hausdorff tvs V of dimension n is isomorphic to $\mathbb{K}^n$ as a tvs via any vector space isomorphism. Thus, by the first part of the proof any linear map $V \to W$, where W is an arbitrary tvs must be continuous. □

Proposition 3.20. *Let X be a Hausdorff tvs. Then, any finite dimensional subspace of X is complete and closed.*

Proof. Let $A \subseteq X$ be a subspace of dimension n. By Theorem 3.19, A as a tvs is isomorphic to $\mathbb{K}^n$. In particular, A is complete and thus closed in X by Proposition 3.14. □

Proposition 3.21. *Let X be a Hausdorff tvs, C a closed subspace of X and F a finite-dimensional subspace of X. Then, $F + C$ is closed in X.*

Proof. Since C is closed X/C is a Hausdorff tvs. Let $p : X \to X/C$ be the continuous projection. Then, $p(F)$ is finite-dimensional, hence complete and closed in X/C by Proposition 3.20. Thus, $F + C = p^{-1}(p(F))$ is closed. $\square$

Proposition 3.22. *Let C be a bounded subset of $\mathbb{K}^n$ with the standard topology. Then C is totally bounded.*

Proof. **Exercise.** $\square$

Theorem 3.23 (Riesz)**.** *Let V be a Hausdorff tvs. Then, V is locally compact iff it is finite dimensional.*

Proof. If V is a finite dimensional Hausdorff tvs, then its is isomorphic to $\mathbb{K}^n$ for some n by Theorem 3.19. But closed balls around 0 are compact neighborhoods of 0 in $\mathbb{K}^n$, i.e., $\mathbb{K}^n$ is locally compact.

Now assume that V is a locally compact Hausdorff tvs. Let K be a compact and balanced neighborhood of 0. We can always find this since given a compact neighborhood by Proposition 2.10 we can find a balanced and closed subneighborhood which by Proposition 1.39 must then also be compact. Now let U be an open subneighborhood of $\frac{1}{2}K$. By compactness of K, there exists a finite set of points $\{x_1, \ldots, x_n\}$ such that $K \subseteq \bigcup_{i=1}^n (x_i + U)$. Let W be the finite dimensional subspace of V spanned by $\{x_1, \ldots, x_n\}$. By Theorem 3.19 W is isomorphic to $\mathbb{K}^m$ for some $m \in \mathbb{N}$ and hence complete and closed in V by Proposition 3.14. So by Proposition 2.19 the quotient space V/W is a Hausdorff tvs. Let $\pi : V \to V/W$ be the projection. Observe that, $K \subseteq W + U \subseteq W + \frac{1}{2}K$. Thus, $\pi(K) \subseteq \pi(\frac{1}{2}K)$, or equivalently $\pi(2K) \subseteq \pi(K)$. Iterating, we find $\pi(2^k K) \subseteq \pi(K)$ for all $k \in \mathbb{N}$ and hence $\pi(V) = \pi(K)$ since $V = \bigcup_{k=1}^\infty 2^k K$ as K is balanced. Since π is continuous $\pi(K) = \pi(V) = V/W$ is compact. But since V/W is Hausdorff any one dimensional subspace of it is isomorphic to $\mathbb{K}$ by Theorem 3.19 and hence complete and closed and would have to be compact. But $\mathbb{K}$ is not compact, so V/W cannot have any one-dimensional subspace, i.e., must have dimension zero. Thus, $W = V$ and V is finite dimensional. $\square$

Exercise 17. (a) Show that a finite dimensional tvs is always locally compact, even if it is not Hausdorff. (b) Give an example of an infinite dimensional tvs that is locally compact.

3.4 Equicontinuity

Definition 3.24. Let S be a topological space, T a tvs and $F \subseteq \mathrm{C}(S,T)$. Then, F is called *equicontinuous* at $a \in S$ iff for all neighborhoods W of 0 in T there exists a neighborhood U of a in S such that $f(U) \subseteq f(a) + W$ for all $f \in F$. Moreover, F is called *equicontinuous* iff F is equicontinuous for all $a \in S$.

Exercise 18. Let S be a topological space and $F \subseteq \mathrm{C}(S,\mathbb{K})$. (a) Show that F is bounded in $\mathrm{C}(S,\mathbb{K})$ with the topology of pointwise convergence iff for each $x \in S$ there exists $c > 0$ such that $|f(x)| < c$ for all $f \in F$. (b) Show that F is bounded in $\mathrm{C}(S,\mathbb{K})$ with the topology of compact convergence iff for each $K \subseteq S$ compact there exists $c > 0$ such that $|f(x)| < c$ for all $x \in K$ and for all $f \in F$.

Lemma 3.25. *Let S be a topological space and $F \subseteq \mathrm{C}(S,\mathbb{K})$ equicontinuous. Then, F is bounded with respect to the topology of pointwise convergence iff it is bounded with respect to the topology of compact convergence.*

Proof. **Exercise.** □

Lemma 3.26. *Let S be a topological space and $F \subseteq \mathrm{C}(S,\mathbb{K})$ equicontinuous. Then, the closures of F in the topology of pointwise convergence and in the topology of compact convergence are equicontinuous.*

Proof. **Exercise.** □

Proposition 3.27. *Let S be a topological space and $F \subseteq \mathrm{C}(S,\mathbb{K})$ equicontinuous. If F is closed then it is complete, both in the topology of pointwise convergence and in the topology of compact convergence.*

Proof. We first consider the topology of pointwise convergence. Let $\mathcal{F}$ be a Cauchy filter in F. For each $x \in S$ induce a filter $\mathcal{F}_x$ generated by $e_x(\mathcal{F})$ on $\mathbb{K}$ through the evaluation map $e_x : \mathrm{C}(S,\mathbb{K}) \to \mathbb{K}$ given by $e_x(f) := f(x)$. Then each $\mathcal{F}_x$ is a Cauchy filter on $\mathbb{K}$ and thus convergent to a uniquely defined $g(x) \in \mathbb{K}$. This defines a function $g : S \to \mathbb{K}$. We proceed to show that g is continuous. Fix $a \in S$ and $\epsilon > 0$. By equicontinuity, there exists a neighborhood U of a such that $f(U) \subseteq B_\epsilon(f(a))$ for all $f \in F$ and hence $|f(x) - f(y)| < 2\epsilon$ for all $x, y \in U$ and $f \in F$. Fix $x, y \in U$. Then, there exists $f \in F$ such that $|f(x) - g(x)| < \epsilon$ and $|f(y) - g(y)| < \epsilon$. Hence

$$|g(x) - g(y)| \le |g(x) - f(x)| + |f(x) - f(y)| + |f(y) - g(y)| < 4\epsilon,$$

showing that g is continuous. Thus, $\mathcal{F}$ converges to g and $g \in F$ if F is closed.

We proceed to consider the topology of compact convergence. Let $\mathcal{F}$ be a Cauchy filter in F (now with respect to compact convergence). Then, $\mathcal{F}$ is also a Cauchy filter with respect to pointwise convergence and the previous part of the proof shows that there exists a function $g \in \mathrm{C}(S, \mathbb{K})$ to which $\mathcal{F}$ converges pointwise. But since $\mathcal{F}$ is Cauchy with respect to compact convergence it must convergence to g also compactly. Then, if F is closed we have $g \in F$ and F is complete. □

Theorem 3.28 (generalized Arzela-Ascoli). *Let S be a topological space. Let $F \subseteq \mathrm{C}(S, \mathbb{K})$ be equicontinuous and bounded in the topology of pointwise convergence. Then, F is relatively compact in $\mathrm{C}(S, \mathbb{K})$ with the topology of compact convergence.*

Proof. We consider the topology of compact convergence on $\mathrm{C}(S, \mathbb{K})$. By Lemma 3.25, F is bounded in this topology. The closure $\overline{F}$ of F is bounded by Proposition 2.11.c, equicontinuous by Lemma 3.26 and complete by Proposition 3.27. Due to Proposition 3.17 it suffices to show that $\overline{F}$ is totally bounded. Let U be a neighborhood of 0 in V. Then, there exists $K \subseteq S$ compact and $\epsilon > 0$ such that $U_{K,3\epsilon} \subseteq U$, where

$$U_{K,\delta} := \{f \in V : |f(x)| < \delta\ \forall x \in K\}.$$

By equicontinuity we can choose for each $a \in K$ a neighborhood W of a such that $|f(x) - f(a)| < \epsilon$ for all $x \in W$ and all $f \in \overline{F}$. By compactness of K there is a finite set of points $\{a_1, \ldots, a_n\}$ such that the associated neighborhoods $\{W_1, \ldots, W_n\}$ cover S. Now consider the continuous linear map $p : \mathrm{C}(S, \mathbb{K}) \to \mathbb{K}^n$ given by $p(f) := (f(a_1), \ldots, f(a_n))$. Since $\overline{F}$ is bounded, $p(\overline{F})$ is bounded in $\mathbb{K}^n$ (due to Proposition 2.16.b) and hence totally bounded (Proposition 3.22). Thus, there exists a finite subset $\{f_1, \ldots, f_m\} \subseteq \overline{F}$ such that $p(\overline{F})$ is covered by balls of radius ϵ centered at the points $p(f_1), \ldots, p(f_m)$. In particular, for any $f \in \overline{F}$ there is then $k \in \{1, \ldots, m\}$ such that $|f(a_i) - f_k(a_i)| < \epsilon$ for all $i \in \{1, \ldots, n\}$. Specifying also $x \in K$ there is $i \in \{1, \ldots, n\}$ such that $x \in W_i$. We obtain the estimate

$$|f(x) - f_k(x)| \leq |f(x) - f(a_i)| + |f(a_i) - f_k(a_i)| + |f_k(a_i) - f_k(x)| < 3\epsilon.$$

Since $x \in K$ was arbitrary this implies $f \in f_k + U_{K,3\epsilon} \subseteq f_k + U$. We conclude that $\overline{F}$ is covered by the set $\{f_1, \ldots, f_m\} + U$. Since U was an arbitrary neighborhood of 0 this means that $\overline{F}$ is totally bounded. □

Proposition 3.29. *Let S be a locally compact space. Let $F \subseteq \mathrm{C}(S, \mathbb{K})$ be totally bounded in the topology of compact convergence. Then, F is equicontinuous.*

Proof. **Exercise.** □

3.5 The Hahn-Banach Theorem

Theorem 3.30 (Hahn-Banach)**.** *Let V be a vector space over $\mathbb{K}$, p be a seminorm on V, $A \subseteq V$ a vector subspace. Let $f : A \to \mathbb{K}$ be a linear map such that $|f(x)| \le p(x)$ for all $x \in A$. Then, there exists a linear map $\tilde{f} : V \to \mathbb{K}$, extending f (i.e., $\tilde{f}(x) = f(x)$ for all $x \in A$) and such that $|f(x)| \le p(x)$ for all $x \in V$.*

Proof. We first consider the case $\mathbb{K} = \mathbb{R}$. Suppose that A is a proper subspace of V. Let $v \in V \setminus A$ and define B to be the subspace of V spanned by A and v. In a first step we show that there exists a linear map $\tilde{f} : B \to \mathbb{R}$ such that $\tilde{f}(x) = f(x)$ for all $x \in A$ and $|f(y)| \le p(y)$ for all $y \in B$. Since any vector $y \in B$ can be uniquely written as $y = x + \lambda v$ for some $x \in A$ and some $\lambda \in \mathbb{R}$, we have $\tilde{f}(y) = f(x) + \lambda \tilde{f}(v)$, i.e, $\tilde{f}$ is completely determined by its value on v. For all $x, x' \in A$ we have

$$f(x) + f(x') = f(x + x') \le p(x + x') \le p(x - v) + p(x' + v)$$

and thus,

$$f(x) - p(x - v) \le p(x' + v) - f(x').$$

In particular, defining a to be the supremum for $x \in A$ on the left and b to be the infimum for $y \in A$ on the right we get

$$a = \sup_{x \in A}\{f(x) - p(x - v)\} \le \inf_{x' \in A}\{p(x' + v) - f(x')\} = b.$$

Now choose $c \in [a, b]$ arbitrary. We claim that by setting $\tilde{f}(v) := c$, $\tilde{f}$ is bounded by p as required. For $x \in A$ and $\lambda > 0$ we get

$$\tilde{f}(x + \lambda v) = \lambda \left(\tilde{f}\left(\lambda^{-1} x\right) + c\right) \le \lambda\, p\left(\lambda^{-1} x + v\right) = p\left(x + \lambda v\right)$$
$$\tilde{f}(x - \lambda v) = \lambda \left(\tilde{f}\left(\lambda^{-1} x\right) - c\right) \le \lambda\, p\left(\lambda^{-1} x - v\right) = p\left(x - \lambda v\right).$$

Thus, we get $\tilde{f}(x) \le p(x)$ for all $x \in B$. Replacing x by $-x$ and using that $p(-x) = p(x)$ we obtain also $-\tilde{f}(x) \le p(x)$ and thus $|\tilde{f}(x)| \le p(x)$ as required.

We proceed to the second step of the proof, showing that the desired linear form $\tilde{f}$ exists on V. We will make use of Zorn's Lemma. Consider the set of pairs $(W, \tilde{f})$ of vector subspaces $A \subseteq W \subseteq V$ with linear forms $\tilde{f} : W \to \mathbb{R}$ that extend f and are bounded by p. These pairs are partially ordered by extension, i.e., $(W, \tilde{f}) \le (W', \tilde{f}')$ iff $W \subseteq W'$ and $\tilde{f}'|_W = \tilde{f}$. Moreover, for any totally ordered subset of pairs $\{(W_i, \tilde{f}_i)\}_{i \in I}$ there is an upper bound given by $(W_I, \tilde{f}_I)$ where $W_I := \bigcup_{i \in I} W_i$ and $\tilde{f}_I(x) := \tilde{f}_i(x)$ for $x \in W_i$. Thus, by Zorn's Lemma there exists a maximal pair $(W, \tilde{f})$. Since the first part of the proof has shown that for any proper vector subspace of V we can construct an extension, i.e., a pair that is strictly greater with respect to the ordering, we must have $W = V$. This concludes the proof in the case $\mathbb{K} = \mathbb{R}$.

We turn to the case $\mathbb{K} = \mathbb{C}$. Let $f_r(x) := \Re f(x)$ for all $x \in A$ be the real part of the linear form $f : A \to \mathbb{C}$. Since the complex vector spaces A and V are also real vector spaces and p reduces to a real seminorm, we can apply the real version of the proof to f_r to get a real linear map $\tilde{f}_r : V \to \mathbb{R}$ extending f_r and being bounded by p. We claim that $\tilde{f} : V \to \mathbb{C}$ given by

$$\tilde{f}(x) := \tilde{f}_r(x) - \mathrm{i}\tilde{f}_r(\mathrm{i}x) \quad \forall x \in V$$

is then a solution to the complex problem. We first verify that $\tilde{f}$ is complex linear. Let $x \in V$ and $\lambda \in \mathbb{C}$. Then, $\lambda = a + ib$ with $a, b \in \mathbb{R}$ and

$$\begin{aligned}
\tilde{f}(\lambda x) &= a\tilde{f}(x) + b\tilde{f}(\mathrm{i}x) \\
&= a\tilde{f}_r(x) - a\mathrm{i}\tilde{f}_r(\mathrm{i}x) + b\tilde{f}_r(\mathrm{i}x) + b\mathrm{i}\tilde{f}_r(x) \\
&= (a + \mathrm{i}b)\left(\tilde{f}_r(x) - \mathrm{i}\tilde{f}_r(\mathrm{i}x)\right) \\
&= \lambda\tilde{f}(x).
\end{aligned}$$

We proceed to verify that $\tilde{f}(x) = f(x)$ for all $x \in A$. For all $x \in A$,

$$\tilde{f}(x) = \Re f(x) - \mathrm{i}\Re f(\mathrm{i}x) = \Re f(x) - \mathrm{i}\Re(\mathrm{i}f(x)) = \Re f(x) + \mathrm{i}\Im(f(x)) = f(x).$$

It remains to show that $\tilde{f}$ is bounded by p. Let $x \in V$. Choose $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ such that $\lambda\tilde{f}(x) \in \mathbb{R}$. Then,

$$\left|\tilde{f}(x)\right| = \left|\lambda\tilde{f}(x)\right| = \left|\tilde{f}(\lambda x)\right| = \left|\tilde{f}_r(\lambda x)\right| \le p(\lambda x) = p(x).$$

This completes the proof. □

Corollary 3.31. *Let V be a seminormed vector space, $c > 0$, $A \subseteq V$ a vector subspace and $f : A \to \mathbb{K}$ a linear form satisfying $|f(x)| \le c\|x\|$ for all $x \in A$. Then, there exists a linear form $\tilde{f} : V \to \mathbb{K}$ that coincides with f on A and satisfies $|\tilde{f}(x)| \le c\|x\|$ for all $x \in V$.*

Proof. Immediate. $\square$

Theorem 3.32. *Let V be a locally convex tvs, $A \subseteq V$ a vector subspace and $f : A \to \mathbb{K}$ a continuous linear form. Then, there exists a continuous linear form $\tilde{f} : V \to \mathbb{K}$ that coincides with f on A.*

Proof. Since f is continuous on A, the set $U := \{x \in A : |f(x)| \le 1\}$ is a neighborhood of 0 in A. Since A carries the subset topology, there exists a neighborhood $\tilde{U}$ of 0 in V such that $\tilde{U} \cap A \subseteq U$. By local convexity, there exists a convex and balanced subneighborhood $W \subseteq \tilde{U}$ of 0 in V. The associated Minkowski functional $\|\cdot\|_W$ is a seminorm on V according to Proposition 2.35 and we have $|f(x)| \le \|x\|_W$ for all $x \in A$. Thus, we may apply the Hahn-Banach Theorem 3.30 to obtain a linear form $\tilde{f} : V \to \mathbb{K}$ that coincides with f on the subspace A and is bounded by $\|\cdot\|_W$. Since $\|\cdot\|_W$ is continuous this implies that $\tilde{f}$ is continuous. $\square$

Corollary 3.33. *Let V be a locally convex Hausdorff tvs. Then,* $\mathrm{CL}(V, \mathbb{K})$ *separates points in V. That is, for any pair $x, y \in V$ such that $x \neq y$, there exists $f \in \mathrm{CL}(V, \mathbb{K})$ such that $f(x) \neq f(y)$.*

Proof. **Exercise.** $\square$

Proposition 3.34. *Let X be a locally convex Hausdorff tvs. Then, any finite dimensional subspace of X admits a closed complement.*

Proof. We proceed by induction in dimension. Let $A \subseteq X$ be a subspace of dimension 1 and $v \in A \setminus \{0\}$. Define the linear map $\lambda : A \to \mathbb{K}$ by $\lambda(v) = 1$. Then, the Hahn-Banach Theorem in the form of Theorem 3.32 ensures that λ extends to a continuous map $\tilde{\lambda} : X \to \mathbb{K}$. Then, clearly $\ker \tilde{\lambda}$ is a closed complement of A in X. Now suppose we have shown that for any subspace of dimension n a closed complement exists in X. Let N be a subspace of X of dimension $n+1$. Choose an n-dimensional subspace $M \subset N$. This has a closed complement C by assumption. Moreover, C is a locally convex Hausdorff tvs in its own right. Let $A = N \cap C$. Then, A is a one-dimensional subspace of C and we can apply the initial part of the proof to conclude that it has a closed complement D in C. But D is closed also in X since C is closed in X and it is a complement of N. $\square$

3.6 More examples of function spaces

Definition 3.35. Let T be a locally compact space. A continuous function $f : T \to \mathbb{K}$ is said to *vanish at infinity* iff for any $\epsilon > 0$ the subset $\{x \in$

$T : |f(x)| \geq \epsilon\}$ is compact in T. The set of such functions is denoted by $\mathrm{C}_0(T, \mathbb{K})$.

Exercise 19. Let T be a locally compact space. Show that $\mathrm{C}_0(T, \mathbb{K})$ is complete in the topology of uniform convergence, but not in general complete in the topology of compact convergence.

Definition 3.36. Let U be a non-empty open subset of $\mathbb{R}^n$. For a multi-index $l \in \mathbb{N}_0^n$ we denote the corresponding partial derivative of a function $f : \mathbb{R}^n \to \mathbb{K}$ by

$$D^l f := \frac{\partial^{l_1} \dots \partial^{l_n}}{\partial x_1^{l_1} \dots \partial x_n^{l_n}} f.$$

Let $k \in \mathbb{N}_0$. If all partial derivatives with $|l| := l_1 + \dots + l_n \leq k$ for a function f exist and are continuous, we say that f is k times continuously differentiable. We denote the vector space of k times *continuously differentiable* functions on U with values in $\mathbb{K}$ by $\mathrm{C}^k(U, \mathbb{K})$. We say a function $f : U \to \mathbb{K}$ is *infinitely differentiable* or *smooth* if it is k times continuously differentiable for any $k \in \mathbb{N}_0$. The corresponding vector space is denoted by $\mathrm{C}^\infty(U, \mathbb{K})$.

Definition 3.37. Let U be a non-empty open and bounded subset of $\mathbb{R}^n$ and $k \in \mathbb{N}_0$. We denote by $\mathrm{C}^k(\overline{U}, \mathbb{K})$ the set of continuous functions $f : \overline{U} \to \mathbb{K}$ that are k times continuously differentiable on U, and such that any partial derivative $D^l f$ with $|l| \leq k$ extends continuously to $\overline{U}$. Similarly, we denote by $\mathrm{C}^\infty(\overline{U}, \mathbb{K})$ the set of continuous functions $f : \overline{U} \to \mathbb{K}$, smooth in U and such that any partial derivative extends continuously to $\overline{U}$.

Example 3.38. Let U be a non-empty open and bounded subset of $\mathbb{R}^n$. Let $l \in \mathbb{N}_0^n$ and define the seminorm $p_l : \mathrm{C}^k(\overline{U}, \mathbb{K}) \to \mathbb{R}_0^+$ via

$$p_l(f) := \sup_{x \in \overline{U}} \left| \left(D^l f\right)(x) \right|$$

for $k \in \mathbb{N}_0$ with $k \geq |l|$ or for $k = \infty$. For any $k \in \mathbb{N}_0$ the set of seminorms $\{p_l : l \in \mathbb{N}_0^n, |l| \leq k\}$ makes $\mathrm{C}^k(\overline{U}, \mathbb{K})$ into a normable vector space. Similarly, the set of seminorms $\{p_l : l \in \mathbb{N}_0^n\}$ makes $\mathrm{C}^\infty(\overline{U}, \mathbb{K})$ into a locally convex mvs.

Exercise 20. Let U be a non-empty open and bounded subset of $\mathbb{R}^n$. Show that $\mathrm{C}^\infty(\overline{U}, \mathbb{K})$ with the topology defined above is complete, but not normable.

Proposition 3.39. *Let T be a σ-compact space. Then, $\mathrm{C}(T,\mathbb{K})$ with the topology of compact convergence is metrizable.*

Proof. **Exercise.** $\square$

Example 3.40. Let U be a non-empty open subset of $\mathbb{R}^n$ and $k \in \mathbb{N}_0 \cup \{\infty\}$. Let W be an open and bounded subset of $\mathbb{R}^n$ such that $\overline{W} \subseteq U$ and let $l \in \mathbb{N}_0^n$ such that $|l| \le k$. Define the seminorm $p_{\overline{W},l} : \mathrm{C}^k(U,\mathbb{K}) \to \mathbb{R}_0^+$ via

$$p_{\overline{W},l}(f) := \sup_{x \in \overline{W}} \left| \left(D^l f\right)(x) \right|.$$

The set of these seminorms makes $\mathrm{C}^k(U,\mathbb{K})$ into a locally convex tvs.

Exercise 21. Let $U \subseteq \mathbb{R}^n$ be non-empty and open and let $k \in \mathbb{N}_0 \cup \{\infty\}$. Show that $\mathrm{C}^k(U,\mathbb{K})$ is complete and metrizable, but not normable.

Exercise 22. Let $0 \le k < m \le \infty$. (a) Let $U \subset \mathbb{R}^n$ be non-empty, open and bounded. Show that the inclusion map $\mathrm{C}^m(\overline{U},\mathbb{K}) \to \mathrm{C}^k(\overline{U},\mathbb{K})$ is injective and continuous, but does not in general have closed image. (b) Let $U \subseteq \mathbb{R}^n$ be non-empty and open. Show that the inclusion map $\mathrm{C}^m(U,\mathbb{K}) \to \mathrm{C}^k(U,\mathbb{K})$ is injective and continuous, but is in general neither bounded nor has closed image.

Exercise 23. Let $U \subset \mathbb{R}^n$ be non-empty, open and bounded, let $k \in \mathbb{N}_0 \cup \{\infty\}$. Show that the inclusion map $\mathrm{C}^k(\overline{U},\mathbb{K}) \to \mathrm{C}^k(U,\mathbb{K})$ is injective and continuous. Show also that its image is in general not closed.

Exercise 24. Let $k \in \mathbb{N}_0 \cup \{\infty\}$. For $f \in \mathrm{C}^1(\mathbb{R},\mathbb{K})$ consider the operator $D(f) := f'$. (a) Show that $D : \mathrm{C}^{k+1}([0,1],\mathbb{K}) \to \mathrm{C}^k([0,1],\mathbb{K})$ is continuous. (b) Show that $D : \mathrm{C}^{k+1}(\mathbb{R},\mathbb{K}) \to \mathrm{C}^k(\mathbb{R},\mathbb{K})$ is continuous.

Exercise 25. Let $k \in \mathbb{N}_0 \cup \{\infty\}$. For $f \in \mathrm{C}(\mathbb{R},\mathbb{K})$ consider the operator

$$(I(f))(y) := \int_0^y f(x)\,\mathrm{d}x.$$

(a) Show that $I : \mathrm{C}^k([0,1],\mathbb{K}) \to \mathrm{C}^{k+1}([0,1],\mathbb{K})$ is continuous. (b) Show that $I : \mathrm{C}^k(\mathbb{R},\mathbb{K}) \to \mathrm{C}^{k+1}(\mathbb{R},\mathbb{K})$ is continuous.

Definition 3.41. Let D be a non-empty, open and connected subset of $\mathbb{C}$. We denote by $\mathcal{O}(D)$ the vector space of holomorphic functions on D. If D is also bounded we denote by $\mathcal{O}(\overline{D})$ the vector space of complex continuous functions on $\overline{D}$ that are holomorphic in D.

<u>Exercise</u> 26. (a) Show that $\mathcal{O}(\overline{D})$ is complete with the topology of uniform convergence. (b) Show that $\mathcal{O}(D)$ is complete with the topology of compact convergence.

Theorem 3.42 (Montel)**.** *Let $D \subseteq \mathbb{C}$ be non-empty, open and connected and $F \subseteq \mathcal{O}(D)$. Then, the following are equivalent:*

1. *F is relatively compact.*

2. *F is totally bounded.*

3. *F is bounded.*

Proof. 1.⇒2. $\overline{F}$ is compact and hence totally bounded by Proposition 1.80. Since F is a subset of $\overline{F}$ it must also be totally bounded. 2.⇒3. This follows from Proposition 2.14. 3.⇒1. Since D is locally compact, it is easy to see that boundedness is equivalent to the following property: For each point $z \in D$ there exists a neighborhood $U \subseteq D$ and a constant $M > 0$ such that $|f(x)| \le M$ for all $x \in U$ and all $f \in F$. It can then be shown that F is equicontinuous [Notes on Complex Analysis, Theorem 5.28]. The Arzela-Ascoli Theorem 3.28 then ensures that F is relatively compact. □

Definition 3.43. Let X be a measurable space, μ a measure on X and $p > 0$. Define

$$\mathcal{L}^p(X, \mu, \mathbb{K}) := \{f : X \to \mathbb{K} \text{ measurable} : |f|^p \text{ integrable}\}.$$

Also define

$$\mathcal{L}^\infty(X, \mu, \mathbb{K}) := \{f : X \to \mathbb{K} \text{ measurable} : |f| \text{ bounded almost everywhere}\}.$$

We recall the following facts from real analysis.

Example 3.44. The set $\mathcal{L}^p(X, \mu, \mathbb{K})$ for $p \in (0, \infty]$ is a vector space.

1. $\|\cdot\|_\infty : \mathcal{L}^\infty(X, \mu, \mathbb{K}) \to \mathbb{R}^+_0$ given by

   $$\|f\|_\infty := \inf\{\|g\|_{\sup} : g = f \text{ a.e. and } g : X \to \mathbb{K} \text{ bounded measurable}\}$$

   defines a seminorm on $\mathcal{L}^\infty(X, \mu, \mathbb{K})$, making it into a complete seminormed space.

2. If $1 \le p < \infty$, then $\|\cdot\|_p : \mathcal{L}^p(X,\mu,\mathbb{K}) \to \mathbb{R}_0^+$ given by
$$\|f\|_p := \left(\int_X |f|^p\right)^{1/p}$$
defines a seminorm on $\mathcal{L}^p(X,\mu,\mathbb{K})$, making it into a complete seminormed space.

3. If $p \le 1$, then $s_p : \mathcal{L}^p(X,\mu,\mathbb{K}) \to \mathbb{R}_0^+$ given by
$$s_p(f) := \int_X |f|^p$$
defines a pseudo-seminorm on $\mathcal{L}^p(X,\mu,\mathbb{K})$, making it into a complete pseudometrizable space.

Example 3.45. For any $p \in (0,\infty]$, the closure $N := \overline{\{0\}}$ of zero in $\mathcal{L}^p(X,\mu,\mathbb{K})$ is the set of measurable functions that vanish almost everywhere. The quotient space $\mathrm{L}^p(X,\mu,\mathbb{K}) := \mathcal{L}^p(X,\mu,\mathbb{K})/N$ is a complete mvs. It carries a norm (i.e., is a Banach space) for $p \ge 1$ and a pseudo-norm otherwise. In the case $p = 2$ the norm comes from an inner product making the space into a Hilbert space.

3.7 The Banach-Steinhaus Theorem

Definition 3.46. Let S be a topological space. A subset $C \subseteq S$ is called *nowhere dense* iff its closure $\overline{C}$ does not contain any non-empty open set. A subset $C \subseteq S$ is called *meager* iff it is the countable union of nowhere dense subsets.

Proposition 3.47. *Let X and Y be tvs and $A \subseteq \mathrm{CL}(X,Y)$. Then A is equicontinuous iff for any neighborhood U of 0 in Y there exists a neighborhood V of 0 in X such that*
$$f(V) \subseteq W \quad \forall f \in A.$$

Proof. Immediate. □

Theorem 3.48 (Banach-Steinhaus)**.** *Let X and Y be tvs and $A \subseteq \mathrm{CL}(X,Y)$. For $x \in X$ define $A(x) := \{f(x) : f \in A\} \subseteq Y$. Define $B \subseteq X$ as*
$$B := \{x \in X : A(x) \text{ is bounded}\}.$$
If B is not meager in X, then $B = X$ and A is equicontinuous.

Proof. We suppose that B is not meager. Let U be an arbitrary neighborhood of 0 in Y. Choose a closed and balanced subneighborhood W of 0. Set

$$E := \bigcap_{f \in A} f^{-1}(W)$$

and note that E is closed and balanced, being an intersection of closed and balanced sets. If $x \in B$, then $A(x)$ is bounded, there exists $n \in \mathbb{N}$ such that $A(x) \subseteq nW$ and hence $x \in nE$. Therefore,

$$B \subseteq \bigcup_{n=1}^{\infty} nE.$$

If all sets nE were meager, their countable union would be meager and also the subset B. Since by assumption B is not meager, there must be at least one $n \in \mathbb{N}$ such that nE is not meager. But since the topology of X is scale invariant, this implies that E itself is not meager. Thus, the interior $\mathring{E} = \overline{\mathring{E}}$ is not empty. Also, $\mathring{E}$ is balanced since E is balanced and thus must contain 0. In particular, $\mathring{E}$, being open, is therefore a neighborhood of 0 and so is E itself. Thus,

$$f(E) \subseteq W \subseteq U \quad \forall f \in A.$$

This means that A is equicontinuous at 0 and hence equicontinuous by linearity (Proposition 3.47). Let now $x \in X$ arbitrary. Since x is bounded, there exists $\lambda > 0$ such that $x \in \lambda E$. But then, $f(x) \in f(\lambda E) \subseteq \lambda U$ for all $f \in A$. That is, $A(x) \subseteq \lambda U$, i.e., $A(x)$ is bounded and $x \in B$. Since x was arbitrary, $B = X$. □

Proposition 3.49. *Let S be a complete metric space and $C \subseteq S$ a meager subset. Then, C does not contain any non-empty open set. In particular, $C \neq S$.*

Proof. Since C is meager, there exists a sequence $\{C_n\}_{n \in \mathbb{N}}$ of nowhere dense subsets of S such that $C = \bigcup_{n \in \mathbb{N}} C_n$. Define $U_n := S \setminus \overline{C_n}$ for all $n \in \mathbb{N}$. Then, each U_n is open and dense in S. Thus, by Baire's Theorem 1.84 the intersection $\bigcap_{n \in \mathbb{N}} U_n$ is dense in S. Thus, its complement $\bigcup_{n \in \mathbb{N}} \overline{C_n}$ cannot contain any non-empty open set. The same is true for the subset $C \subseteq \bigcup_{n \in \mathbb{N}} \overline{C_n}$. □

Corollary 3.50. *Let X be a complete Hausdorff mvs, Y be a tvs and $A \subseteq \mathrm{CL}(X, Y)$. Suppose that $A(x) := \{f(x) : f \in A\} \subseteq Y$ is bounded for all $x \in X$. Then, A is equicontinuous.*

Proof. **Exercise.** □

Corollary 3.51. *Let X be a Banach space, Y a normed vector space and $A \subseteq \mathrm{CL}(X,Y)$. Suppose that*

$$\sup_{f\in A} \|f(x)\| < \infty \quad \forall x \in X.$$

Then, there exists $M > 0$ such that

$$\|f(x)\| < M\|x\| \quad \forall x \in X, \forall f \in A.$$

Proof. **Exercise.** □

3.8 The Open Mapping Theorem

Theorem 3.52 (Open Mapping Theorem)**.** *Let X be a complete Hausdorff mvs, Y a Hausdorff tvs, $f \in \mathrm{CL}(X,Y)$ and $f(X)$ not meager in Y. Then, Y is a complete Hausdorff mvs and f is open and surjective.*

Proof. Suppose U is a neighborhood of 0 in X. Let $V \subseteq U$ be a balanced subneighborhood of 0. Since every point of X is bounded we have

$$X = \bigcup_{n\in\mathbb{N}} nV \quad \text{and hence} \quad f(X) = \bigcup_{n\in\mathbb{N}} nf(V).$$

But $f(X)$ is not meager, so $nf(V)$ is not meager for at least one $n \in \mathbb{N}$. But then scale invariance of the topology of Y implies that $f(V)$ itself is not meager. Thus, $\overset{\circ}{\overline{f(V)}}$ is not empty, is open and balanced (since V is balanced) and thus forms a neighborhood of 0 in Y. Consequently, $\overline{f(V)}$ is also a neighborhood of 0 in Y and so is $\overline{f(U)}$.

Consider now a compatible pseudonorm on X. Let U be a neighborhood of 0 in X. There exists then $r > 0$ such that $B_r(0) \subseteq U$. Let $y_1 \in \overline{f(B_{r/2}(0))}$. We proceed to construct sequences $\{y_n\}_{n\in\mathbb{N}}$ and $\{x_n\}_{n\in\mathbb{N}}$ by induction. Supposed we are given $y_n \in \overline{f(B_{r/2^n}(0))}$. By the first part of the proof $\overline{f(B_{r/2^{n+1}}(0))}$ is a neighborhood of 0 in Y. Thus,

$$f(B_{r/2^n}(0)) \cap \left(y_n + \overline{f(B_{r/2^{n+1}}(0))}\right) \neq \emptyset.$$

In particular, we can choose $x_n \in B_{r/2^n}(0)$ such that

$$f(x_n) \in y_n + \overline{f(B_{r/2^{n+1}}(0))}.$$

Now set $y_{n+1} := y_n - f(x_n)$. Then, $y_{n+1} \in \overline{f(B_{r/2^{n+1}}(0))}$ as the latter is balanced.

Since in the pseudonorm $\|x_n\| < r/2^n$ for all $n \in \mathbb{N}$, the partial sums $\{\sum_{n=1}^m x_n\}_{m\in\mathbb{N}}$ form a Cauchy sequence. (Use the triangle inequality). Since X is complete, they converge to some $x \in X$ with $\|x\| < r$, i.e., $x \in B_r(0)$. On the other hand

$$f\left(\sum_{n=1}^m x_n\right) = \sum_{n=1}^m f(x_n) = \sum_{n=1}^m (y_n - y_{n+1}) = y_1 - y_{m+1}.$$

Since f is continuous the limit $m \to \infty$ exists and yields

$$f(x) = y_1 - y \quad \text{where} \quad y := \lim_{m\to\infty} y_m.$$

Note that our notation for the limit y implies uniqueness which indeed follows from the fact that Y is Hausdorff.

We proceed to show that $y = 0$. Suppose the contrary. Again using that Y is Hausdorff there exists a closed neighborhood C of 0 in Y that does not contain y. Its preimage $f^{-1}(C)$ is a neighborhood of 0 in X by continuity and must contain a ball $B_{r/2^n}(0)$ for some $n \in N$. But then $f(B_{r/2^n}(0)) \subseteq C$ and $\overline{f(B_{r/2^n}(0))} \subseteq C$ since C is closed. But $y_k \in \overline{f(B_{r/2^n}(0))} \subseteq C$ for all $k \geq n$. So no y_k for $k \geq n$ is contained in the open neighborhood $Y \setminus C$ of y, contradicting convergence of the sequence to y. We have thus established $f(x) = y_1$. But since $x \in B_r(0)$ and $y_1 \in \overline{f(B_{r/2}(0))}$ was arbitrary we may conclude that $\overline{f(B_{r/2}(0))} \subseteq f(B_r(0)) \subseteq f(U)$. By the first part of the proof $\overline{f(B_{r/2}(0))}$ is a neighborhood of 0 in Y. So we may conclude that $f(U)$ is also a neighborhood of 0 in Y. This establishes that f is open at 0 and hence open everywhere by linearity.

Since f is open the image $f(X)$ must be open in Y. On the other hand $f(X)$ is a vector subspace of Y. But the only open vector subspace of a tvs is the space itself. Hence, $f(X) = Y$, i.e., f is surjective.

Let now $C := \ker f$. Since f is surjective, Y is naturally isomorphic to the quotient space X/C as a vector space. Since f is continuous and open Y is also homeomorphic to X/C by Proposition 2.19.3 and hence isomorphic as a tvs. But then Propositions 2.29 and 3.18 imply that Y is metrizable and complete. □

Corollary 3.53. *Let X, Y be complete Hausdorff mvs and $f \in \mathrm{CL}(X, Y)$ surjective. Then, f is open.*

Proof. **Exercise.** □

4 Algebras, Operators and Dual Spaces

4.1 The Stone-Weierstraß Theorem

Definition 4.1. A vector space A over the field $\mathbb{K}$ is called an *algebra* over $\mathbb{K}$ iff it is equipped with an associative bilinear map $\cdot : A \times A \to A$. This map is called *multiplication*.

Definition 4.2. Let A be an algebra over $\mathbb{K}$. A is called a *commutative* algebra iff $a \cdot b = b \cdot a$ for all $a, b \in A$. An element $e \in A$ is called a *unit* iff $e \cdot a = a \cdot e = a$ for all $a \in A$ and $e \neq 0$. Iff A is equipped with a unit it is called a *unital* algebra. Assume now A to be unital and consider $a \in A$. Then, $b \in A$ is called an *inverse* of a iff $b \cdot a = a \cdot b = e$. An element $a \in A$ possessing an inverse is called *invertible*.

It is immediately verified that a unit and an inverse are unique.

Definition 4.3. Let A be an algebra over $\mathbb{K}$ equipped with a topology. Then A is called a *topological algebra* iff vector addition, scalar multiplication and algebra multiplication are continuous.

Proposition 4.4. *Let S be a topological space. Then, $\mathrm{C}(S, \mathbb{K})$ with the topology of compact convergence is a unital topological algebra.*

Proof. **Exercise.** □

Lemma 4.5. *Let $c > 0$. The absolute value function $|\cdot| : \mathbb{R} \to \mathbb{R}$ given by $x \mapsto |x|$ can be approximated uniformly on $[-c, c]$ by polynomials with vanishing constant term.*

Proof. **Exercise.** □

Lemma 4.6. *Let $c > 0$ and $\epsilon > 0$. Then, there exist polynomials $P_{\min}$ and $P_{\max}$ of n variables and without constant term such that for all $a_1, \ldots, a_n \in [-c, c]$,*

$$|P_{\min}(a_1, \ldots, a_n) - \min\{a_1, \ldots, a_n\}| < \epsilon,$$
$$|P_{\max}(a_1, \ldots, a_n) - \max\{a_1, \ldots, a_n\}| < \epsilon.$$

Furthermore, $P_{\min}(a, \ldots, a) = a$ and $P_{\max}(a, \ldots, a) = a$.

Proof. It suffices to show the statement for $n = 2$. Since the minimum and maximum functions can be evaluated iteratively, the general statement follows then by iteration and a multi-ϵ argument. We notice that

$$\max\{a_1, a_2\} = \frac{a_1 + a_2}{2} + \frac{|a_1 - a_2|}{2}$$
$$\min\{a_1, a_2\} = \frac{a_1 + a_2}{2} - \frac{|a_1 - a_2|}{2}.$$

By Lemma 4.5 there exists a polynomial P without constant terms such that $|P(x) - |x|| < 2\epsilon$ for all $x \in [-2c, 2c]$. It is easily verified that

$$P_{\max}(a_1, a_2) := \frac{a_1 + a_2}{2} + \frac{P(a_1 - a_2)}{2},$$
$$P_{\min}(a_1, a_2) := \frac{a_1 + a_2}{2} - \frac{P(a_1 - a_2)}{2}$$

have the desired properties. □

Definition 4.7. Let S be a set and $A \subseteq F(S, \mathbb{K})$. We say that A *separates points* iff for each pair $x, y \in S$ such that $x \neq y$ there exists $f \in A$ such that $f(x) \neq f(y)$. We say that A vanishes nowhere iff for each $x \in S$ there exists $f \in A$ such that $f(x) \neq 0$.

Lemma 4.8. *Let S be a topological space and $A \subseteq \mathrm{C}(S, \mathbb{K})$ a subalgebra. Suppose that A separates points and vanishes nowhere. Then, for any pair $x, y \in S$ with $x \neq y$ and any pair $a, b \in \mathbb{K}$ there exists a function $f \in A$ such that $f(x) = a$ and $f(y) = b$.*

Proof. **Exercise.** □

Theorem 4.9 (real Stone-Weierstraß)**.** *Let K be a compact Hausdorff space and $A \subseteq \mathrm{C}(K, \mathbb{R})$ a subalgebra. Suppose that A separates points and vanishes nowhere. Then, A is dense in $\mathrm{C}(K, \mathbb{R})$ with respect to the topology of uniform convergence.*

Proof. Given $f \in \mathrm{C}(K, \mathbb{R})$, and $\epsilon > 0$ we have to show that there is $k \in A$ such that $k \in B_\epsilon(f)$, i.e.,

$$f(x) - \epsilon < k(x) < f(x) + \epsilon \quad \forall x \in K.$$

Fix $x \in K$. For each $y \in K$ we choose a function $g_{x,y} \in A$ such that $f(x) = g_{x,y}(x)$ and $f(y) = g_{x,y}(y)$. This is possible by Lemma 4.8. By continuity there exists an open neighborhood U_y for each $y \in K$ such that

$g_{x,y}(z) < f(z) + \epsilon/4$ for all $z \in U_y$. Since K is compact there are finitely many points $y_1, \ldots, y_n \in K$ such that the associated open neighborhoods $U_{y_1}, \ldots, U_{y_n}$ cover K. Let

$$g_x := \min\{g_{x,y_1}, \ldots, g_{x,y_n}\}.$$

Since K is compact there exists $c > 0$ such that $|g_{x,y_i}(z)| \le c$ for all $z \in K$ and all $i \in \{1, \ldots, n\}$. Then, by Lemma 4.6 there exists a polynomial $P_{\min}$ such that $h_x := P_{\min}(g_{x,y_1}, \ldots, g_{x,y_n}) \in A$ satisfies $|h_x(z) - g_x(z)| < \epsilon/4$ for all $z \in K$ and $h_x(x) = g_x(x)$. Thus, $h_x(x) = f(x)$ and $h_x(z) < f(z) + \epsilon/2$ for all $z \in K$.

Choose now for each $x \in K$ a function $h_x \in A$ as above. Then, by continuity, for each $x \in K$ there exists an open neighborhood U_x such that $f(z) - \epsilon/2 < h_x(z)$ for all $z \in U_x$. By compactness of K there exists a finite set of points $x_1, \ldots, x_m \in K$ such that the associated neighborhoods $U_{x_1}, \ldots, U_{x_m}$ cover K. Let

$$h := \max\{h_{x_1}, \ldots, h_{x_m}\}.$$

Since K is compact there exists $c > 0$ such that $|h_{x_i}(z)| \le c$ for all $z \in K$ and all $i \in \{1, \ldots, m\}$. By Lemma 4.6 there exists a polynomial $P_{\max}$ such that $k := P_{\max}(h_{x_1}, \ldots, h_{x_m}) \in A$ satisfies $|k(z) - h(z)| < \epsilon/2$ for all $z \in K$. Then, $f(z) - \epsilon < k(z) < f(z) + \epsilon$ for all $z \in K$. This completes the proof. □

Theorem 4.10 (complex Stone-Weierstraß)**.** *Let K be a compact Hausdorff space and $A \subseteq \mathrm{C}(K, \mathbb{C})$ a subalgebra. Suppose that A separates points, vanishes nowhere and is invariant under complex conjugation. Then, A is dense in $\mathrm{C}(K, \mathbb{C})$ with respect to the topology of uniform convergence.*

Proof. Let $A_{\mathbb{R}}$ be the real subalgebra of A given by the functions with values in $\mathbb{R}$. Note that if $f \in A$, then $\Re f \in A_{\mathbb{R}}$ since $\Re f = (f + \overline{f})/2$. Likewise if $f \in A$, then $\Im f \in A_{\mathbb{R}}$ since $\Im f = -\Re(\mathrm{i} f)$. It is then clear that $A_{\mathbb{R}}$ separates points and vanishes nowhere. Applying the real version of the Stone-Weierstraß Theorem 4.9 we find that $A_{\mathbb{R}}$ is dense in $\mathrm{C}(K, \mathbb{R})$. But then $A = A_{\mathbb{R}} + \mathrm{i} A_{\mathbb{R}}$ is dense in $\mathrm{C}(K, \mathbb{C}) = \mathrm{C}(K, \mathbb{R}) + \mathrm{i}\, \mathrm{C}(K, \mathbb{R})$. □

Theorem 4.11. *Let S be a Hausdorff space and $A \subseteq \mathrm{C}(S, \mathbb{K})$ a subalgebra. Suppose that A separates points, vanishes nowhere and is invariant under complex conjugation if $\mathbb{K} = \mathbb{C}$. Then, A is dense in $\mathrm{C}(S, \mathbb{K})$ with respect to the topology of compact convergence.*

Proof. Recall that the sets of the form

$$U_{K,\epsilon} := \{f \in \mathrm{C}(S,\mathbb{K}) : |f(x)| < \epsilon\ \forall x \in K\},$$

where $K \subseteq S$ is compact and $\epsilon > 0$ form a basis of neighborhoods of 0 in $\mathrm{C}(S,\mathbb{K})$. Given $f \in \mathrm{C}(S,\mathbb{K})$, $K \subseteq S$ compact and $\epsilon > 0$ we have to show that there is $g \in A$ such that $g \in f + U_{K,\epsilon}$. Let A_K be the image of A under the projection $p : \mathrm{C}(S,\mathbb{K}) \to \mathrm{C}(K,\mathbb{K})$. Then, A_K is an algebra that separates points, vanishes nowhere and is invariant under complex conjugation if $\mathbb{K} = \mathbb{C}$. By the ordinary Stone-Weierstraß Theorem 4.9 or 4.10, A_K is dense in $\mathrm{C}(K,\mathbb{K})$ with respect to the topology of uniform convergence. Hence, there exists $g \in A$ such that $p(g) \in B_\epsilon(p(f))$. But this is equivalent to $g \in f + U_{K,\epsilon}$. □

Theorem 4.12. *Let S be a locally compact Hausdorff space and $A \subseteq \mathrm{C}_0(S,\mathbb{K})$ a subalgebra. Suppose that A separates points, vanishes nowhere and is invariant under complex conjugation if $\mathbb{K} = \mathbb{C}$. Then, A is dense in $\mathrm{C}_0(S,\mathbb{K})$ with respect to the topology of uniform convergence.*

Proof. **Exercise**.Hint: Let $\tilde{S} = S \cup \{\infty\}$ be the one-point compactification of S (compare Exercise 2). Show that $\mathrm{C}_0(S,\mathbb{K})$ can be identified with the closed subalgebra $\mathrm{C}_{|\infty=0}(\tilde{S},\mathbb{K}) \subseteq \mathrm{C}(\tilde{S},\mathbb{K})$ of those continuous functions on $\tilde{S}$ that vanish at ∞. Denote by $\tilde{A}$ the corresponding extension of A to $\tilde{S}$. Now modify Theorem 4.9 in such a way that $\tilde{A}$ is assumed to vanish nowhere except at ∞ to show that $\tilde{A}$ is dense in $\mathrm{C}_{|\infty=0}(\tilde{S},\mathbb{K})$. □

4.2 Operators

Proposition 4.13. *Let X, Y, Z be tvs. Let $f \in \mathrm{CL}(X,Y)$ and $g \in \mathrm{CL}(Y,Z)$. If f or g is bounded, then $g \circ f$ is bounded. If f or g is compact, then $g \circ f$ is compact.*

Proof. **Exercise.** □

Definition 4.14. Let X, Y be normed vector spaces. Then, the *operator norm* on $\mathrm{CL}(X,Y)$ is given by

$$\|f\| := \sup\left\{\|f(x)\| : x \in \overline{B_1(0)} \subseteq X\right\}.$$

Proposition 4.15. *Let X be a normed vector space and Y a Banach space. Then, $\mathrm{CL}(X,Y)$ with the operator norm is a Banach space.*

Proof. Let $\{f_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathrm{CL}(X,Y)$. This means,

$$\forall \epsilon > 0 : \exists N > 0 : \forall n, m \geq N : \|f_n - f_m\| \leq \epsilon.$$

But by the definition of the operator norm this is equivalent to

$$\forall \epsilon > 0 : \exists N > 0 : \forall n, m \geq N : \forall x \in X : \|f_n(x) - f_m(x)\| \leq \epsilon \|x\|. \quad (1)$$

Since Y is complete, so each of the Cauchy sequences $\{f_n(x)\}_{n\in\mathbb{N}}$ converges to a vector $f(x) \in Y$. This defines a map $f : X \to Y$. f is linear since we have for all $x, y \in X$ and $\lambda, \mu \in \mathbb{K}$,

$$\begin{aligned} f(\lambda x + \mu y) = \lim_{n\to\infty} f_n(\lambda x + \mu y) &= \lim_{n\to\infty} (\lambda f_n(x) + \mu f_n(y)) \\ &= \lambda \lim_{n\to\infty} f_n(x) + \mu \lim_{n\to\infty} f_n(y) = \lambda f(x) + \mu f(y). \end{aligned}$$

Equation (1) implies now

$$\forall \epsilon > 0 : \exists N > 0 : \forall n \geq N : \forall x \in X : \|f_n(x) - f(x)\| \leq \epsilon \|x\|.$$

This implies that f is continuous and is equivalent to

$$\forall \epsilon > 0 : \exists N > 0 : \forall n \geq N : \|f_n - f\| \leq \epsilon.$$

That is, $\{f_n\}_{n\in\mathbb{N}}$ converges to f. □

Exercise 27. Let X, Y be tvs. Let $\mathfrak{S}$ be the set of bounded subsets of X. (a) Show that $\mathrm{CL}(X,Y)$ is a tvs with the $\mathfrak{S}$-topology. (b) Suppose further that X is locally bounded and Y is complete and Hausdorff. Show that then $\mathrm{CL}(X,Y)$ is complete. (c) Show that if X and Y are normed vector spaces the $\mathfrak{S}$-topology coincides with the operator norm topology.

Example 4.16. Let X be a tvs. Then, $\mathrm{CL}(X,X)$ is an algebra over $\mathbb{K}$ and Proposition 4.13 implies that the subsets $\mathrm{BL}(X,X)$ and $\mathrm{KL}(X,X)$ of $\mathrm{CL}(X,X)$ are bi-ideals.

Exercise 28. Let X be a normed vector space. Show that $\mathrm{CL}(X,X)$ with the operator norm and multiplication given by composition is a topological algebra. Moreover, show that $\|A \circ B\| \leq \|A\|\|B\|$ for all $A, B \in \mathrm{CL}(X,X)$.

4.3 Dual spaces

Definition 4.17. Let X be a tvs over $\mathbb{K}$. Then, the space $\mathrm{L}(X, \mathbb{K})$ of linear maps $X \to \mathbb{K}$ is called the *algebraic dual* of X and denoted by $X^{\times}$. The space $\mathrm{CL}(X, \mathbb{K})$ of continuous linear maps $X \to \mathbb{K}$ is called the *(topological) dual* of X and denoted by X^*.

Definition 4.18. Let X be a tvs. Then, the *weak* topology* on X^* is the coarsest topology on X^* such that the evaluation maps $\hat{x} : X^* \to \mathbb{K}$ given by $\hat{x}(f) := f(x)$ are continuous for all $x \in X$.

Exercise 29. Let X be a tvs. Show that the weak* topology on X^* makes it into a locally convex tvs and indeed coincides with the topology of pointwise convergence under the inclusion $\mathrm{CL}(X, \mathbb{K}) \subseteq \mathrm{C}(X, \mathbb{K})$. Moreover, show that $\mathrm{CL}(X, \mathbb{K})$ is closed in $\mathrm{C}(X, \mathbb{K})$.

Proposition 4.19. *Let X be a tvs, $F \subseteq \mathrm{CL}(X, \mathbb{K})$ equicontinuous. Then, F is bounded in the weak* topology.*

Proof. **Exercise.** □

Proposition 4.20. *Let X be a normed vector space. Then, the operator norm topology on X^* is finer than the weak* topology.*

Proof. **Exercise.** □

Indeed, we shall see that the following Banach-Alaoglu Theorem has as a striking consequence a considerable strengthening of the above statement.

Theorem 4.21 (Banach-Alaoglu)**.** *Let X be a tvs, U a neighborhood of 0 in X and V a bounded and closed set in $\mathbb{K}$. Then, the set*

$$M(U, V) := \{f \in X^* : f(U) \subseteq V\}.$$

is compact with respect to the weak topology.*

Proof. We first show that $M(U, V)$ is closed. To this end observe that

$$M(U, V) = \bigcap_{x \in U} M(\{x\}, V) \quad \text{where} \quad M(\{x\}, V) := \{f \in X^* : f(x) \in V\}.$$

Each set $M(\{x\}, V)$ is closed since it is the preimage of the closed set V under the continuous evaluation map $\hat{x} : X^* \to \mathbb{K}$. Thus, $M(U, V)$, being an intersection of closed sets is closed.

Next we show that $M(U,V)$ is equicontinuous and bounded. Let W be a neighborhood of 0 in $\mathbb{K}$. Since V is bounded there exists $\lambda > 0$ such that $V \subseteq \lambda W$, i.e., $\lambda^{-1}V \subseteq W$. But by linearity $M(U,V) = M(\lambda^{-1}U, \lambda^{-1}V)$. This means that $f(\lambda^{-1}U) \subseteq \lambda^{-1}V \subseteq W$ for all $f \in M(U,V)$, showing equicontinuity. By Proposition 4.19 it is also bounded.

Thus, the assumptions of the Arzela-Ascoli Theorem 3.28 are satisfied and we obtain that $M(U,V)$ is relatively compact with respect to the topology of compact convergence. But since $M(U,V)$ is closed in the topology of pointwise convergence it is also closed in the topology of compact convergence which is finer. Hence, $M(U,V)$ is compact in the topology of compact convergence. But since the topology of pointwise convergence is coarser, $M(U,V)$ must also be compact in this topology. □

Corollary 4.22. *Let X be a normed vector space and $B \subseteq X^*$ the closed unit ball with respect to the operator norm. Then B is compact in the weak* topology.*

Proof. **Exercise.** □

Remark 4.23. Let X be a normed space. Then, X^* with the operator norm topology is complete, i.e., a Banach space (due to Proposition 4.15).

Given a normed vector space X, we shall in the following always equip X^* with the operator norm if not mentioned otherwise.

Definition 4.24. Let X be a normed vector space. The *bidual* space of X, denoted by X^{**} is the dual space of the dual space X^*. Let $x \in X$.

Proposition 4.25. *Let X be a normed vector space. Given $x \in X$ the evaluation map $\hat{x} : X^* \to \mathbb{K}$ given by $\hat{x}(y) := y(x)$ for all $y \in X^*$ is an element of X^{**}. Moreover, the canonical linear map $i_X : X \to X^{**}$ given by $x \mapsto \hat{x}$ is isometric.*

Proof. The continuity of $\hat{x}$ follows from Proposition 4.20. Thus, it is an element of X^{**}. We proceed to show that i_X is isometric. Denote by $\overline{B}_{X^*}$ the closed unit ball in X^*. Then, for all $x \in X$,

$$\|\hat{x}\| = \sup_{f \in \overline{B}_{X^*}} |\hat{x}(f)| = \sup_{f \in \overline{B}_{X^*}} |f(x)| \leq \sup_{f \in \overline{B}_{X^*}} \|f\|\|x\| = \|x\|.$$

On the other hand, given $x \in X$ choose with the help of the Hahn-Banach Theorem (Corollary 3.31) $g \in X^*$ such that $g(x) = \|x\|$ and $\|g\| = 1$. Then,

$$\|\hat{x}\| = \sup_{f \in B_{X^*}} |\hat{x}(f)| \geq |\hat{x}(g)| = |g(x)| = \|x\|.$$

□

Definition 4.26. A Banach space X is called *reflexive* iff the canonical linear map $i_X : X \to X^{**}$ is surjective.

4.4 Adjoint operators

Definition 4.27. Let X, Y be tvs and $f \in \mathrm{CL}(X, Y)$. The *adjoint operator* $f^* \in \mathrm{L}(Y^*, X^*)$ is defined by

$$(f^*(g))(x) := g(f(x)) \quad \forall x \in X, g \in Y^*.$$

Remark 4.28. It is immediately verified that the image of f^* is indeed contained in X^* and not merely in $X^\times$.

Proposition 4.29. *Let X, Y be tvs and $f \in \mathrm{CL}(X, Y)$. Then, $f^* \in \mathrm{CL}(Y^*, X^*)$ if we equip X^* and Y^* with the weak* topology.*

Proof. **Exercise.** □

Proposition 4.30. *Let X, Y be normed vector spaces and $f \in \mathrm{CL}(X, Y)$. Then, $f^* \in \mathrm{CL}(Y^*, X^*)$ if we equip X^* and Y^* with the operator norm topology. Moreover, equipping also $\mathrm{CL}(X, Y)$ and $\mathrm{CL}(Y^*, X^*)$ with the operator norm we get $\|f^*\| = \|f\|$ for all $f \in \mathrm{CL}(X, Y)$. That is, $^* : \mathrm{CL}(X, Y) \to \mathrm{CL}(Y^*, X^*)$ is a linear isometry.*

Proof. **Exercise.**Hint: Use the Hahn-Banach Theorem in the form of Corollary 3.31 to show that $\|f^*\| \geq \|f\|$. □

Lemma 4.31. *Let X, Y be normed vector spaces and $f \in \mathrm{CL}(X, Y)$. Then, $f^{**} \circ i_X = i_Y \circ f$.*

Proof. **Exercise.** □

Proposition 4.32. *Let X, Y be normed vector spaces and $f \in \mathrm{CL}(X, Y)$. Equip X^* and Y^* with the operator norm topology. Then, compactness of f implies compactness of f^*. Supposing in addition that Y is complete, also compactness of f^* implies compactness of f.*

Proof. Suppose first that f is compact. Then, $C := \overline{f(B_1(0))}$ is compact. Let B_{Y^*} be the open unit ball in Y^*. Then, B_{Y^*} is equicontinuous and the restriction of B_{Y^*} to $C \subseteq Y$ is bounded in $\mathrm{C}(C, \mathbb{K})$ (with the topology of pointwise convergence). Thus, by the Arzela-Ascoli Theorem 3.28, B_{Y^*} restricted to C is totally bounded in $\mathrm{C}(C, \mathbb{K})$ (with the topology of uniform convergence). In particular, for any $\epsilon > 0$ there exists a finite set $F \subseteq B_{Y^*}$

such that for any $g \in B_{Y^*}$ there is $\tilde{g} \in F$ with $|g(y) - \tilde{g}(y)| < \epsilon$ for all $y \in C$. But then also $|f^*(g)(x) - f^*(\tilde{g})(x)| < \epsilon$ for all $x \in B_1(0) \subseteq X$. This in turn implies $\|f^*(g) - f^*(\tilde{g})\| \le \epsilon$. That is, $f^*(B_{Y^*})$ is totally bounded and hence relatively compact. Hence, f^* is compact.

Conversely, suppose that f^* is compact. Then, by the same argument as above $f^{**} : X^{**} \to Y^{**}$ is compact. That is, there is a neighborhood U^{**} of 0 in X^{**} such that $f^{**}(U^{**})$ is compact in Y^{**}. Since i_X is continuous $U := i_X^{-1}(U^{**})$ is a neighborhood of 0 in X. Using Lemma 4.31 we get $f^{**}(U^{**}) \supseteq f^{**} \circ i_X(U) = i_Y \circ f(U)$. In particular, this means that $i_Y \circ f(U)$ is totally bounded. Since i_Y is isometric, $f(U)$ is also totally bounded. So, $\overline{F(U)}$ is totally bounded and also complete given completeness of Y, hence compact. Thus, f is compact. □

Proposition 4.33. *Let X, Y be Hausdorff tvs, $A \in \mathrm{CL}(X, Y)$. Then, there are canonical isomorphisms of vector spaces,*

1. $\left(Y/\overline{A(X)}\right)^* \to \ker(A^*)$,
2. $Y^*/\ker(A^*) \to \left(\overline{A(X)}\right)^*$.

Moreover, supposing in addition that Y is locally convex, if we equip dual space with the weak topology, these isomorphisms become isomorphisms of tvs. Similarly, If X and Y are normed vector spaces and we equip dual spaces with the operator norm, the isomorphisms become isometries.*

Proof. Let $q : Y \to Y/\overline{A(X)}$ be the quotient map. The adjoint of q is $q^* : \left(Y/\overline{A(X)}\right)^* \to Y^*$. Since q is surjective, q^* is injective. We claim that the image of q^* is $\ker(A^*) \subseteq Y^*$ proving 1. Let $f \in \left(Y/\overline{A(X)}\right)^*$. Then, $A^*(q^*(f)) = f \circ q \circ A = 0$ since already $q \circ A = 0$. Now suppose $f \in \ker(A^*) \subseteq Y^*$. Then, $f \circ A = 0$, i.e., $f|_{A(X)} = 0$. Since f is continuous, we must actually have $f|_{\overline{A(X)}} = 0$. But this means there is a well defined $g : Y/\overline{A(X)} \to \mathbb{K}$ such that $f = g \circ q$. Moreover, the continuity of f implies continuity of g by the definition of the quotient topology on $Y/\overline{A(X)}$. This completes the proof of 1.

Consider the inclusion $i : \overline{A(X)} \to Y$. The adjoint of i is $i^* : Y^* \to \left(\overline{A(X)}\right)^*$. Since i is injective, i^* is surjective. We claim that the kernel of i^* is precisely $\ker(A^*)$ so that quotienting it leads the isomorphism 2. Indeed, let $f \in Y^*$. $f \in \ker(A^*)$ iff $0 = A^*(f) = f \circ A$. But this is equivalent to $f|_{A(X)} = 0$. Since f is continuous this is in turn equivalent to $f|_{\overline{A(X)}} = 0$. But this is in turn equivalent to $0 = f \circ i = i^*(f)$, completing the proof of 2.

Exercise. Complete the topological part of the proof. $\square$

4.5 Approximating Compact Operators

Definition 4.34. Let X, Y be tvs. We denote the space of continuous linear maps $X \to Y$ with finite dimensional image by $\mathrm{CL}_{\mathrm{fin}}(X, Y)$.

Proposition 4.35. *Let X, Y be tvs such that Y is Hausdorff. Then, $\mathrm{CL}_{\mathrm{fin}}(X, Y) \subseteq \mathrm{KL}(X, Y)$.*

Proof. **Exercise.** $\square$

Proposition 4.36. *Let X be a normed vector space, Y a Banach space. Then, $\overline{\mathrm{CL}_{\mathrm{fin}}(X, Y)} \subseteq \mathrm{KL}(X, Y)$ with respect to the operator norm topology.*

Proof. Let $f \in \overline{\mathrm{CL}_{\mathrm{fin}}(X, Y)}$ and $\epsilon > 0$. Then, there exists $g \in \mathrm{CL}_{\mathrm{fin}}(X, Y)$ such that $\|f - g\| < \epsilon$. In particular, $(f - g)(\overline{B_1(0)}) \subseteq B_\epsilon(0)$. This implies $f(\overline{B_1(0)}) \subseteq g(\overline{B_1(0)}) + B_\epsilon(0)$. But $g(\overline{B_1(0)})$ is a bounded subset of the finite dimensional subspace $g(X)$ and hence totally bounded. Thus, there exists a finite subset $F \subseteq g(\overline{B_1(0)})$ such that $g(\overline{B_1(0)}) \subseteq F + B_\epsilon(0)$. But then, $f(\overline{B_1(0)}) \subseteq F + B_\epsilon(0) + B_\epsilon(0) \subseteq F + B_{2\epsilon}(0)$. That is, $f(\overline{B_1(0)})$ is covered by a finite number of balls of radius 2ϵ. Since ϵ was arbitrary this means that $f(\overline{B_1(0)})$ is totally bounded and hence relatively compact. $\square$

Proposition 4.37. *Let X, Y be normed vector spaces. Suppose there exists a bounded sequence $\{s_n\}_{n\in\mathbb{N}}$ of operators $s_n \in \mathrm{CL}_{\mathrm{fin}}(Y, Y)$ such that $\lim_{n\to\infty} s_n(y) = y$ for all $y \in Y$. Then, $\mathrm{KL}(X, Y) \subseteq \overline{\mathrm{CL}_{\mathrm{fin}}(X, Y)}$ with respect to the operator norm topology.*

Proof. **Exercise.**Hint: For $f \in \mathrm{KL}(X, Y)$ and $\epsilon > 0$ show that there exists $n \in \mathbb{N}$ such that $\|s_n \circ f - f\| < \epsilon$. $\square$

4.6 Fredholm Operators

Proposition 4.38. *Let X be a Hausdorff tvs and $T \in \mathrm{KL}(X, X)$. Then, the kernel of $S := \mathbf{1} - T \in \mathrm{CL}(X, X)$ is finite-dimensional.*

Proof. Note that T acts as the identity on the subspace $\ker S$. Denote this induced operator by $\tilde{T} : \ker S \to \ker S$. Since T is compact so is $\tilde{T}$. Thus, there exists a neighborhood of 0 in $\ker S$ that is compact. In particular, $\ker S$ is locally compact. By Theorem 3.23, $\ker S$ is finite dimensional. $\square$

Proposition 4.39. *Let X, Y be Banach spaces and $f \in \mathrm{CL}(X,Y)$ injective. Then, $f(X)$ is closed iff there exists $c > 0$ such that $\|f(x)\| \geq c\|x\|$ for all $x \in X$.*

Proof. Suppose first that $f(X)$ is closed. Then, $f(X)$ is complete since Y is complete. Thus, by Corollary 3.53, f is open as a map $X \to f(X)$. In particular, $f(B_1(0))$ is an open neighborhood of 0 in $f(X)$. Thus, there exists $c > 0$ such that $B_c(0) \subseteq f(B_1(0)) \subseteq f(X)$. By injectivity of f this implies that $\|f(x)\| \geq c$ for all $x \in X$ with $\|x\| \geq 1$. This implies in turn $\|f(x)\| \geq c\|x\|$ for all $x \in X$.

Conversely, assume that there is $c > 0$ such that $\|f(x)\| \geq c\|x\|$ for all $x \in X$. Let $y \in \overline{f(X)}$. Then there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ in X such that $\{f(x_n)\}_{n\in\mathbb{N}}$ converges to y. In particular, $\{f(x_n)\}_{n\in\mathbb{N}}$ is a Cauchy sequence. But as is easy to see the assumption then implies that $\{x_n\}_{n\in\mathbb{N}}$ is also a Cauchy sequence. Since X is complete this sequence converges, say to $x \in X$. But since f is continuous we must have

$$y = \lim_{n\to\infty} f(x_n) = f\left(\lim_{n\to\infty} x_n\right) = f(x).$$

In particular, $y \in f(X)$, i.e., $f(X)$ is closed. □

Proposition 4.40. *Let X be a Banach space and $T \in \mathrm{KL}(X,X)$. Then, the image of $S := \mathbf{1} - T \in \mathrm{CL}(X,X)$ is closed and has finite codimension, i.e., $X/S(X)$ is finite dimensional.*

Proof. We first show that $S(X)$ is a closed subspace of X. Since S is continuous $\ker S$ is a closed subspace of X. The quotient map $q : X \to X/\ker(S)$ is thus a continuous and open linear map between Banach spaces. S factorizes through q via $S = \tilde{S} \circ q$, where $\tilde{S} : X/\ker(S) \to X$ is linear, continuous and injective. We equip $X/\ker(S)$ with the quotient norm. By Propositions 2.44 and 3.18 this space is a Banach space. By Proposition 4.39 the image of $\tilde{S}$ (and thus that of S) is closed iff there exists a constant $c > 0$ such that $\|\tilde{S}(y)\| \geq c\|y\|$ for all $y \in S/\ker(S)$. Hence, we have to demonstrate the existence of such a constant. Suppose it does not exist. Then, there is a sequence $\{y_n\}_{n\in\mathbb{N}}$ of elements of $X/\ker(S)$ with $\|y_n\| = 1$ and such that $\lim_{n\to\infty} \tilde{S}(y_n) = 0$. Now choose a preimages x_n of the y_n in X with $1 \leq \|x_n\| < 2$. Then, $\{x_n\}_{n\in\mathbb{N}}$ is bounded so that $\{T(x_n)\}_{n\in\mathbb{N}}$ is compact. In particular, there is a subsequence $\{x_k\}_{k\in\mathbb{N}}$ so that $\{T(x_k)\}_{k\in\mathbb{N}}$ converges, say to $z \in X$. Since on the other hand $\lim_{k\to\infty} S(x_k) = 0$ we find with $S + T = \mathbf{1}$ that $\lim_{k\to\infty} x_k = z$. So by continuity of S we get $S(z) = 0$, i.e., $z \in \ker(S)$ and hence $z \in \ker q$. By continuity of q this implies, $\lim_{k\to\infty} \|q(x_k)\| = 0$, contradicting $\|q(x_k)\| = \|y_k\| = 1$ for all $k \in \mathbb{N}$.

This completes the proof of the existence of c and hence of the closedness of the image of S.

The compactness of T implies the compactness of T^* by Proposition 4.32. Thus, by Proposition 4.38, $S^* = \mathbf{1}^* - T^*$ has finite dimensional kernel. But Proposition 4.33.1 implies then that the codimension of $\overline{S(X)}$ in X, i.e., the dimension of $X/\overline{S(X)}$ is also finite. Since we have seen above that $\overline{S(X)} = S(X)$, this completes the proof. □

Definition 4.41. Let X, Y be normed vector spaces and $A \in \mathrm{CL}(X, Y)$. A is called a *Fredholm operator* iff the kernel of A is finite dimensional and its image is closed and of finite codimension. Then, we define the *index* of a A to be

$$\operatorname{ind} A = \dim(\ker A) - \dim(Y/A(Y)).$$

We denote by $\mathrm{FL}(X, Y)$ the set of Fredholm operators.

Lemma 4.42 (Riesz)**.** *Let X be a normed vector space and C a closed subspace. Then, for any $1 > \epsilon > 0$ there exists $x \in X \setminus C$ with $\|x\| = 1$ such that for all $y \in C$,*

$$\|x - y\| \geq 1 - \epsilon.$$

Proof. Choose $x_0 \in X \setminus C$ arbitrary. Now choose $y_0 \in C$ such that

$$\|x_0 - y_0\| \leq \|x_0 - y\| \frac{1}{1 - \epsilon}$$

for all $y \in C$. We claim that

$$x := \frac{x_0 - y_0}{\|x_0 - y_0\|}$$

has the desired property. Indeed, for all $y \in C$,

$$\|x - y\| = \frac{\|x_0 - y_0 - (\|x_0 - y_0\|)y\|}{\|x_0 - y_0\|} \geq \frac{\|x_0 - y_0\|(1 - \epsilon)}{\|x_0 - y_0\|}.$$

□

Proposition 4.43. *Let X, Y be Banach space. Then, the subset $\mathrm{CL}_{\mathrm{inv}}(X, Y)$ of continuously invertible maps is open in $\mathrm{CL}(X, Y)$.*

Proof. Let $f : X \to Y$ be linear and continuous and with continuous inverse f^{-1}. By Proposition 4.39 there is a constant $c > 0$ such that $\|f(x)\| \geq c\|x\|$

for all $x \in X$. Now consider $g \in \mathrm{CL}(X,Y)$ such that $\|f-g\| < c/2$. We claim that g has a continuous inverse. First, observe

$$\|g(x)\| \geq \|f(x)\| - \|f(x)-g(x)\| \geq c\|x\| - \frac{c}{2}\|x\| = \frac{c}{2}\|x\| \quad \forall x \in X. \quad (2)$$

This implies that g is injective and moreover has closed image by Proposition 4.39. Suppose now that $g(x) \neq Y$. By Lemma 4.42 there exists then $y_0 \in Y \setminus g(X)$ with $\|y_0\| = 1$ such that $\|y_0 - y\| \geq 1/2$ for all $y \in g(X)$. Let $x_0 := f^{-1}(y_0)$. Then,

$$\frac{1}{2} = \frac{1}{2}\|f(x_0)\| \geq \frac{c}{2}\|x_0\| > \|f(x_0) - g(x_0)\| \geq \frac{1}{2},$$

a contradiction. Thus, $g(X) = Y$ and g is invertible. But g^{-1} is continuous since (2) now implies $\|g^{-1}(y)\| \leq (2/c)\|y\|$ for all $y \in Y$. □

Proposition 4.44. *Let X, Y be Banach spaces. Then,* $\mathrm{FL}(X,Y)$ *is open in* $\mathrm{CL}(X,Y)$*. Moreover,* $\mathrm{ind} : \mathrm{FL}(X,Y) \to \mathbb{Z}$ *is continuous.*

Proof. Let $S : X \to Y$ be Fredholm. Since $\ker S$ is finite dimensional, there exists a closed complement $C \subseteq X$ by Proposition 3.34. Then, $S|_C : C \to Y$ is injective and has closed image $S(C) = S(X)$. Also, let $D \subseteq Y$ be a complement of $S(X)$. Since S is Fredholm, D is finite-dimensional and thus also closed. Note that $C \oplus D$ is a Banach space. It will be convenient to equip it with the norm $\|x+y\| := \|x\| + \|y\|$ for $x \in C$, $y \in D$. Define the map $\tilde{S} : C \oplus D \to Y$ by $\tilde{S}(x,y) := S(x) + y$. $\tilde{S}$ is the product of two continuously invertible maps and hence continuously invertible. By Proposition 4.43 there is thus $r > 0$ such that $B_r(\tilde{S}) \subseteq \mathrm{CL}_{\mathrm{inv}}(C \oplus D, Y)$. Let $T \in \mathrm{CL}(X,Y)$ such that $\|T - S\| < r$. Define $\tilde{T} : C \oplus D \to Y$ as $\tilde{T}(x,y) := T(x) + y$. Then,

$$\|\tilde{T} - \tilde{S}\| = \sup_{\|x+y\| \leq 1} \|T(x) - S(x)\| = \sup_{\|x\| \leq 1} \|T(x) - S(x)\| \leq \|T - S\|,$$

where $x \in C$ and $y \in D$. In particular, $\|\tilde{T} - \tilde{S}\| < r$, so $\tilde{T}$ has a continuous inverse.

Note that $\ker T \cap C = \{0\}$, so there is a subspace $N \subseteq X$ such that $X = C \oplus N \oplus \ker T$. In particular, $\ker T$ is finite-dimensional. Since $\tilde{T}$ is homeomorphism, $T(C) = \tilde{T}(C)$ is closed and thus complete. On the other hand $T(N)$ being finite-dimensional is also complete. Thus $T(X) = T(N) \oplus T(C)$ is also complete and thus closed in Y. Also, $T(C) + D = Y$, so in particular $T(C)$ has finite codimension and so does $T(X)$. Thus, T is Fredholm.

__**Exercise.**__Complete the proof by showing $\mathrm{ind}\, T = \mathrm{ind}\, S$. □

Corollary 4.45. *Let X be a Banach space and $T \in \mathrm{KL}(X,X)$. Then, $S := \mathbf{1} - T \in \mathrm{FL}(X,X)$. Moreover,* $\operatorname{ind} S = 0$*.*

Proof. **Exercise.**Hint: For the second assertion consider the family of operators $S_t := \mathbf{1} - tT$ for $t \in [0,1]$ and use the continuity of ind. □

Proposition 4.46 (Fredholm alternative)**.** *Let X be a Banach spaces, $T \in \mathrm{KL}(X,X)$ and $\lambda \in \mathbb{K} \setminus \{0\}$. Then, either the equation*

$$\lambda x - Tx = y$$

has one unique solution $x \in X$ for each $y \in X$, or it has no solution for some $y \in X$ and infinitely many solutions for all other $y \in X$.

Proof. **Exercise.** □

4.7 Eigenvalues and Eigenvectors

Definition 4.47. Let X be a tvs and $A \in \mathrm{CL}(X,X)$. Then, $\lambda \in \mathbb{K}$ is called an *eigenvalue* of A iff there exists $x \in X \setminus \{0\}$ such that $\lambda x - Ax = 0$. Then x is called an *eigenvector* for the eigenvalue λ. Moreover, the vector space of eigenvectors for the eigenvalue λ is called the *eigenspace* of λ.

Proposition 4.48. *Let X be a Banach space and $T \in \mathrm{KL}(X,X)$. Then, $\lambda \in \mathbb{K} \setminus \{0\}$ is an eigenvalue of T iff $\lambda \mathbf{1} - T$ does not have a continuous inverse.*

Proof. **Exercise.** □

Lemma 4.49. *Let X be a Banach space, $T \in \mathrm{KL}(X,X)$ and $c > 0$. Then, the set of eigenvalues λ such that $|\lambda| > c$ is finite.*

Proof. Suppose the assertion is not true. Thus, there exists a sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ of distinct eigenvalues of T such that $|\lambda_n| > c$ for all $n \in \mathbb{N}$. Let $\{v_n\}_{n\in\mathbb{N}}$ be a sequence of associated eigenvectors. Observe that the set of these eigenvectors is linearly independent. For all $n \in \mathbb{N}$ let A_n be the vector space spanned by $\{v_1, \ldots, v_n\}$. Thus $\{A_n\}_{n\in\mathbb{N}}$ is a strictly ascending sequence of finite-dimensional subspaces of X. Set $y_1 := v_1/\|v_1\|$. Using Lemma 4.42 we choose for each $n \in \mathbb{N}$ a vector $y_{n+1} \in A_{n+1}$ such that $\|y_{n+1}\| = 1$ and $\|y_{n+1} - y\| > 1/2$ for all $y \in A_n$. Now let $n > m \geq 1$. Then,

$$\begin{aligned}\|Ty_n - Ty_m\| &= \|\lambda_n y_n - (\lambda_n y_n - Ty_n + Ty_m)\| \\ &= |\lambda_n|\|y_n - |\lambda_n|^{-1}(\lambda_n y_n - Ty_n + Ty_m)\| > |\lambda_n|\frac{1}{2} > \frac{1}{2}c.\end{aligned}$$

We have used here that $\lambda_n y_n - T y_n \in A_{n-1}$ and that $T y_m \in A_m \subseteq A_{n-1}$. This shows that the image of the bounded set $\{y_n\}_{n\in\mathbb{N}}$ under T is not totally bounded. But this contradicts the compactness of T. $\square$

Definition 4.50. Let X be a Banach space and $A \in \mathrm{CL}(X, X)$. Then, the set $\sigma(A) := \{\lambda \in \mathbb{K} : \lambda\mathbf{1} - A \text{ is not continuously invertible}\}$ is called the *spectrum* of A.

Theorem 4.51. *Let X be a Banach space and $T \in \mathrm{KL}(X, X)$.*

1. *If X is infinite-dimensional, then $0 \in \sigma(T)$.*
2. *The set $\sigma(T)$ is bounded.*
3. *The set $\sigma(T)$ is countable.*
4. *$\sigma(T)$ has at most one accumulation point, 0.*

Proof. **Exercise.** $\square$

5 Banach Algebras

5.1 Invertibility and the Spectrum

Suppose X is a Banach space. Then we are often interested in (continuous) operators on this space, i.e, elements of the space $\mathrm{CL}(X,X)$. We have already seen that this is again a Banach space. However, operators can also be composed with each other, which gives us more structure, namely that of an algebra. It is often useful to study this abstractly, i.e., forgetting about the original space on which the operators X act. This leads us to the concept of a *Banach algebra*. In the following of this section we work exclusively over the field $\mathbb{C}$ of complex numbers.

Definition 5.1 (Banach Algebra)**.** A is called a *Banach algebra* iff it is a complete normable topological algebra.

Proposition 5.2. *Let A be a complete normable tvs and an algebra. Then, A is a Banach algebra iff there exists a compatible norm on A such that $\|a \cdot b\| \le \|a\| \cdot \|b\|$ for all $a, b \in A$. Moreover, if A is unital then it is a Banach algebra iff there exists a compatible norm that satisfies in addition $\|e\| = 1$.*

Proof. Suppose that A admits a norm generating the topology and satisfying $\|a \cdot b\| \le \|a\| \cdot \|b\|$ for all $a, b \in A$. Fix $a, b \in A$ and let $\epsilon > 0$. Choose $\delta > 0$ such that

$$(\|a\| + \|b\|)\delta + \delta^2 < \epsilon.$$

Then,

$$\begin{aligned}\|(a + x) \cdot (b + y) - a \cdot b\| = \|a \cdot y + x \cdot b + x \cdot y\| &\le \|a \cdot y\| + \|x \cdot b\| + \|x \cdot y\| \\ &\le \|a\| \cdot \|y\| + \|x\| \cdot \|b\| + \|x\| \cdot \|y\| < \epsilon\end{aligned}$$

if $x, y \in B_\delta(0)$, showing continuity of multiplication.

Now suppose that A is a Banach algebra. Let $\|\cdot\|'$ be a norm generating the topology. By continuity there exists $\delta > 0$ such that $\|a \cdot b\|' \le 1$ for all $a, b \in B_\delta(0)$. But this implies $\|a \cdot b\|' \le \delta^{-2}\|a\|' \cdot \|b\|'$ for all $a, b \in A$. It is then easy to see that $\|a\| := \delta^{-2}\|a\|'$ for all $a \in A$ defines a norm that is topologically equivalent and satisfies $\|a \cdot b\| \le \|a\| \cdot \|b\|$ for all $a, b \in A$.

Now suppose that A is a unital Banach algebra. Let $\|\cdot\|'$ be a norm generating the topology. As we have just seen there exists a constant $c > 0$ such that $\|a \cdot b\|' \le c\|a\|' \cdot \|b\|'$ for all $a, b \in A$. We claim that

$$\|a\| := \sup_{\|b\|' \le 1} \|a \cdot b\|' \quad \forall a \in A$$

defines a topologically equivalent norm with the desired properties. It is easy to see that $\|\cdot\|$ is a seminorm. Now note that

$$\|a\| = \sup_{\|b\|'\le 1} \|a\cdot b\|' \le c \sup_{\|b\|'\le 1} \|a\|'\cdot\|b\|' = c\|a\|' \quad \forall a\in A.$$

On the other hand we have

$$\|a\| = \sup_{\|b\|'\le 1} \|a\cdot b\|' \ge \frac{\|a\cdot e\|'}{\|e\|'} = \frac{\|a\|'}{\|e\|'} \quad \forall a\in A.$$

This shows that $\|\cdot\|$ is indeed a norm and generates the same topology as $\|\cdot\|'$. The proof of the property $\|a\cdot b\| \le \|a\|\cdot\|b\|$ for all $a,b\in A$ now proceeds as in Exercise 28. Finally, it is easy to see that $\|e\| = 1$. □

We have already seen the prototypical example of a Banach algebra in Exercise 28: The algebra of continuous linear operators $\mathrm{CL}(X,X)$ on a Banach space X.

Exercise 30. Let T be a compact topological space. Show that $\mathrm{C}(T,\mathbb{C})$ with the supremum norm is a unital commutative Banach algebra.

Exercise 31. Consider the space $l^1(\mathbb{Z})$, i.e., the space of complex sequences $\{a_n\}_{n\in\mathbb{Z}}$ with $\|a\| := \sum_{n\in\mathbb{Z}} |a_n| < \infty$. 1. Show that this is a Banach space. 2. Define a multiplication by convolution, i.e., $(a\star b)_n := \sum_{k\in\mathbb{Z}} a_k b_{n-k}$. Show that this is well defined and yields a commutative Banach algebra.

Proposition 5.3. *Let A be a unital Banach algebra and $a\in A$. If $\|e-a\| < 1$ then a is invertible. Moreover, in this case*

$$a^{-1} = \sum_{n=0}^{\infty} (e-a)^n \quad \textit{and} \quad \|a^{-1}\| \le \frac{1}{1-\|e-a\|}.$$

Proof. **Exercise.** □

Proposition 5.4. *Let A be a unital Banach algebra. Denote the subset of invertible elements of A by I_A. Then, I_A is open. Moreover, the map $I_A \to I_A : a \mapsto a^{-1}$ is continuous.*

Proof. Consider an invertible element $a\in I_A$ and choose $\epsilon > 0$. Set

$$\delta := \min\left\{\frac{1}{2}\|a^{-1}\|^{-1}, \frac{1}{2}\epsilon\|a^{-1}\|^{-2}\right\}.$$

Take $b \in B_\delta(a)$. Then $b = a(e + a^{-1}(b - a))$. But

$$\|a^{-1}(b-a)\| \le \|a^{-1}\|\|b-a\| < \|a^{-1}\|\delta \le \frac{1}{2}.$$

So by Proposition 5.3 the element $e + a^{-1}(b-a)$ is invertible. Consequently, b is a product of invertible elements and hence itself invertible. Therefore, $B_\delta(a) \subseteq I_A$ and I_A is open. Furthermore, using the same inequality we find by Proposition 5.3 that

$$\|(e + a^{-1}(b-a))^{-1}\| \le \frac{1}{1 - \|a^{-1}(b-a)\|} < 2.$$

This implies

$$\|b^{-1}\| \le \|a^{-1}\|\|(e + a^{-1}(b-a))^{-1}\| < 2\|a^{-1}\|.$$

Hence,

$$\|a^{-1} - b^{-1}\| = \|a^{-1}(b-a)b^{-1}\| \le \|a^{-1}\|\|b^{-1}\|\|b-a\| < 2\|a^{-1}\|^2\delta \le \epsilon.$$

This shows the continuity of the inversion map, completing the proof. $\square$

Definition 5.5. Let A be a unital Banach algebra and $a \in A$. Then, the set $\sigma_A(a) := \{\lambda \in \mathbb{C} : \lambda e - a \text{ not invertible}\}$ is called the *spectrum* of a.

Proposition 5.6. *Let A be a unital Banach algebra and $a \in A$. Then the spectrum $\sigma_A(a)$ of a is a compact subset of $\mathbb{C}$. Moreover, $|\lambda| \le \|a\|$ if $\lambda \in \sigma_A(a)$.*

Proof. Consider $\lambda \in \mathbb{C}$ such that $|\lambda| > \|a\|$. Then, $\|\lambda^{-1}a\| = |\lambda^{-1}|\|a\| < 1$. So, $e - \lambda^{-1}a$ is invertible by Proposition 5.3. Equivalently, $\lambda e - a$ is invertible and hence $\lambda \notin \sigma_A(a)$. This proves the second statement and also implies that $\sigma_A(a)$ is bounded.

It remains to show that $\sigma_A(a)$ is closed. Take $\lambda \notin \sigma_A(a)$. Set $\epsilon := \|(\lambda e - a)^{-1}\|^{-1}$. We claim that for all $\lambda' \in B_\epsilon(\lambda)$ the element $\lambda' e - a$ is invertible. Note that $\|(\lambda - \lambda')(\lambda e - a)^{-1}\| = |\lambda - \lambda'|\|(\lambda e - a)^{-1}\| < \epsilon\|(\lambda e - a)^{-1}\| = 1$. So by Proposition 5.3 the element $e - (\lambda - \lambda')(\lambda e - a)^{-1}$ is invertible. But the product of invertible elements is invertible and so is hence $\lambda' e - a = (\lambda e - a)(e - (\lambda - \lambda')(\lambda e - a)^{-1})$, proving the claim. Thus, $\mathbb{C} \setminus \sigma_A(a)$ is open and $\sigma_A(a)$ is closed, completing the proof. $\square$

Lemma 5.7. *Let A be a unital algebra and $a, b \in A$. Suppose that $a \cdot b$ and $b \cdot a$ are invertible. Then, a and b are separately invertible.*

Proof. **Exercise.** □

Theorem 5.8 (Spectral Mapping Theorem)**.** *Let A be a unital Banach algebra, p a complex polynomial in one variable and $a \in A$. Then, $\sigma_A(p(a)) = p(\sigma_A(a))$.*

Proof. If p is a constant the statement is trivially satisfied. We thus assume in the following that p has degree at least 1.

We first prove that $p(\sigma_A(a)) \subseteq \sigma_A(p(a))$. Let $\lambda \in \mathbb{C}$. Then the polynomial in t given by $p(t) - p(\lambda)$ can be decomposed as $p(t) - p(\lambda) = q(t)(t - \lambda)$ for some polynomial q. In particular, $p(a) - p(\lambda) = q(a)(a - \lambda)$ in A. Suppose $p(\lambda) \notin \sigma_A(p(a))$. Then the left hand side is invertible and so must be the right hand side. By Lemma 5.7 each of the factors must be invertible. In particular, $a - \lambda$ is invertible and so $\lambda \notin \sigma_A(a)$. We have thus shown that $\lambda \in \sigma_A(a)$ implies $p(\lambda) \in \sigma_A(p(a))$.

We proceed to prove that $\sigma_A(p(a)) \subseteq p(\sigma_A(a))$. Let $\mu \in \mathbb{C}$ and factorize the polynomial in t given by $p(t) - \mu$, i.e., $p(t) - \mu = c(t - \gamma_1) \cdots (t - \gamma_n)$, where $c \neq 0$. We apply this to a to get $p(a) - \mu = c(a - \gamma_1) \cdots (a - \gamma_n)$. Now if $\mu \in \sigma_A(p(a))$, then the left hand side is not invertible. Hence, at least one factor $a - \gamma_k$ must be non-invertible on the right hand side. So, $\gamma_k \in \sigma_A(a)$ and also $\mu = p(\gamma_k)$. Thus, $\mu \in p(\sigma_A(a))$. This completes the proof. □

Definition 5.9. Let A be a Banach algebra and $a \in A$. We define the *spectral radius* of a as

$$r_A(a) := \inf_{n \in \mathbb{N}} \|a^n\|^{1/n}.$$

Lemma 5.10. *Let $\{c_n\}_{n \in \mathbb{N}}$ be a sequence of non-negative real numbers such that $c_{n+m} \leq c_n c_m$ for all $n, m \in \mathbb{N}$. Then $\{c_n^{1/n}\}_{n \in \mathbb{N}}$ converges to $\inf_{n \in \mathbb{N}} c_n^{1/n}$.*

Proof. Define $c_0 := 1$. For fixed m decompose any positive integer $n = k(n)m + r(n)$ such that $r(n), k(n) \in \mathbb{N}_0$ and $r(n) < m$. Then,

$$c_n^{1/n} \leq c_{k(n)m}^{1/n} c_{r(n)}^{1/n} \leq c_m^{k(n)/n} c_{r(n)}^{1/n}.$$

Since $r(n)$ is bounded and $k(n)/n$ converges to $1/m$ for large n the right hand side tends to $c_m^{1/m}$ for large n. This implies,

$$\limsup_{n \to \infty} c_n^{1/n} \leq c_m^{1/m}.$$

Since m was arbitrary we conclude,

$$\limsup_{n \to \infty} c_n^{1/n} \leq \inf_{n \in \mathbb{N}} c_n^{1/n} \leq \liminf_{n \to \infty} c_n^{1/n}.$$

This completes the proof. □

Proposition 5.11. *Let A be a Banach algebra and $a \in A$. Then,*

$$\lim_{n\to\infty} \|a^n\|^{1/n} \quad \textit{exists and is equal to} \quad \inf_{n\in\mathbb{N}} \|a^n\|^{1/n}.$$

Proof. If a is nilpotent (i.e., $a^n = 0$ for some n) the statement is trivial. Assume otherwise and set $c_n := a^n$. Applying Lemma 5.10 yields the result. □

Lemma 5.12. *Let A be a unital Banach algebra, $\psi : A \to \mathbb{C}$ linear and continuous, $a \in A$. Then the map $f : \mathbb{C} \setminus \sigma_A(a) \to \mathbb{C}$ given by $f(z) := \psi\left((a - ze)^{-1}\right)$ is holomorphic in all its domain.*

Proof. Let $z \in \mathbb{C} \setminus \sigma_A(a)$. Since $\sigma_A(a)$ is closed, there exists $r > 0$ such that $\xi \mapsto (a-(z+\xi)e)^{-1}$ is well defined if $\xi \in B_r(0)$. For $\xi \in B_r(0)$ we thus have

$$\begin{aligned}
&(a - (z+\xi)e)^{-1} - (a - ze)^{-1} \\
&= (a - ze)(a - (z+\xi)e)^{-1}(a - ze)^{-1} \\
&\quad - (a - (z+\xi)e)(a - (z+\xi)e)^{-1}(a - ze)^{-1} \\
&= (a - ze - a + (z+\xi)e)(a - (z+\xi)e)^{-1}(a - ze)^{-1} \\
&= \xi(a - (z+\xi)e)^{-1}(a - ze)^{-1}.
\end{aligned}$$

In the first equality we have used the commutativity of the subalgebra of A that is generated by polynomials in a. Supposing $\xi \neq 0$ we divide by ξ and apply ψ on both sides yielding,

$$\frac{f(z+\xi) - f(z)}{\xi} = \psi\left((a - (z+\xi)e)^{-1}(a - ze)^{-1}\right).$$

Since inversion in A is continuous (Proposition 5.4), the right hand side of this equality is continuous in ξ and we may take the limit,

$$\lim_{|\xi|\to 0} \frac{f(z+\xi) - f(z)}{\xi} = \psi\left((a - ze)^{-1}(a - ze)^{-1}\right).$$

This shows that f is complex differentiable at z. Since z was arbitrary in $\mathbb{C} \setminus \sigma_A(a)$, this implies that f is holomorphic in $\mathbb{C} \setminus \sigma_A(a)$. □

Theorem 5.13. *Let A be a unital Banach algebra and $a \in A$. Then*

$$r_A(a) = \sup_{\lambda\in\sigma_A(a)} |\lambda|.$$

In particular, $\sigma_A(a) \neq \emptyset$.

Proof. Choose $\lambda \in \mathbb{C}$ such that $|\lambda| > r_A(a)$. Then there exists $n \in \mathbb{N}$ such that $|\lambda| > \|a^n\|^{1/n}$ and hence $|\lambda^n| > \|a^n\|$. By Proposition 5.6 we know that $\lambda^n \notin \sigma_A(a^n)$. By Theorem 5.8 with $p(t) = t^n$ this implies $\lambda \notin \sigma_A(a)$. This shows $|\lambda| \le r_A(a)$ for all $\lambda \in \sigma_A(a)$.

Applying Proposition 5.3 to $e - a/z$ yields the power series expansion

$$(a - ze)^{-1} = \sum_{n=0}^{\infty} -a^n z^{-n-1}$$

for $|z| > \|a\|$. Given a continuous linear functional $\psi : A \to \mathbb{C}$ we obtain the Laurent series

$$\psi\left((a - ze)^{-1}\right) = \sum_{n=0}^{\infty} -\psi(a^n) z^{-n-1}.$$

However, the left hand side is holomorphic in $\mathbb{C} \setminus \sigma_A(a)$ due to Lemma 5.12. Thus, the inner radius of convergence of the Laurent series is at most $\rho := \sup_{\lambda \in \sigma_A(a)} |\lambda|$, supposing that $\sigma_A(a) \neq \emptyset$. This is equivalent to the statement

$$\limsup_{n \to \infty} |\psi(a^n)|^{1/n} \le \rho.$$

Given $\mu > \rho$ we obtain

$$\limsup_{n \to \infty} \left|\psi\left(\left(\frac{a}{\mu}\right)^n\right)\right|^{1/n} \le \frac{\rho}{\mu} < 1.$$

This in turn implies

$$\sup_{n \in \mathbb{N}} \left|\psi\left(\left(\frac{a}{\mu}\right)^n\right)\right| < \infty.$$

Define now the subset $B \subseteq A$ given by

$$B := \left\{\frac{a}{\mu}, \left(\frac{a}{\mu}\right)^2, \left(\frac{a}{\mu}\right)^3, \dots\right\}.$$

Identifying A as a Banach space isometrically with the corresponding subspace of A^{**} according to Proposition 4.25 allows to view B as a subset of $\mathrm{CL}(A^*, \mathbb{C})$. We may thus apply the Banach-Steinhaus Theorem in the form of Corollary 3.51 to conclude that there is a constant $M > 0$ such that for all $n \in \mathbb{N}$,

$$\left|\psi\left(\left(\frac{a}{\mu}\right)^n\right)\right| \le M\|\psi\|.$$

This in turn implies, for all $n \in \mathbb{N}$,

$$\left\|\left(\frac{a}{\mu}\right)^n\right\| \le M.$$

From this we conclude,

$$\limsup_{n\to\infty} \|a^n\|^{1/n} \le \mu.$$

Due to the existence of the ordinary limit (Proposition 5.11) together with the fact that $\mu > \rho$ was arbitrary we obtain,

$$r_A(a) = \lim_{n\to\infty} \|a^n\|^{1/n} \le \rho.$$

This completes the proof that $r_A(a) = \rho$.

Exercise. Complete the proof by showing that $\sigma_A(a) \neq \emptyset$. □

Theorem 5.14 (Gelfand-Mazur)**.** *Let A be a unital Banach algebra such that all its non-zero elements are invertible. Then A is isomorphic to $\mathbb{C}$ as a Banach algebra.*

Proof. **Exercise.** □

5.2 The Gelfand Transform

Suppose we have some topological space T. Then, this space gives rise to a commutative algebra, namely the algebra of continuous functions on T (with complex values say). A natural question arises thus: If we are given a commutative algebra, is the algebra of continuous functions on some topological space? We might refine the question, considering more specific spaces such as Hausdorff spaces, manifolds etc. On the other hand we could also consider other classes of functions, e.g., differentiable ones etc. The Gelfand transform goes towards answering this question in the context of unital commutative Banach algebras on the one hand and compact Hausdorff spaces on the other.

5.2.1 Ideals

Definition 5.15. Let A be an algebra. An *ideal* in A is a vector subspace J of A such that $aJ \subseteq J$ and $Ja \subseteq J$ for all $a \in A$. An ideal is called *proper* iff it is not equal to A. An ideal is called *maximal* iff it is proper and it is not contained in any other proper ideal.

The special significance of maximal ideals for our present purposes is revealed by the following Exercise. This also provides a preview of what we are going to show.

Exercise 32. Consider the Banach algebra $\mathrm{C}(T,\mathbb{C})$ of Exercise 30. Assume in addition that T is Hausdorff. 1. Show that for any non-empty subset U of T the set $\{f \in \mathrm{C}(T,\mathbb{C}) : f(U) = 0\}$ forms a proper closed ideal. 2. Show that the maximal ideals are in one-to-one correspondence to points of T.

Proposition 5.16. *Let A be a Banach algebra. Then, the closure of an ideal is an ideal.*

Proof. Let J be an ideal. We already know that $\overline{J}$ is a vector subspace. It remains to show the property $a\overline{J} \subseteq \overline{J}$ and $\overline{J}a \subseteq \overline{J}$ for all $a \in A$. Consider $b \in \overline{J}$. Then, there is a sequence $\{b_n\}_{n\in\mathbb{N}}$ with $b_n \in J$ converging to b. Take now $a \in A$ and consider the sequences $\{ab_n\}_{n\in\mathbb{N}}$ and $\{b_n a\}_{n\in\mathbb{N}}$. Since J is an ideal the elements of these sequences are all in J. And since multiplication by a fixed element is continuous the sequences converge to ab and ba respectively. So $ba \in \overline{J}$ and $ab \in \overline{J}$. This completes the proof. □

Proposition 5.17. *Let A be a unital Banach algebra.*

1. *If $a \in A$ is invertible it is not contained in any proper ideal.*
2. *Maximal ideals are closed.*
3. *Any proper ideal is contained in a maximal ideal.*

Proof. Suppose J is an ideal containing an invertible element $a \in A$. Then, $a^{-1}a = e \in J$ and thus $J = A$. This proves 1. Suppose J is a proper ideal. Then, $\overline{J}$ is an ideal by Proposition 5.16. On the other hand, by 1. the intersection of the set I_A of invertible elements of A with J is empty. But by Proposition 5.4 this set is open, so $I_A \cap \overline{J} = \emptyset$. Since $e \in I_A$, $\overline{J} \neq A$, i.e., $\overline{J}$ is proper. So we get an inclusion of proper ideals, $J \subseteq \overline{J}$. If J is maximal we must therefore have $J = \overline{J}$. This proves 2. The proof of 3 is a standard application of Zorn's Lemma. □

Proposition 5.18. *Let A be a Banach algebra and J a closed proper ideal. Then, A/J is a Banach algebra with the quotient norm. If A is unital then so is A/J. If A is commutative then so is A/J.*

Proof. **Exercise.** □

Definition 5.19. Let A be a Banach algebra. The set of maximal ideals of A is called the *maximal ideal space* and denoted by M_A. The set of maximal ideals with codimension 1 is denoted by M_A^1.

Proposition 5.20. *Let A be a commutative unital Banach algebra. Then, maximal ideals have codimension* 1*. In particular, $M_A = M_A^1$.*

Proof. Let J be a maximal ideal. By Proposition 5.17.2, J is closed. Hence, by Proposition 5.18, A/J is a unital commutative Banach algebra. We show that every non-zero element of A/J is invertible. For $a \in A \setminus J$ set $J_a := \{ab + c : b \in A \text{ and } c \in J\}$. It is easy to see that J_a is an ideal and $J \subset J_a$ as well as $J_a \neq J$. Since J is maximal we find $J_a = A$. But his means there is a $b \in A$ such that $[a][b] = [e]$ in A/J, i.e., $[a]$ is invertible in A/J. But every non-zero element of A/J arises as $[a]$ with $a \in A \setminus J$, so they are all invertible. By the Theorem 5.14 of Gelfand-Mazur we find that A/J is isomorphic to $\mathbb{C}$ and hence 1-dimensional. So, J must have codimension 1. ☐

5.2.2 Characters

Definition 5.21. Let A be a Banach algebra. An algebra homomorphism $\phi : A \to \mathbb{C}$ is called a *character* of A.

Proposition 5.22. *Let A be a Banach algebra. Then, any character $\phi : A \to \mathbb{C}$ is continuous. Moreover, $\|\phi\| \leq 1$. If A is also unital and $\phi \neq 0$ then $\phi(e) = 1$ and $\|\phi\| = 1$.*

Proof. Consider an algebra homomorphism $\phi : A \to \mathbb{C}$. Suppose $|\phi(a)| > \|a\|$ for some $a \in A$. Then we can find $\lambda \in \mathbb{C}$ such that $\phi(\lambda a) = 1$ while $\|\lambda a\| < 1$. Set $b := \sum_{n=1}^{\infty} (\lambda a)^n$. Then $b = \lambda a + \lambda a b$ and we obtain the contradiction $\phi(b) = \phi(\lambda a) + \phi(\lambda a)\phi(b) = 1 + \phi(b)$. Thus, $|\phi(a)| \leq \|a\|$ for all $a \in A$ and ϕ must be continuous. Also, $\|\phi\| \leq 1$.

Now assume in addition that A is unital and $\phi \neq 0$. Then there exists $a \in A$ such that $\phi(a) \neq 0$. We deduce $\phi(e) = 1$ since $\phi(a) = \phi(ea) = \phi(e)\phi(a)$ and thus $\|\phi\| \geq 1$. ☐

Definition 5.23. Let A be a Banach algebra. The set of non-zero characters on A is called the *character space* or *Gelfand space* of A, denoted by Γ_A. We view Γ_A as a subset of A^*, but equipped with the weak* topology. Define the map $A \to \mathrm{C}(\Gamma_A, \mathbb{C})$ given by $a \mapsto \hat{a}$ where $\hat{a}(\phi) := \phi(a)$. This map is called the *Gelfand transform*.

Proposition 5.24. *Let A be a unital Banach algebra. Then, Γ_A is a compact Hausdorff space.*

Proof. Since A^* is Hausdorff with the weak* topology so is its subset Γ_A. Let $\phi \in \Gamma_A$. By Proposition 5.22, ϕ is contained in the unit ball $\overline{B_1(0)} \subset A^*$. But by Corollary 4.22, $\overline{B_1(0)}$ is compact in the weak* topology so Γ_A is relatively compact. It remains to show that Γ_A is closed in the weak* topology. Suppose $\phi \in \overline{\Gamma_A}$. Pick two arbitrary elements $a, b \in A$. We know that the Gelfand transforms $\hat{a}, \hat{b}, \widehat{ab}$ are continuous functions on A^* with the weak* topology. Hence, choosing an arbitrary $\epsilon > 0$ we can find $\phi' \in \Gamma_A$ such that $|\phi'(a) - \phi(a)| < \epsilon$ and $|\phi'(b) - \phi(b)| < \epsilon$ and $|\phi'(ab) - \phi(ab)| < \epsilon$. **<u>Exercise</u>.**Explain! Then, $|\phi'(a)\phi'(b) - \phi(a)\phi(b)| < \epsilon(|\phi(a)| + |\phi(b)| + \epsilon)$. But, ϕ' is a character, so $\phi'(a)\phi'(b) = \phi'(ab)$. Thus, $|\phi(a)\phi(b) - \phi(ab)| < \epsilon(1 + |\phi(a)| + |\phi(b)| + \epsilon)$. Since ϵ was arbitrary we conclude that $\phi(a)\phi(b) = \phi(ab)$. This argument holds for any a, b so ϕ is a character. We have thus shown that either $\overline{\Gamma_A} = \Gamma_A$ or $\overline{\Gamma_A} = \Gamma_A \cup \{0\}$. To exclude the second possibility we need the unitality of A. Consider the subset $E := \{\phi \in A^* : \phi(e) = 1\} \subset A^*$. This subset is closed in the weak* topology since it is the preimage of the closed set $\{1\} \subset \mathbb{C}$ under the Gelfand transform $\hat{e}$ of the unit e of A. Now, $\Gamma_A \subseteq E$, but $\{0\} \notin E$, so $\{0\} \notin \overline{\Gamma_A}$. □

We are now ready to link the character space with the maximal ideal space introduced earlier. They are (essentially) the same!

Theorem 5.25. *Let A be a Banach algebra. There is a natural map $\gamma : \Gamma_A \to M_A^1$ given by $\phi \mapsto \ker\phi$. If A is unital, this map is bijective.*

Proof. Consider $\phi \in \Gamma_A$. Suppose $a \in \ker\phi$. Then, for any $b \in A$ we have $ab \in \ker\phi$ and $ba \in \ker\phi$ since $\phi(ab) = \phi(a)\phi(b) = 0$ and $\phi(ba) = \phi(b)\phi(a) = 0$. Thus, $\ker\phi$ is an ideal. It is proper since $\phi \neq 0$. Now choose $a \in A$ such that $\phi(a) \neq 0$. For arbitrary $b \in A$ there is then a $\lambda \in \mathbb{C}$ such that $\phi(b) = \phi(\lambda a)$, i.e., $\phi(b - \lambda a) = 0$ and $b - \lambda a \in \ker\phi$. In particular, $b \in \lambda a + \ker\phi$. So $\ker\phi$ has codimension 1 in A and must be maximal. This shows that γ is well defined.

Suppose now that A is unital and that J is a maximal ideal of codimension 1. Note that we can write any element a of A uniquely as $a = \lambda e + b$ where $\lambda \in \mathbb{C}$ and $b \in J$. In order for $J = \ker\phi$ for some $\phi \in \Gamma_A$ we must then have $\phi(\lambda e + b) = \lambda\phi(e) + \phi(b) = \lambda$. This determines ϕ uniquely. Hence, γ is injective. On the other hand, this formula defines a non-zero linear map $\phi : A \to \mathbb{C}$. It is easily checked that it is multiplicative and thus a character. Hence, γ is surjective. □

Proposition 5.26. *Let A be a unital Banach algebra and $a \in A$. Then, $\{\phi(a) : \phi \in \Gamma_A\} \subseteq \sigma_A(a)$. If A is commutative, then even $\{\phi(a) : \phi \in \Gamma_A\} = \sigma_A(a)$. In particular, $\Gamma_A \neq 0$.*

Proof. Suppose $\lambda = \phi(a)$ for some $\phi \in \Gamma_A$. Then, $\phi(\lambda e - a) = 0$, i.e., $\lambda e - a \in \ker \phi$. But by Theorem 5.25, $\ker \phi$ is a maximal ideal which by Proposition 5.17.1 cannot contain an invertible element. So $\lambda e - a$ is not invertible and $\lambda \in \sigma_A(a)$. This proves the first statement.

Suppose now that A is commutative and let $\lambda \in \sigma_A(a)$. Define $J := \{b(\lambda e - a) : b \in A\}$. It is easy to see that J defines an ideal. It is proper, since $\lambda e - a$ is not invertible. So, by Proposition 5.17.3 it is contained in a maximal ideal J'. This maximal ideal has codimension 1 by Proposition 5.20 and induces by Theorem 5.25 a non-zero character ϕ with $\ker \phi = J'$. Hence, $\phi(\lambda e - a) = 0$ and $\phi(a) = \lambda$. This completes the proof. □

When Γ_A is compact, then the set of continuous functions of Γ_A forms a unital commutative Banach algebra by Exercise 30. We then have the following Theorem.

Theorem 5.27 (Gelfand Representation Theorem)**.** *Let A be a unital Banach algebra. The Gelfand transform $A \to \mathrm{C}(\Gamma_A, \mathbb{C})$ is a continuous unital algebra homomorphism. The image of A under the Gelfand transform, denoted $\hat{A}$, is a normed subalgebra of $\mathrm{C}(\Gamma_A, \mathbb{C})$. Moreover, $\|\hat{a}\| \le r_A(a) \le \|a\|$ and $\sigma_{\hat{A}}(\hat{a}) \subseteq \sigma_A(a)$ for all $a \in A$. If A is commutative we have the sharper statements $\|\hat{a}\| = r_A(a)$ and $\sigma_{\hat{A}}(\hat{a}) = \sigma_A(a)$.*

Proof. The property of being a unital algebra homomorphism is clear. For $a \in A$ we have $\|\hat{a}\| = \sup_{\phi \in \Gamma_A} |\phi(a)|$. By Proposition 5.26 combined with Theorem 5.13 we then find $\|\hat{a}\| \le r_A(a)$ and in the commutative case $\|\hat{a}\| = r_A(a)$. On the other hand Proposition 5.6 combined with Theorem 5.13 implies $r_A(a) \le \|a\|$. Thus, the Gelfand transform is bounded by 1 and hence continuous. Since the Gelfand transform is a unital algebra homomorphism, invertible elements are mapped to invertible elements, so $\sigma_{\hat{A}}(\hat{a}) \subseteq \sigma_A(a)$. Let $a \in A$ and consider $\lambda \in \mathbb{C}$. If $\phi(a) = \lambda$ for some $\phi \in \Gamma_A$ then $\lambda \hat{e} - \hat{a}$ vanishes on this ϕ and cannot be invertible in $\hat{A}$, i.e., $\lambda \in \sigma_{\hat{A}}(\hat{a})$. Using Proposition 5.26 we conclude $\sigma_{\hat{A}}(\hat{a}) \supseteq \sigma_A(a)$ in the commutative case. □

Proposition 5.28. *Let A be a unital commutative Banach algebra. Suppose that $\|a^2\| = \|a\|^2$ for all $a \in A$. Then, the Gelfand transform $A \to \mathrm{C}(\Gamma_A, \mathbb{C})$ is isometric. In particular, it is injective and its image $\hat{A}$ is a Banach algebra.*

Proof. Under the assumption $\lim_{n\to\infty} \|a^n\|^{1/n}$, which exists by Proposition 5.11, is equal to $\|a\|$ for all $a \in A$. By the same Proposition then $r_A(a) = \|a\|$. So by Theorem 5.27, $\|\hat{a}\| = r_A(a) = \|a\|$. Isometry implies of course injectivity. Moreover, it implies that the image is complete since the domain is complete. So $\hat{A}$ is a Banach algebra. $\square$

Exercise 33. Let $A = \mathrm{C}(T, \mathbb{C})$ be the Banach algebra of Exercises 30 and 32. Show that $\Gamma_A = T$ as topological spaces in a natural way and that the Gelfand transform is the identity under this identification.

6 Hilbert Spaces

6.1 The Fréchet-Riesz Representation Theorem

Definition 6.1. Let X be an inner product space. A pair of vectors $x, y \in X$ is called *orthogonal* iff $\langle x, y\rangle = 0$. We write $x \perp y$. A pair of subsets $A, B \subseteq X$ is called *orthogonal* iff $x \perp y$ for all $x \in A$ and $y \in B$. Moreover, if $A \subseteq X$ is some subset we define its *orthogonal complement* to be

$$A^\perp := \{y \in X : x \perp y\ \forall x \in A\}.$$

Exercise 34. Let X be an inner product space.

1. Let $x, y \in X$. If $x \perp y$ then $\|x\|^2 + \|y\|^2 = \|x + y\|^2$.
2. Let $A \subseteq X$ be a subset. Then $A^\perp$ is a closed subspace of X.
3. $A \subseteq (A^\perp)^\perp$.
4. $A^\perp = \overline{(\operatorname{span} A)}^\perp$.
5. $A \cap A^\perp \subseteq \{0\}$.

Proposition 6.2. *Let H be a Hilbert space, $F \subseteq H$ a closed and convex subset and $x \in H$. Then, there exists a unique element $\tilde{x} \in F$ such that*

$$\|\tilde{x} - x\| = \inf_{y\in F} \|y - x\|.$$

Proof. Define $a := \inf_{y\in F} \|y-x\|$. Let $\{y_n\}_{n\in\mathbb{N}}$ be a sequence in F such that $\lim_{n\to\infty} \|y_n - x\| = a$. Let $\epsilon > 0$ and choose $n_0 \in \mathbb{N}$ such that $\|y_n - x\|^2 \le a^2 + \epsilon$ for all $n \ge n_0$. Now let $n, m \ge n_0$. Then, using the parallelogram equality of Theorem 2.50 we find

$$\begin{aligned}\|y_n - y_m\|^2 &= 2\|y_n - x\|^2 + 2\|y_m - x\|^2 - \|y_n + y_m - 2x\|^2\\ &= 2\|y_n - x\|^2 + 2\|y_m - x\|^2 - 4\left\|\frac{y_n + y_m}{2} - x\right\|^2\\ &\le 2(a^2 + \epsilon) + 2(a^2 + \epsilon) - 4a^2 = 4\epsilon\end{aligned}$$

This shows that $\{y_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence which must converge to some vector $\tilde{x} \in F$ with the desired properties since F is complete.

It remains to show that $\tilde{x}$ is unique. Suppose $\tilde{x}, \tilde{x}' \in F$ both satisfy the condition. Then, by a similar use of the parallelogram equation as above,

$$\|\tilde{x}-\tilde{x}'\|^2 = 2\|\tilde{x}-x\|^2+2\|\tilde{x}'-x\|^2-4\left\|\frac{\tilde{x} + \tilde{x}'}{2} - x\right\|^2 \le 2a^2+2a^2-4a^2 = 0.$$

That is, $\tilde{x}' = \tilde{x}$, completing the proof. $\square$

Lemma 6.3. *Let H be a Hilbert space, $F \subseteq H$ a closed and convex subset, $x \in H$ and $\tilde{x} \in F$. Then, the following are equivalent:*

1. $\|\tilde{x} - x\| = \inf_{y \in F} \|y - x\|$
2. $\Re\langle \tilde{x} - y, \tilde{x} - x\rangle \le 0 \;\forall y \in F$

Proof. Suppose 2. holds. Then, for any $y \in F$ we have

$$\begin{aligned}\|y - x\|^2 &= \|(y - \tilde{x}) + (\tilde{x} - x)\|^2 \\ &= \|y - \tilde{x}\|^2 + 2\Re\langle y - \tilde{x}, \tilde{x} - x\rangle + \|\tilde{x} - x\|^2 \ge \|\tilde{x} - x\|^2.\end{aligned}$$

Conversely, suppose 1. holds. Fix $y \in F$ and consider the continuous map $[0,1] \to F$ given by $t \mapsto y_t := (1-t)\tilde{x} + ty$. Then,

$$\begin{aligned}\|\tilde{x} - x\|^2 \le \|y_t - x\|^2 &= \|t(y - \tilde{x}) + (\tilde{x} - x)\|^2 \\ &= t^2\|y - \tilde{x}\|^2 + 2t\Re\langle y - \tilde{x}, \tilde{x} - x\rangle + \|\tilde{x} - x\|^2.\end{aligned}$$

Subtracting $\|\tilde{x} - x\|^2$ and dividing for $t \in (0,1]$ by t leads to,

$$\frac{1}{2}t\|y - \tilde{x}\|^2 \ge \Re\langle \tilde{x} - y, \tilde{x} - x\rangle.$$

This implies 2. □

Lemma 6.4. *Let H be a Hilbert space, $F \subseteq H$ a closed subspace, $x \in H$ and $\tilde{x} \in F$. Then, the following are equivalent:*

1. $\|\tilde{x} - x\| = \inf_{y \in F} \|y - x\|$
2. $\langle y, \tilde{x} - x\rangle = 0 \;\forall y \in F$

Proof. **Exercise**. □

Proposition 6.5. *Let H be a Hilbert space, $F \subseteq H$ a closed proper subspace. Then, $F^\perp \ne \{0\}$.*

Proof. Since F is proper, there exists $x \in H \setminus F$. By Proposition 6.2 there exists an element $\tilde{x} \in F$ such that $\|\tilde{x} - x\| = \inf_{y \in F} \|y - x\|$. By Lemma 6.4, $\langle y, \tilde{x} - x\rangle = 0$ for all $y \in F$. That is, $\tilde{x} - x \in F^\perp$. □

Theorem 6.6 (Fréchet-Riesz Representation Theorem)**.** *Let H be a Hilbert space. Then, the map $\Phi : H \to H^*$ given by $(\Phi(x))(y) := \langle y, x\rangle$ for all $x, y \in H$ is anti-linear, bijective and isometric.*

Proof. The anti-linearity of Φ follows from the properties of the scalar product. Observe that for all $x \in H$, $\|\Phi(x)\| = \sup_{\|y\|=1} |\langle y, x\rangle| \le \|x\|$ because of the Schwarz inequality (Theorem 2.46). On the other hand, $(\Phi(x))(x/\|x\|) = \|x\|$ for all $x \in H \setminus \{0\}$. Hence, $\|\Phi(x)\| = \|x\|$ for all $x \in H$, i.e., Φ is isometric. It remains to show that Φ is surjective. Let $f \in H^* \setminus \{0\}$. Then $\ker f$ is a closed proper subspace of H and by Proposition 6.5 there exists a vector $v \in (\ker f)^\perp \setminus \{0\}$. Observe that for all $x \in H$,

$$x - \frac{f(x)}{f(v)} v \in \ker f.$$

Hence,

$$\langle x, v\rangle = \left\langle x - \frac{f(x)}{f(v)} v + \frac{f(x)}{f(v)} v, v \right\rangle = \frac{f(x)}{f(v)} \langle v, v\rangle$$

In particular, setting

$$w := \frac{\overline{f(v)}}{\|v\|^2} v$$

we see that $\Phi(w) = f$. □

Corollary 6.7. *Let H be a Hilbert space. Then, H^* is also a Hilbert space. Moreover, H is reflexive, i.e., H^{**} is naturally isomorphic to H.*

Proof. By Theorem 6.6 the spaces H and H^* are isometric. This implies in particular, that H^* is complete and that its norm satisfies the parallelogram equality, i.e., that it is a Hilbert space. Indeed, it is easily verified that the inner product is given by

$$\langle \Phi(x), \Phi(y)\rangle_{H^*} = \langle y, x\rangle_H \quad \forall x, y \in H.$$

Consider the canonical linear map $i_H : H \to H^{**}$. It is easily verified that $i_H = \Psi \circ \Phi$, where $\Psi : H^* \to H^{**}$ is the corresponding map of Theorem 6.6. Thus, i_H is a linear bijective isometry, i.e., an isomorphism of Hilbert spaces. □

6.2 Orthogonal Projectors

Theorem 6.8. *Let H be a Hilbert space and $F \subseteq H$ a closed subspace such that $F \neq \{0\}$. Then, there exists a unique operator $P_F \in \mathrm{CL}(H, H)$ with the following properties:*

1. $P_F|_F = \mathbf{1}_F$.

2. $\ker P_F = F^\perp$.

Moreover, P_F also has the following properties:

3. $P_F(H) = F$.
4. $P_F \circ P_F = P_F$.
5. $\|P_F\| = 1$.
6. *Given $x \in H$, $P_F(x)$ is the unique element of F such that $\|P_F(x) - x\| = \inf_{y \in F} \|y - x\|$.*
7. *Given $x \in H$, $P_F(x)$ is the unique element of F such that $x - P_F(x) \in F^\perp$.*

Proof. We define P_F to be the map $x \mapsto \tilde{x}$ given by Proposition 6.2. Then, clearly $P_F(H) = F$ and $P_F(x) = x$ if $x \in F$ and thus $P_F \circ P_F = P_F$. By Lemma 6.4 we have $P_F(x) - x \in F^\perp$ for all $x \in H$. Since $F^\perp$ is a subspace we have

$$(\lambda_1 P_F(x_1) - \lambda_1 x_1) + (\lambda_2 P_F(x_2) - \lambda_2 x_2) \in F^\perp$$

for $x_1, x_2 \in H$ and $\lambda_1, \lambda_2 \in \mathbb{K}$ arbitrary. Rewriting this we get,

$$(\lambda_1 P_F(x_1) + \lambda_2 P_F(x_2)) - (\lambda_1 x_1 + \lambda_2 x_2) \in F^\perp.$$

But Lemma 6.4 also implies that if given $x \in H$ we have $z - x \in F^\perp$ for some $z \in F$, then $z = P_F(x)$. Thus,

$$\lambda_1 P_F(x_1) + \lambda_2 P_F(x_2) = P_F(\lambda_1 x_1 + \lambda_2 x_2).$$

That is, P_F is linear. Using again that $x - P_F(x) \in F^\perp$ we have $x - P_F(x) \perp P_F(x)$ and hence the Pythagoras equality (Exercise 34.1)

$$\|x - P_F(x)\|^2 + \|P_F(x)\|^2 = \|x\|^2 \quad \forall x \in H.$$

This implies $\|P_F(x)\| \leq \|x\|$ for all $x \in H$. In particular, P_F is continuous. On the other hand $\|P_F(x)\| = \|x\|$ if $x \in F$. Therefore, $\|P_F\| = 1$. Now suppose $x \in \ker P_F$. Then, $\langle y, x\rangle = -\langle y, P_F(x) - x\rangle = 0$ for all $y \in F$ and hence $x \in F^\perp$. That is, $\ker P_F \subseteq F^\perp$. Conversely, suppose now $x \in F^\perp$. Then, $\langle y, P_F(x)\rangle = \langle y, P_F(x) - x\rangle = 0$ for all $y \in F$. Thus, $P_F(x) \in F^\perp$. But we know already that $P_F(x) \in F$. Since, $F \cap F^\perp = \{0\}$ we get $P_F(x) = 0$, i.e., $x \in \ker P_F$. Then, $F^\perp \subseteq \ker P_F$. Thus, $\ker P_F = F^\perp$. This concludes the proof the the existence of P_F with properties 1, 2, 3, 4, 5, 6 and 7.

Suppose now there is another operator $Q_F \in \mathrm{CL}(H, H)$ which also has the properties 1 and 2. We proceed to show that $Q_F = P_F$. Let $x \in H$ arbitrary. Since $P_F(x) - x \in F^\perp$, property 2 of Q_F implies $Q_F(x) = Q_F(P_F(x))$. On the other hand $P_F(x) \in F$ so by property 1 of Q_F we have $Q_F(P_F(x)) = P_F(x)$. Hence $Q_F(x) = P_F(x)$. Since x was arbitrary we have $Q_F = P_F$, completing the proof. □

Definition 6.9. Given a Hilbert space H and a closed subspace F, the operator $P_F \in \mathrm{CL}(H, H)$ constructed in Theorem 6.8 is called the *orthogonal projector* onto the subspace F.

Corollary 6.10. *Let H be a Hilbert space and F a closed subspace. Let P_F be the associated orthogonal projector. Then $\mathbf{1} - P_F$ is the orthogonal projector onto $F^\perp$. That is, $P_{F^\perp} = \mathbf{1} - P_F$.*

Proof. Let $x \in F^\perp$. Then, $(\mathbf{1} - P_F)(x) = x$ since $\ker P_F = F^\perp$ by Theorem 6.8.1. That is, $(\mathbf{1} - P_F)|_{F^\perp} = \mathbf{1}_{F^\perp}$. On the other hand, suppose $(\mathbf{1} - P_F)(x) = 0$. By Theorem 6.8.1. and 3. this is equivalent to $x \in F$. That is, $\ker(\mathbf{1} - P_F) = F$. Applying Theorem 6.8 to $F^\perp$ yields the conclusion $P_{F^\perp} = \mathbf{1} - P_F$ due to the uniqueness of $P_{F^\perp}$. □

Corollary 6.11. *Let H be a Hilbert space and F a closed subspace. Then, $F = (F^\perp)^\perp$.*

Proof. **Exercise.** □

Definition 6.12. Let H_1 and H_2 be inner product spaces. Then, $H_1 \oplus_2 H_2$ denotes the direct sum as a vector space with the inner product

$$\langle x_1 + x_2, y_1 + y_2 \rangle := \langle x_1, y_1 \rangle + \langle x_2, y_2 \rangle \quad \forall x_1, x_2 \in H_1, \forall y_1, y_2 \in H_2.$$

Proposition 6.13. *Let H_1 and H_2 be inner product spaces. Then, the topology of $H_1 \oplus_2 H_2$ agrees with the topology of the direct sum of H_1 and H_2 as tvs. That is, it agrees with the product topology of $H_1 \times H_2$. In particular, if H_1 and H_2 are complete, then $H_1 \oplus_2 H_2$ is complete.*

Proof. **Exercise.** □

Corollary 6.14. *Let H be a Hilbert space and F a closed subspace. Then, $H = F \oplus_2 F^\perp$.*

Proof. **Exercise.** □

6.3 Orthonormal Bases

Definition 6.15. Let H be a Hilbert space and $S \subseteq H$ a subset such that $\|s\| = 1$ for all $s \in S$ and such that $\langle s, t\rangle \neq 0$ for $s, t \in S$ implies $s = t$. Then, S is called an *orthonormal system* in H. Suppose furthermore that S is maximal, i.e., that for any orthonormal system T in H such that $S \subseteq T$ we have $S = T$. Then, S is called an *orthonormal basis* of H.

Proposition 6.16. *Let H be a Hilbert space and S an orthonormal system in H. Then, S is linearly independent.*

Proof. **Exercise.** □

Proposition 6.17 (Gram-Schmidt)**.** *Let H be a Hilbert space and $\{x_n\}_{n\in I}$ be a linearly independent subset, indexed by the countable set I. Then, there exists an orthonormal system $\{s_n\}_{n\in I}$, also indexed by I and such that* $\operatorname{span}\{s_n : n \in I\} = \operatorname{span}\{x_n : n \in I\}$.

Proof. If I is finite we identify it with $\{1, \ldots, m\}$ for some $m \in \mathbb{N}$. Otherwise we identify I with $\mathbb{N}$. We construct the set $\{s_n\}_{n\in I}$ iteratively. Set $s_1 := x_1/\|x_1\|$. (Note that $x_n \neq 0$ for any $n \in I$ be the assumption of linear independence.) We now suppose that $\{s_1, \ldots, s_k\}$ is an orthonormal system and that $\operatorname{span}\{s_1, \ldots, s_k\} = \operatorname{span}\{x_1, \ldots, x_k\}$. Set $X_k := \operatorname{span}\{x_1, \ldots, x_k\}$. By linear independence $y_{k+1} := x_{k+1} - P_{X_k}(x_{k+1}) \neq 0$. Set $s_{k+1} := y_{k+1}/\|y_{k+1}\|$. Clearly, $s_{k+1} \perp X_K$, i.e., $\{s_1, \ldots, s_{k+1}\}$ is an orthonormal system. Moreover, $\operatorname{span}\{s_1, \ldots, s_{k+1}\} = \operatorname{span}\{x_1, \ldots, x_{k+1}\}$. If I is finite this process terminates, leading to the desired result. If I is infinite, it is clear that this process leads to $\operatorname{span}\{s_n : n \in \mathbb{N}\} = \operatorname{span}\{x_n : n \in \mathbb{N}\}$. □

Proposition 6.18 (Bessel's inequality)**.** *Let H be a Hilbert space, $m \in \mathbb{N}$ and $\{s_1, \ldots, s_m\}$ an orthonormal system in H. Then, for all $x \in H$,*

$$\sum_{n=1}^{m} |\langle x, s_n\rangle|^2 \leq \|x\|^2$$

Proof. Define $y := x - \sum_{n=1}^{m}\langle x, s_n\rangle s_n$. Then, $y \perp s_n$ for all $n \in \{1, \ldots, m\}$. Thus, applying Pythagoras we obtain

$$\|x\|^2 = \|y\|^2 + \left\|\sum_{n=1}^{m}\langle x, s_n\rangle s_n\right\|^2 = \|y\|^2 + \sum_{n=1}^{m} |\langle x, s_n\rangle|^2.$$

This implies the inequality. □

Lemma 6.19. *Let H be a Hilbert space, $S \subset H$ an orthonormal system and $x \in H$. Then, $S_x := \{s \in S : \langle x, s\rangle \neq 0\}$ is countable.*

Proof. **Exercise.**Hint: Use Bessel's Inequality (Proposition 6.18). □

Proposition 6.20 (Generalized Bessel's inequality)**.** *Let H be a Hilbert space, $S \subseteq H$ an orthonormal system and $x \in H$. Then*

$$\sum_{s\in S} |\langle x, s\rangle|^2 \leq \|x\|^2.$$

Proof. By Lemma 6.19, the subset $S_x := \{s \in S : \langle x, s\rangle \neq 0\}$ is countable. If S_x is finite we are done due to Proposition 6.18. Otherwise let $\alpha : \mathbb{N} \to S_x$ be a bijection. Then, by Proposition 6.18

$$\sum_{n=1}^{m} |\langle x, s_{\alpha(n)}\rangle|^2 \leq \|x\|^2$$

For any $m \in \mathbb{N}$. Thus, we may take the limit $m \to \infty$ on the left hand side, showing that the series converges absolutely and satisfies the inequality. □

Definition 6.21. Let X be a tvs and $\{x_i\}_{i\in I}$ an indexed set of elements of X. We say that the series $\sum_{i\in I} x_i$ *converges unconditionally* to $x \in X$ iff $I_0 := \{i \in I : x_i \neq 0\}$ is countable and for any bijection $\alpha : \mathbb{N} \to I$ the sum $\sum_{n=1}^{\infty} x_{\alpha(n)}$ converges to x.

Proposition 6.22. *Let H be a Hilbert space and $S \subset H$ an orthonormal system. Then, $P(x) := \sum_{s\in S}\langle x, s\rangle s$ converges unconditionally. Moreover, $P : x \mapsto P(x)$ defines an orthogonal projector onto $\overline{\operatorname{span} S}$.*

Proof. Fix $x \in H$. We proceed to show that $\sum_{s\in S}\langle x, s\rangle s$ converges unconditionally. The set S can be replaced by the set $S_x := \{s \in S : \langle x, s\rangle \neq 0\}$, which is countable due to Lemma 6.19. If S_x is even finite we are done. Otherwise, let $\alpha : \mathbb{N} \to S_x$ be a bijection. Then, given $\epsilon > 0$ by Proposition 6.20 there is $n_0 \in \mathbb{N}$ such that

$$\sum_{n=n_0+1}^{\infty} |\langle x, s_{\alpha(n)}\rangle|^2 < \epsilon^2.$$

For $m > k \geq n_0$ this implies using Pythagoras,

$$\left\| \sum_{n=1}^{m} \langle x, s_{\alpha(n)}\rangle s_{\alpha(n)} - \sum_{n=1}^{k} \langle x, s_{\alpha(n)}\rangle s_{\alpha(n)} \right\|^2 = \left\| \sum_{n=k+1}^{m} \langle x, s_{\alpha(n)}\rangle s_{\alpha(n)} \right\|^2 = \sum_{n=k+1}^{m} |\langle x, s_{\alpha(n)}\rangle|^2 < \epsilon^2.$$

So the sequence $\{\sum_{n=1}^m \langle x, s_{\alpha(n)}\rangle s_{\alpha(n)}\}_{m\in\mathbb{N}}$ is Cauchy and must converge to some element $y_\alpha \in H$ since H is complete. Now let $\beta : \mathbb{N} \to S_x$ be another bijection. Then, $\sum_{n=1}^\infty \langle x, s_{\beta(n)}\rangle s_{\beta(n)} = y_\beta$ for some $y_\beta \in H$. We need to show that $y_\beta = y_\alpha$. Let $m_0 \in \mathbb{N}$ such that $\{\alpha(n) : n \le n_0\} \subseteq \{\beta(n) : n \le m_0\}$. Then, for $m \ge m_0$ we have (again using Pythagoras)

$$\left\| \sum_{n=1}^{m} \langle x, s_{\beta(n)}\rangle s_{\beta(n)} - \sum_{n=1}^{n_0} \langle x, s_{\alpha(n)}\rangle s_{\alpha(n)} \right\|^2 \le \sum_{n=n_0+1}^{\infty} |\langle x, s_{\alpha(n)}\rangle|^2 < \epsilon^2.$$

Taking the limit $m \to \infty$ we find

$$\left\| y_\beta - \sum_{n=1}^{n_0} \langle x, s_{\alpha(n)}\rangle s_{\alpha(n)} \right\| < \epsilon.$$

But on the other hand we have,

$$\left\| y_\alpha - \sum_{n=1}^{n_0} \langle x, s_{\alpha(n)}\rangle s_{\alpha(n)} \right\| < \epsilon.$$

Thus, $\|y_\beta - y_\alpha\| < 2\epsilon$. Since ϵ was arbitrary this shows $y_\beta = y_\alpha$ proving the unconditional convergence.

It is now clear that $x \mapsto P(x)$ yields a well defined map $P : H \to H$. From the definition it is also clear that $P(H) \subseteq \overline{\operatorname{span} S}$. Let $s \in S$. Then,

$$\langle x - P(x), s\rangle = \langle x, s\rangle - \langle P(x), s\rangle = \langle x, s\rangle - \langle x, s\rangle = 0.$$

That is, $x - P(x) \in S^\perp = \overline{\operatorname{span} S}^\perp$. By Theorem 6.8.7 this implies that P is the orthogonal projector onto $\overline{\operatorname{span} S}$. $\square$

Proposition 6.23. *Let H be a Hilbert space and $S \subset H$ an orthonormal system. Then, the following are equivalent:*

1. *S is an orthonormal basis.*
2. *Suppose $x \in H$ and $x \perp S$. Then, $x = 0$.*
3. *$H = \overline{\operatorname{span} S}$.*
4. *$x = \sum_{s\in S} \langle x, s\rangle s \quad \forall x \in H$.*
5. *$\langle x, y\rangle = \sum_{s\in S} \langle x, s\rangle\langle s, y\rangle \quad \forall x, y \in H$.*
6. *$\|x\|^2 = \sum_{s\in S} |\langle x, s\rangle|^2 \quad \forall x \in H$.*

Proof. 1.⇒2.: If there exists $x \in S^\perp \setminus \{0\}$ then $S \cup \{x/\|x\|\}$ would be an orthonormal system strictly containing S, contradicting the maximality of S. 2.⇒3.: Note that $H = \{0\}^\perp = (S^\perp)^\perp = (\overline{\operatorname{span} S}^\perp)^\perp = \overline{\operatorname{span} S}$. 3.⇒4.: $\mathbf{1}(x) = P_{\overline{\operatorname{span} S}}(x) = \sum_{s\in S}\langle x, s\rangle s$ by Proposition 6.22. 4.⇒5.: Apply $\langle \cdot, y\rangle$. Since the inner product is continuous in the left argument, its application commutes with the limit taken in the sum. 5.⇒6.: Insert $y = x$. 6.⇒1.: Suppose S was not an orthonormal basis. Then there exists $y \in H \setminus \{0\}$ such that $y \in S^\perp$. But then $\|y\|^2 = \sum_{s\in S} |\langle y, s\rangle|^2 = 0$, a contradiction. □

Proposition 6.24. *Let H be a Hilbert space. Then, H admits an orthonormal basis.*

Proof. **Exercise.**Hint: Use Zorn's Lemma. □

Proposition 6.25. *Let H be a Hilbert space and $S \subset H$ an orthonormal basis of H. Then, S is countable iff H is separable.*

Proof. Suppose S is countable. Let $\mathbb{Q}S$ denote the set of linear combinations of elements of S with coefficients in $\mathbb{Q}$. Then, $\mathbb{Q}S$ is countable and also dense in H by using Proposition 6.23.3, showing that H is separable. Conversely, suppose that H is separable. Observe that $\|s - t\| = \sqrt{2}$ for $s, t \in S$ such that $s \neq t$. Thus, the open balls $B_{\sqrt{2}/2}(s)$ for different $s \in S$ are disjoint. Since H is separable there must be a countable subset of H with at least one element in each of these balls. In particular, S must be countable. □

In the following, we denote by $|S|$ the cardinality of a set S.

Proposition 6.26. *Let H be a Hilbert space and $S, T \subset H$ orthonormal basis of H. Then, $|S| = |T|$.*

Proof. If S or T is finite this is clear from linear algebra. Thus, suppose that $|S| \geq |\mathbb{N}|$ and $|T| \geq |\mathbb{N}|$. For $s \in S$ define $T_s := \{t \in T : \langle s, t\rangle \neq 0\}$. By Lemma 6.19, $|T_s| \leq |\mathbb{N}|$. Proposition 6.23.2 implies that $T \subseteq \bigcup_{s\in S} T_s$. Hence, $|T| \leq |S| \cdot |\mathbb{N}| = |S|$. Using the same argument with S and T interchanged yields $|S| \leq |T|$. Therefore, $|S| = |T|$. □

Proposition 6.27. *Let H_1 be a Hilbert space with orthonormal basis $S_1 \subset H_1$ and H_2 a Hilbert space with orthonormal basis $S_2 \subset H_2$. Then, H_1 is isometrically isomorphic to H_2 iff $|S_1| = |S_2|$.*

Proof. **Exercise.** □

Exercise 35. Let S be a set. Define $\ell^2(S)$ to be the set of maps $f : S \to \mathbb{K}$ such that $\sum_{s\in S} |f(s)|^2$ converges absolutely. (a) Show that $\ell^2(S)$ forms a Hilbert space with the inner product $\langle f, g\rangle := \sum_{s\in S} f(s)\overline{g(s)}$. (b) Let H be a Hilbert space with orthonormal basis $S \subset H$. Show that H is isomorphic to $\ell^2(S)$ as a Hilbert space.

Example 6.28. Recall the Banach spaces of Example 3.45, where X is a measurable space with measure μ. The space $\mathrm{L}^2(X, \mu, \mathbb{K})$ is a Hilbert space with inner product

$$\langle f, g\rangle := \int_X f\overline{g}.$$

Exercise 36. Let S^1 be the unit circle with the algebra of Borel sets and μ the Lebesgue measure on S^1. Parametrize S^1 with an angle $\phi \in [0, 2\pi)$ in the standard way. Show that $\{\phi \mapsto e^{\mathrm{i}n\phi}/\sqrt{2\pi}\}_{n\in\mathbb{Z}}$ is an orthonormal basis of $\mathrm{L}^2(S^1, \mu, \mathbb{C})$.

Exercise 37. Equip the closed interval $[-1, 1]$ with the algebra of Borel sets and the Lebesgue measure μ. Consider the set of monomials $\{x^n\}_{n\in\mathbb{N}}$ as functions $[-1, 1] \to \mathbb{C}$ in $\mathrm{L}^2([-1, 1], \mu, \mathbb{C})$. (a) Show that the set $\{x^n\}_{n\in\mathbb{N}}$ is linearly independent and dense. (b) Suppose an orthonormal basis $\{s_n\}_{n\in\mathbb{N}}$ of functions $s_n \in \mathrm{L}^2([-1, 1], \mu, \mathbb{C})$ is constructed using the algorithm of Gram-Schmidt (Proposition 6.17) applied to $\{x^n\}_{n\in\mathbb{N}}$. Define $p_n := \sqrt{2/(2n + 1)}s_n$. Show that

$$(n + 1)p_{n+1}(x) = (2n + 1)xp_n(x) - np_{n-1}(x) \quad \forall x \in [-1, 1], \forall n \in \mathbb{N} \setminus \{1\}.$$

6.4 Operators on Hilbert Spaces

Definition 6.29. Let H_1, H_2 be Hilbert spaces and $\Phi_i : H_i \to H_i^*$ the associated anti-linear bijective isometries from Theorem 6.6. Let $A \in \mathrm{CL}(H_1, H_2)$ and $A^* : H_2^* \to H_1^*$ its adjoint according to Definition 4.27. We say that $A^\star \in \mathrm{CL}(H_2, H_1)$ given by $A^\star := \Phi_1^{-1} \circ A^* \circ \Phi_2$ is the *adjoint operator of A in the sense of Hilbert spaces.*

In the following of this section, *adjoint* will always refer to the adjoint in the sense of Hilbert spaces.

Proposition 6.30. *Let H_1, H_2 be Hilbert spaces and $A \in \mathrm{CL}(H_1, H_2)$. Then, $A^\star$ is the adjoint of A iff*

$$\langle Ax, y\rangle_{H_2} = \langle x, A^\star y\rangle_{H_1} \quad \forall x \in H_1, y \in H_2.$$

Proof. **Exercise.** □

In the following, we will omit subscripts indicating to which Hilbert space a given inner product belongs as long as no confusion can arise.

Proposition 6.31. *Let H_1, H_2, H_3 be Hilbert spaces, $A, B \in \mathrm{CL}(H_1, H_2)$, $C \in \mathrm{CL}(H_2, H_3)$, $\lambda \in \mathbb{K}$.*

1. $(A + B)^\star = A^\star + B^\star$.
2. $(\lambda A)^\star = \overline{\lambda} A^\star$.
3. $(C \circ A)^\star = A^\star \circ C^\star$.
4. $(A^\star)^\star = A$.
5. $\|A^\star\| = \|A\|$.
6. $\|A \circ A^\star\| = \|A^\star \circ A\| = \|A\|^2$.
7. $\ker A = (A^\star(H_2))^\perp$ *and* $\ker A^\star = (A(H_1))^\perp$.

Proof. **Exercise.** □

Definition 6.32. Let H_1, H_2 be Hilbert spaces and $A \in \mathrm{CL}(H_1, H_2)$. Then, A is called *unitary* iff A is an isometric isomorphism.

Remark 6.33. It is clear that $A \in \mathrm{CL}(H_1, H_2)$ is unitary iff

$$\langle Ax, Ay \rangle = \langle x, y \rangle \quad \forall x, y \in H_1.$$

Equivalently, $A^\star \circ A = \mathbf{1}_{H_1}$ or $A \circ A^\star = \mathbf{1}_{H_2}$.

Definition 6.34. Let H be a Hilbert space and $A \in \mathrm{CL}(H, H)$. A is called *self-adjoint* iff $A = A^\star$. A is called *normal* iff $A^\star \circ A = A \circ A^\star$.

Proposition 6.35. *Let H be a Hilbert space and $A \in \mathrm{CL}(H, H)$ self-adjoint. Then,*

$$\|A\| = \sup_{\|x\| \le 1} |\langle Ax, x \rangle|.$$

Proof. Set $M := \sup_{\|x\| \le 1} |\langle Ax, x \rangle|$. Since $|\langle Ax, x \rangle| \le \|Ax\| \|x\| \le \|A\| \|x\|^2$, it is clear that $\|A\| \ge M$. We proceed to show that $\|A\| \le M$. Given $x, y \in H$ arbitrary we have

$$\begin{aligned}\langle A(x+y), x+y \rangle - \langle A(x-y), x-y \rangle &= 2\langle Ax, y \rangle + 2\langle Ay, x \rangle \\ &= 2\langle Ax, y \rangle + 2\langle y, Ax \rangle = 4\Re\langle Ax, y \rangle.\end{aligned}$$

Thus,

$$\begin{aligned}4\Re\langle Ax, y\rangle &\le |\langle A(x+y), x+y\rangle| + |\langle A(x-y), x-y\rangle| \\ &\le M(\|x+y\|^2 + \|x-y\|^2) = 2M(\|x\|^2 + \|y\|^2).\end{aligned}$$

The validity of this for all $x, y \in H$ in turn implies

$$\Re\langle Ax, y\rangle \le M\|x\|\|y\| \quad \forall x, y \in H.$$

Replacing x with λx for a suitable $\lambda \in \mathbb{K}$ with $|\lambda| = 1$ yields

$$|\langle Ax, y\rangle| \le M\|x\|\|y\| \quad \forall x, y \in H.$$

Inserting now $y = Ax$ we can infer

$$\|Ax\| \le M\|x\|\forall x \in H,$$

and hence $\|A\| \le M$, concluding the proof. □

Proposition 6.36. *Let H be a complex Hilbert space and $A \in \mathrm{CL}(H, H)$. Then, the following are equivalent:*

1. *A is self-adjoint.*
2. *$\langle Ax, x\rangle \in \mathbb{R}$ for all $x \in H$.*

Proof. 1.⇒2.: For all $x \in H$ we have $\langle Ax, x\rangle = \langle x, Ax\rangle = \overline{\langle Ax, x\rangle}$. 2.⇒1.: Let $x, y \in H$ and $\lambda \in \mathbb{C}$. Then,

$$\langle A(x+\lambda y), x+\lambda y\rangle = \langle Ax, x\rangle + \overline{\lambda}\langle Ax, y\rangle + \lambda\langle Ay, x\rangle + |\lambda|^2\langle Ay, y\rangle.$$

By assumption, the left-hand side as well as the first and the last term on the right-hand side are real. Thus, we may equating the right hand side with its complex conjugate yielding,

$$\overline{\lambda}\langle Ax, y\rangle + \lambda\langle Ay, x\rangle = \lambda\langle y, Ax\rangle + \overline{\lambda}\langle x, Ay\rangle.$$

Since $\lambda \in \mathbb{C}$ is arbitrary, the terms proportional to λ and those proportional to $\overline{\lambda}$ have to be equal separately, showing that A must be self-adjoint. □

Corollary 6.37. *Let H be a complex Hilbert space and $A \in \mathrm{CL}(H, H)$ such that $\langle Ax, x\rangle = 0$ for all $x \in H$. Then, $A = 0$.*

Proof. By Proposition 6.36, A is self-adjoint. Then, by Proposition 6.35, $\|A\| = 0$. □

Exercise 38. Give a counter example to the above statement for the case of a real Hilbert space.

Proposition 6.38. *Let H be a Hilbert space and $A \in \mathrm{CL}(H,H)$ normal. Then,*

$$\|Ax\| = \|A^\star x\| \quad \forall x \in H.$$

Proof. For all $x \in H$ we have,

$$0 = \langle (A^\star \circ A - A \circ A^\star)x, x\rangle = \langle Ax, Ax\rangle - \langle A^\star x, A^\star x\rangle = \|Ax\|^2 - \|A^\star x\|^2.$$

□

Proposition 6.39. *Let H be a Hilbert space and $A \in \mathrm{CL}(H,H)$ with $A \neq 0$ a projection operator, i.e., $A \circ A = A$. Then, the following are equivalent:*

1. *A is an orthogonal projector.*
2. *$\|A\| = 1$.*
3. *A is self-adjoint.*
4. *A is normal.*
5. *$\langle Ax, x\rangle \geq 0$ for all $x \in H$.*

Proof. 1.⇒2.: This follows from Theorem 6.8.5. 2.⇒1.: Let $x \in \ker A$, $y \in F := A(H)$ and $\lambda \in \mathbb{K}$. Then,

$$\|\lambda y\|^2 = \|A(x + \lambda y)\|^2 \leq \|x + \lambda y\|^2 = \|x\|^2 + 2\Re\langle x, \lambda y\rangle + \|\lambda y\|^2.$$

Since $\lambda \in \mathbb{K}$ is arbitrary we may conclude $\langle x, y\rangle = 0$. That is, $\ker A \subseteq F^\perp$. On the other hand set $\tilde{F} := (\mathbf{1} - A)(H)$ and note that $\tilde{F} \subseteq \ker A$. But since $\mathbf{1} = A + (\mathbf{1} - A)$ we must have $F + \tilde{F} = H$. Given $\tilde{F} \subseteq F^\perp$ this implies $\tilde{F} = F^\perp$ and hence $\ker A = F^\perp$. Observe also that F is closed since A is a projector and hence $F = \ker(\mathbf{1} - A)$. By Theorem 6.8, A is an orthogonal projector. 1.⇒3.: Using Theorem 6.8.2 and 6.8.7, observe for $x, y \in H$:

$$\langle Ax, y\rangle = \langle Ax, Ay - (Ay - y)\rangle = \langle Ax, Ay\rangle = \langle Ax - (Ax - x), Ay\rangle = \langle x, Ay\rangle.$$

3.⇒4.: Immediate. 4.⇒1.: Combining Proposition 6.38 with Proposition 6.31 we have $\ker A = \ker A^\star = (A(H))^\perp$. Note also that $A(H)$ is closed since A is a projector. Thus, by Theorem 6.8, A is an orthogonal projection. 3.⇒5.: For $x \in H$ observe

$$\langle Ax, x\rangle = \langle A \circ Ax, x\rangle = \langle Ax, Ax\rangle = \|Ax\|^2 \geq 0.$$

5.⇒1.: Let $x \in \ker A$ and $y \in F := A(H)$. Then,

$$0 \leq \langle A(x+y), x+y \rangle = \langle y, x+y \rangle = \|y\|^2 + \langle y, x \rangle.$$

Since x can be scaled arbitrarily, we must have $\langle y, x \rangle = 0$. Thus, $\ker A \subseteq F^\perp$. As above we may conclude that A is an orthogonal projector. □

Exercise 39. Let X be a normed vector space and Y a separable Hilbert space. Show that $\mathrm{KL}(X,Y) = \overline{\mathrm{CL}_{\mathrm{fin}}}(X,Y)$. [Hint: Use Proposition 4.36 and show that the assumptions of Proposition 4.37 can be satisfied.]

Exercise 40. Let $w \in \mathrm{C}([0,1],\mathbb{R})$ and consider the map $\langle \cdot,\cdot \rangle_w : \mathrm{C}([0,1],\mathbb{C}) \times \mathrm{C}([0,1],\mathbb{C}) \to \mathbb{C}$ given by

$$\langle f, g \rangle_w := \int_0^1 f(x)\overline{g(x)}w(x)\mathrm{d}x.$$

1. Give necessary and sufficient conditions for $\langle \cdot,\cdot \rangle_w$ to be a scalar product.

2. When is the norm induced by $\langle \cdot,\cdot \rangle_w$ equivalent to the norm induced by the usual scalar product

$$\langle f, g \rangle := \int_0^1 f(x)\overline{g(x)}\mathrm{d}x?$$

Exercise 41. Let S be a set and $H \subseteq F(S,\mathbb{K})$ a subspace of the functions on S with values in $\mathbb{K}$. Suppose that an inner product is given on H that makes it into a Hilbert space. Let $K : S \times S \to \mathbb{K}$ and define $K_x : S \to \mathbb{K}$ by $K_x(y) := K(y,x)$. Then, K is called a *reproducing kernel* iff $K_x \in H$ for all $x \in S$ and $f(x) = \langle f, K_x \rangle$ for all $x \in S$ and $f \in H$. Show the following:

1. If a reproducing kernel exists, it is unique.

2. A reproducing kernel exists iff the topology of H is finer than the topology of pointwise convergence.

3. If K is a reproducing kernel, then $\mathrm{span}(\{K_x\}_{x \in S})$ is dense in H.

4. Let H be the two-dimensional subspace of $\mathrm{L}^2([0,1],\mathbb{K})$ consisting of functions of the form $x \mapsto ax+b$. Determine its reproducing kernel.

7 C*-Algebras

7.1 The commutative Gelfand-Naimark Theorem

In the same sense as Banach algebras may be seen as an abstraction of the space of continuous operators on a Banach space, we can abstract the concept of continuous operators on a Hilbert space. Of course, a Hilbert space is in particular a Banach space. So the algebras we are looking for are in particular Banach algebras. The additional structure of interest coming from Hilbert spaces is that of an *adjoint*. As in the section about Banach algebras we work in the following exclusively over the field of complex numbers.

Definition 7.1. Let A be an algebra over $\mathbb{C}$. Consider a map $^\star : A \to A$ with the following properties:

- $(a+b)^\star = a^\star + b^\star$ for all $a, b \in A$.
- $(\lambda a)^\star = \overline{\lambda} a^\star$ for all $\lambda \in \mathbb{C}$ and $a \in A$.
- $(ab)^\star = b^\star a^\star$ for all $a, b \in A$.
- $(a^\star)^\star = a$ for all $a \in A$.

Then, * is called an *(anti-linear anti-multiplicative) involution*.

Definition 7.2. Let A be a Banach algebra with involution $^\star : A \to A$ such that $\|a^\star a\| = \|a\|^2$. Then, A is called a *C*-algebra*. For an element $a \in A$, the element $a^\star$ is called its *adjoint*. If $a^\star = a$, then a is called *self-adjoint*. If $a^\star a = a a^\star$, then a is called *normal*.

Exercise 42. Let A be a C*-algebra. (a) Show that $\|a^\star\| = \|a\|$ and $\|aa^\star\| = \|a\|^2$ for all $a \in A$. (b) If $e \in A$ is a unit, show that $e^\star = e$. (c) If $a \in A$ is invertible, show that $a^\star$ is also invertible.

Exercise 43. Let A be a unital C*-algebra and $a \in A$. Show that $\sigma_A(a^\star) = \overline{\sigma_A(a)}$.

Exercise 44. Let X be a Hilbert space. (a) Show that $\mathrm{CL}(X, X)$ is a unital C*-algebra. (b) Show that $\mathrm{KL}(X, X)$ is a C*-ideal in $\mathrm{CL}(X, X)$.

Exercise 45. Let A be a C*-algebra and $a \in A$. Show that there is a unique way to write $a = b + ic$ so that b and c are self-adjoint.

Exercise 46. Let T be a compact topological space. Show that the Banach algebra $\mathrm{C}(T, \mathbb{C})$ of Exercise 30 is a C*-algebra, where the involution is given by complex conjugation.

Proposition 7.3. *Let A be a $C^\star$-algebra and $a \in A$ normal. Then, $\|a^2\| = \|a\|^2$ and $r_A(a) = \|a\|$.*

Proof. We have $\|a^2\|^2 = \|(a^2)^\star(a^2)\| = \|(a^\star a)^\star(a^\star a)\| = \|a^\star a\|^2 = (\|a\|^2)^2$. This implies the first statement. Also, this implies $\|a^{2^k}\| = \|a\|^{2^k}$ for all $k \in \mathbb{N}$ and hence $\lim_{n\to\infty} \|a^n\|^{1/n} = \|a\|$ if the limit exists. But by Proposition 5.11 the limit exists and is equal to $r_A(a)$. □

Proposition 7.4. *Let A be a $C^\star$-algebra and $a \in A$ self-adjoint. Then, $\sigma_A(a) \subset \mathbb{R}$.*

Proof. Take $\alpha + \mathrm{i}\beta \in \sigma_A(a)$, where $\alpha, \beta \in \mathbb{R}$. Thus, for any $\lambda \in \mathbb{R}$ we have $\alpha + \mathrm{i}(\beta + \lambda) \in \sigma_A(a + \mathrm{i}\lambda e)$. By Proposition 5.6 we have $|\alpha + \mathrm{i}(\beta + \lambda)| \le \|a + \mathrm{i}\lambda e\|$. We deduce

$$\begin{aligned}
\alpha^2 + (\beta + \lambda)^2 &= |\alpha + \mathrm{i}(\beta + \lambda)|^2 \\
&\le \|a + \mathrm{i}\lambda e\|^2 \\
&= \|(a + \mathrm{i}\lambda e)^\star(a + \mathrm{i}\lambda e)\| \\
&= \|(a - \mathrm{i}\lambda e)(a + \mathrm{i}\lambda e)\| \\
&= \|a^2 + \lambda^2 e\| \\
&\le \|a^2\| + \lambda^2
\end{aligned}$$

Subtracting λ^2 on both sides we are left with $\alpha^2 + \beta^2 + 2\beta\lambda \le \|a^2\|$. Since this is satisfied for all $\lambda \in \mathbb{R}$ we conclude $\beta = 0$. □

Proposition 7.5. *Let A be a unital $C^\star$-algebra. Then, the Gelfand transform $A \to \mathrm{C}(\Gamma_A, \mathbb{C})$ is a continuous unital $C^\star$-algebra homomorphism. Moreover, its image is dense in $\mathrm{C}(\Gamma_A, \mathbb{C})$.*

Proof. By Theorem 5.27, the Gelfand transform is a continuous unital algebra homomorphism. We proceed to show that it respects the *-structure. Let $a \in A$ be self-adjoint. Then, combining Proposition 5.26 with Proposition 7.4 we get $\hat{a}(\phi) = \phi(a) \in \sigma_A(a) \subset \mathbb{R}$ for all $\phi \in \Gamma_A$. So $\hat{a}$ is real-valued, i.e., self-adjoint. In particular, $\widehat{a^\star} = \hat{a}^\star$. Using the decomposition of Exercise 45 this follows for general elements of A. **(Explain!)**

It remains to show that the image $\hat{A}$ of the Gelfand transform is dense. It is clear that $\hat{A}$ separates points of Γ_A by construction, vanishes nowhere (as it contains a unit) and is invariant under complex conjugation (as it is the image of a *-algebra homomorphism). Thus, the Stone-Weierstrass Theorem 4.10 ensures that $\hat{A}$ is dense in $\mathrm{C}(\Gamma_A, \mathbb{C})$. □

Theorem 7.6 (Gelfand-Naimark)**.** *Let A be a unital commutative $C^\star$-algebra. Then, the Gelfand transform $A \to \mathrm{C}(\Gamma_A, \mathbb{C})$ is an isometric isomorphism of unital commutative $C^\star$-algebras.*

Proof. Using Proposition 7.5 it remains to show that the Gelfand transform is isometric. Surjectivity then follows from the fact that the isometric image of a complete set is complete and hence closed. Since A is commutative all its elements are normal. Then, by Proposition 7.3, $\|a^2\| = \|a\|^2$ and we can apply Proposition 5.28 to conclude isometry. □

The Gelfand-Naimark Theorem 7.6 (in view of Exercise 33) gives rise to a one-to-one correspondence between compact Hausdorff spaces and unital commutative C*-algebras.

Theorem 7.7. *The category of compact Hausdorff spaces is naturally equivalent to the category of unital commutative $C^\star$-algebras.*

Proof. **Exercise.** □

Before we proceed we need a few more results about C*-algebras.

Proposition 7.8. *Let A be a unital $C^\star$-algebra and $a \in A$ normal. Define B to be the unital $C^\star$-subalgebra of A generated by a. Then, B is commutative and the Gelfand transform $\hat{a}$ of a defines a homeomorphism onto its image, $\Gamma_B \to \sigma_B(a)$ which we denote by $\tilde{a}$.*

Proof. B consists of possibly infinite linear combinations of elements of the form $(a^\star)^m a^n$ where $n, m \in \mathbb{N}_0$ (and $a^0 = (a^\star)^0 = e$). In particular, B is commutative. Consider the Gelfand transform $\hat{a} : \Gamma_B \to \mathbb{C}$ of a in B. Suppose $\hat{a}(\phi) = \hat{a}(\psi)$ for $\phi, \psi \in \Gamma_B$. Then, $\phi(a) = \psi(a)$, but also

$$\phi(a^\star) = \hat{a^\star}(\phi) = \overline{\hat{a}(\phi)} = \overline{\hat{a}(\psi)} = \hat{a^\star}(\psi) = \psi(a^\star),$$

using Proposition 7.5. Thus, ϕ is equal to ψ on monomials $(a^\star)^m a^n$ by multiplicativity and hence on all of B by linearity and continuity. This shows that $\hat{a}$ is injective. By Proposition 5.26 the image of $\hat{a}$ is $\sigma_B(a)$. Thus, $\hat{a}$ is a continuous bijective map $\hat{a} : \Gamma_B \to \sigma_B(a)$. With Lemma 1.41 it is even a homeomorphism. □

Proposition 7.9. *Let A be a unital $C^\star$-algebra and $a \in A$. Let B be a unital $C^\star$-subalgebra containing a. Then, $\sigma_B(a) = \sigma_A(a)$.*

Proof. It is clear that $\sigma_A(a) \subseteq \sigma_B(a)$. It remains to show that if $b := \lambda e - a$ for any $\lambda \in \mathbb{C}$ has an inverse in A then this inverse is also contained in B.

Assume first that a (and hence b) is normal. We show that b^{-1} is even contained in the unital C*-subalgebra C of B that is generated by b. Suppose that b^{-1} is not contained in C and hence $0 \in \sigma_C(b)$. Choose $m > \|b^{-1}\|$ and define a continuous function $f : \sigma_C(b) \to \mathbb{C}$ such that $f(0) = m$ and $|f(x)x| \le 1$ for all $x \in \sigma_C(b)$. Using Theorem 7.6 and Proposition 7.8 there is a unique element $c \in C$ such that $\hat{c} = f \circ \tilde{b}$. Observe also that $\hat{b} = i \circ \tilde{b}$, where $i : \sigma_C(b) \to \mathbb{C}$ is the inclusion map $x \mapsto x$ and hence $\hat{c}\hat{b} = (f \cdot i) \circ \tilde{b}$. Using Theorem 7.6 we find

$$m \le \|f\| = \|c\| = \|cbb^{-1}\| \le \|cb\|\|b^{-1}\| = \|f \cdot i\|\|b^{-1}\| \le \|b^{-1}\|.$$

This contradicts $m > \|b^{-1}\|$. So $0 \notin \sigma_C(b)$ and $b^{-1} \in C$ as was to be demonstrated. This concludes the proof for the case that a is normal.

Consider now the general case. If b is not invertible in B then by Lemma 5.7 at least one of the two elements $b^\star b$ or $bb^\star$ is not invertible in B. Suppose $b^\star b$ is not invertible in B (the other case proceeds analogously). $b^\star b$ is self-adjoint and in particular normal so the version of the proposition already proofed applies and $\sigma_A(b^\star b) = \sigma_B(b^\star b)$. In particular, $b^\star b$ is not invertible in A and hence b cannot be invertible in A. This completes the proof. □

7.2 Spectral decomposition of normal operators

Proposition 7.10 (Spectral Theorem for Normal Elements)**.** *Let A be a unital C*-algebra and $a \in A$ normal. Then, there exists an isometric homomorphism of unital *-algebras $\phi : \mathrm{C}(\sigma_A(a), \mathbb{C}) \to A$ such that $\phi(\mathbf{1}) = a$.*

Proof. **Exercise.**Hint: Combine Proposition 7.8 with Theorem 7.6. □

Of course, an important application of this is the case when A is the algebra of continuous operators on some Hilbert space and a is a normal operator.

In the context of this proposition we also use the notation $f(a) := \phi(f)$ for $f \in \mathrm{C}(\sigma_A(a), \mathbb{C})$. We use the same notation if f is defined on a larger subset of the complex plane.

Corollary 7.11 (Continuous Spectral Mapping Theorem)**.** *Let A be a unital C*-algebra, $a \in A$ normal and $f : T \to \mathbb{C}$ continuous such that $\sigma_A(a) \subseteq T$. Then, $\sigma_A(f(a)) = f(\sigma_A(a))$.*

Proof. **<u>Exercise</u>.** □

Corollary 7.12. *Let A be a unital C*-algebra and $a \in A$ normal. Furthermore, let $f : \sigma_A(a) \to \mathbb{C}$ and $g : f(\sigma_A(a)) \to \mathbb{C}$ continuous. Then $(g \circ f)(a) = g(f(a))$.*

Proof. **<u>Exercise</u>.** □

Definition 7.13. Let A be a unital C*-algebra. If $u \in A$ is invertible and satisfies $u^\star = u^{-1}$ we call u *unitary*. If $p \in A$ is self-adjoint and satisfies $p^2 = p$ we call it an *orthogonal projector*. (**<u>Exercise</u>.**Justify this terminology!)

<u>Exercise</u>. Let A be a unital C*-algebra.

1. Let $u \in A$ be unitary. What can you say about $\sigma_A(u)$?
2. Let $p \in A$ be an orthogonal projector. Show that $\sigma_A(p) \subseteq \{0, 1\}$.
3. Let $a \in A$ be normal and $\sigma_A(a) \subset \mathbb{R}$. Show that a is self-adjoint.

Proposition 7.14. *Let A be a unital C*-algebra and $a \in A$ normal. Suppose the spectrum of a is the disjoint union of two non-empty subsets $\sigma_A(a) = s_1 \cup s_2$. Then, there exist $a_1, a_2 \in A$ normal, such that $\sigma_A(a_1) = s_1$ and $\sigma_A(a_2) = s_2$ and $a = a_1 + a_2$. Moreover, $a_1 a_2 = a_2 a_1 = 0$ and a commutes both with a_1 and a_2.*

Proof. **<u>Exercise</u>.** □

Proposition 7.15. *Let H be a Hilbert space, $A := \mathrm{CL}(H, H)$ and $k \in \mathrm{KL}(H, H)$ normal. Then, there exists an orthogonal projector $p_\lambda \in A$ for each $\lambda \in \sigma_A(k)$ such that $p_\lambda p_{\lambda'} = 0$ if $\lambda \neq \lambda'$ and*

$$k = \sum_{\lambda \in \sigma_A(k)} \lambda p_\lambda \quad \textit{and} \quad e = \sum_{\lambda \in \sigma_A(k)} p_\lambda .$$

Proof. **<u>Exercise</u>.** (Explain also in which sense the sums converge!) □

7.3 Positive elements and states

We now move towards a characterization of noncommutative C*-algebras. We are going to show that any unital C*-algebra is isomorphic to a C*-subalgebra of the algebra of continuous operators on some Hilbert space.

Definition 7.16. Let A be a unital C*-algebra. A self-adjoint element $a \in A$ is called *positive* iff $\sigma_A(a) \subset [0, \infty)$.

Exercise 47. Let T be a compact Hausdorff space and consider the C*-algebra $\mathrm{C}(T,\mathbb{C})$. Show that the self-adjoint elements are precisely the real valued functions and the positive elements are the functions with non-negative values.

Proposition 7.17. *Let A be a unital C*-algebra and $a, b \in A$ positive. Then, $a + b$ is positive.*

Proof. Suppose $\lambda \in \sigma_A(a + b)$. Since a and b are self-adjoint so is $a + b$. In particular, $\sigma_A(a + b) \subset \mathbb{R}$ and λ is real. Set $\alpha := \|a\|$ and $\beta := \|b\|$. Then, $(\alpha+\beta)-\lambda \in \sigma_A((\alpha+\beta)e-(a+b))$ and thus $|(\alpha+\beta)-\lambda| \le r_A((\alpha+\beta)e-(a+b))$ by Theorem 5.13. But the element $(\alpha + \beta)e - (a + b)$ is normal (and even self-adjoint), so Proposition 7.3 applies and we have $r_A((\alpha+\beta)e-(a+b)) = \|(\alpha + \beta)e - (a + b)\| \le \|\alpha e - a\| + \|\beta e - b\|$. Again using Proposition 7.3 we find $\|\alpha e - a\| = r_A(\alpha e - a)$ and $\|\beta e - b\| = r_A(\beta e - b)$. But $\sigma_A(a) \subseteq [0, \alpha]$ by positivity and Proposition 5.6. Thus, $\sigma_A(\alpha e - a) \subseteq [0, \alpha]$. Hence, by Theorem 5.13, $r_A(\alpha e - a) \le \alpha$. In the same way we find $r_A(\beta e - b) \le \beta$. We have thus demonstrated the inequality $|(\alpha + \beta) - \lambda| \le \alpha + \beta$. This implies $\lambda \ge 0$, completing the proof. □

Proposition 7.18. *Let A be a unital C*-algebra and $a \in A$ self-adjoint. Then, there exist positive elements $a_+, a_- \in A$ such that $a = a_+ - a_-$ and $a_+a_- = a_-a_+ = 0$.*

Proof. **Exercise.** Hint: Consider the unital C*-subalgebra generated by a. □

Proposition 7.19. *Let A be a unital C*-algebra and $a \in A$. Then, a is positive iff there exists $b \in A$ such that $a = b^\star b$.*

Proof. **Exercise.** □

Lemma 7.20. *Let A be a unital C*-algebra and $a \in A$ positive and such that $\|a\| \le 1$. Then, $e - a$ is positive and $\|e - a\| \le 1$.*

Proof. **Exercise.** □

A similar role to that played by the characters in the theory of commutative C*-algebras is now played by *states*.

Definition 7.21. Let A be a unital C*-algebra. A continuous linear functional $\omega : A \to \mathbb{C}$ is called *positive* iff $\omega(a) \ge 0$ for all positive elements $a \in A$. A positive functional $\omega : A \to \mathbb{C}$ is called a *state* iff it is normalized, i.e., iff $\|\omega\| = 1$. The set Σ_A of states of A is called the *state space* of A.

Exercise 48. Let A be a unital C*-algebra. Show that $\Gamma_A \subseteq \Sigma_A$, i.e., each character is in particular a state.

Proposition 7.22. *Let A be a unital C*-algebra and ω a positive functional on A. Then $\omega(a^\star) = \overline{\omega(a)}$ for all a in A. In particular, $\omega(a) \in \mathbb{R}$ if a is self-adjoint.*

Proof. **Exercise.** □

Proposition 7.23. *Let A be a unital C*-algebra and ω a positive functional on A. Consider the map $[\cdot,\cdot]_\omega : A \times A \to \mathbb{C}$ given by $[a,b]_\omega = \omega(b^\star a)$. It has the following properties:*

1. *$[\cdot,\cdot]_\omega$ is a sesquilinear form on A.*
2. *$[a,b]_\omega = \overline{[b,a]_\omega}$ for all $a, b \in A$.*
3. *$[a,a]_\omega \geq 0$ for all $a \in A$.*

Proof. **Exercise.** □

This shows that we almost have a scalar product, only the definiteness condition is missing. Nevertheless we have the Cauchy-Schwarz inequality.

Proposition 7.24. *Let A be a unital C*-algebra and ω a non-zero positive functional on A. The following is true:*

1. *$|[a,b]_\omega|^2 \leq [a,a]_\omega [b,b]_\omega$ for all $a, b \in A$.*
2. *Let $a \in A$. Then, $[a,a]_\omega = 0$ iff $[a,b]_\omega = 0$ for all $b \in A$.*
3. *$[ab,ab]_\omega \leq \|a\|^2 [b,b]_\omega$ for all $a, b \in A$.*

Proof. **Exercise.** □

Proposition 7.25. *Let A be a unital C*-algebra and $\omega : A \to \mathbb{C}$ continuous and linear. Then, ω is a positive functional iff $\|\omega\| = \omega(e)$.*

Proof. Suppose that ω is a positive functional. Given $\epsilon > 0$ there exists $a \in A$ with $\|a\| = 1$ such that $\|\omega(a)\|^2 \geq \|\omega\|^2 - \epsilon$. Using the Cauchy-Schwarz inequality (Proposition 7.24.1) with $b = e$ we find

$$\|\omega(a)\|^2 \leq \omega(a^\star a)\omega(e) \leq \|\omega\| \|a^\star a\| \omega(e) = \|\omega\| \omega(e).$$

Combining this with the first inequality we get $\|\omega\|^2 - \epsilon \leq \|\omega\| \omega(e)$. Since ϵ was arbitrary this implies $\|\omega\| \leq \omega(e)$. On the other hand, the inequality $\omega(e) \leq \|\omega\|$ is clear.

Conversely, suppose now that ω is a continuous linear functional with the property $\|\omega\| = \omega(e)$. Without loss of generality we normalize ω such that $\omega(e) = 1 = \|\omega\|$. We first show that $\omega(a) \in \mathbb{R}$ if $a \in A$ is self-adjoint. Assume the contrary, i.e., assume there exists $a \in A$ such that $\omega(a) = x + \mathrm{i}y$ with $x, y \in \mathbb{R}$ and $y \neq 0$. Set $b := a - xe$. Then, b is self-adjoint and $\omega(b) = \mathrm{i}y$. For $\lambda \in \mathbb{R}$ we get,

$$|\omega(b + \mathrm{i}\lambda e)|^2 = |\mathrm{i}y + \mathrm{i}\lambda\omega(e)|^2 = y^2 + 2\lambda y + \lambda^2.$$

One the other hand,

$$|\omega(b + \mathrm{i}\lambda e)|^2 \leq \|\omega\|^2 \|b + \mathrm{i}\lambda e\|^2 = \|(b + \mathrm{i}\lambda e)^\star (b + \mathrm{i}\lambda e)\| \leq \|b\|^2 + \lambda^2.$$

The resulting inequality is equivalent to,

$$y^2 + 2\lambda y \leq \|b\|^2,$$

which obviously cannot be fulfilled for arbitrary $\lambda \in \mathbb{R}$ (recall that $y \neq 0$), giving a contradiction. This shows that $\omega(a) \in \mathbb{R}$ if $a \in A$ is self-adjoint.

We proceed to show that $\omega(a) \geq 0$ if $a \in A$ is positive. Assume the contrary, i.e., assume there is $a \in A$ positive such that $\omega(a) < 0$. (Note that $\omega(a) \in \mathbb{R}$ by the previous part of the proof.) By suitable normalization we can achieve $\|a\| \leq 1$ as well. By Lemma 7.20 we have $\|e - a\| \leq 1$ and thus $|\omega(e - a)| \leq 1$ since $\|\omega\| = 1$. On the other hand, $|\omega(e - a)| = |1 - \omega(a)| > 1$, a contradiction. This shows that ω must be positive. □

Proposition 7.26. *Let A be a unital $C^\star$-algebra and $a \in A$ positive. Then, there exists a state $\omega \in \Sigma_A$ such that $\omega(a) = \|a\|$.*

Proof. Since a is positive we have $\sigma_A(a) \subseteq [0, \infty)$. Moreover, a is normal, so by Proposition 7.3 we have $r_A(a) = \|a\|$. Thus, $\|a\| \in \sigma_A(a)$. Let B be the unital C*-subalgebra of A generated by a. By Proposition 7.9 we have $\sigma_B(a) = \sigma_A(a)$ and in particular $\|a\| \in \sigma_B(a)$. By Proposition 7.8, $\hat{a}$ induces a homeomorphism $\Gamma_B \to \sigma_B(a)$. In particular, there exists a character $\phi \in \Gamma_B$ such that $\|a\| = \hat{a}(\phi) = \phi(a)$. Recall that $\phi(e) = 1$ and $\|\phi\| = 1$ by Proposition 5.22. By the Hahn-Banach Theorem (Corollary 3.31) there exists an extension of ϕ to a linear functional $\omega : A \to \mathbb{C}$ such that $\omega|_B = \phi$ and $\|\omega\| = 1$. Note in particular that $\omega(e) = 1 = \|\omega\|$. So by Proposition 7.25, $\omega \in \Sigma_A$. □

7.4 The GNS construction

Proposition 7.27. *Let A be a unital C*-algebra and ω a state on A. Define $I_\omega := \{a \in A : [a,a]_\omega = 0\} \subseteq A$. Then, I_ω is a left ideal of the algebra A. In particular, the quotient vector space A/I_ω is an inner product space with the induced sesquilinear form.*

Proof. **Exercise.** □

Definition 7.28. Let A be a unital C*-algebra and ω a state on A. We call the completion of the inner product space A/I_ω the *Hilbert space associated with the state* ω and denote it by H_ω. We denote its scalar product by $\langle\cdot,\cdot\rangle_\omega : H_\omega \times H_\omega \to \mathbb{C}$.

A consequence of the fact that A/I_ω is a left ideal is that we have a representation of A on this space and its completion from the left.

Definition 7.29. Let A be a unital C*-algebra and H a Hilbert space. A homomorphism of unital *-algebras $A \to \mathrm{CL}(H,H)$ is called a *representation of A*. A representation that is injective is called *faithful*. A representation that is surjective is called *full*.

Proposition 7.30. *Let A, B be unital C*-algebras and $\phi : A \to B$ a homomorphism of unital *-algebras.*

1. *$\|\phi(a)\| \le \|a\|$ for all $a \in A$. In particular, ϕ is continuous.*
2. *If ϕ is injective then it is isometric.*

Proof. **Exercise.** □

Theorem 7.31. *Let A be a unital C*-algebra and ω a state on A. Then, there is a natural representation $\pi_\omega : A \to \mathrm{CL}(H_\omega, H_\omega)$. Moreover,*

$$\|\pi_\omega(a)\|^2 \ge \omega(a^\star a) \quad \forall a \in A,$$

and $\|\pi_\omega\| = 1$.

Proof. Define the linear maps $\tilde{\pi}_\omega(a) : A/I_\omega \to A/I_\omega$ by left multiplication, i.e., $\tilde{\pi}_\omega(a) : [b] \mapsto [ab]$. That $\tilde{\pi}_\omega(a)$ is well defined follows from Proposition 7.27 (I_ω is a left ideal). By definition we have then $\tilde{\pi}_\omega(ab) = \tilde{\pi}_\omega(a) \circ \tilde{\pi}_\omega(b)$ and $\tilde{\pi}_\omega(e) = \mathbf{1}_{A/I_\omega}$. Furthermore, $\|\tilde{\pi}_\omega(a)\| \le \|a\|$ due to

Proposition 7.24.3 and hence $\tilde{\pi}_\omega(a)$ is continuous. So we have a homomorphism of unital algebras $\tilde{\pi}_\omega : A \to \mathrm{CL}(A/I_\omega, A/I_\omega)$. Also, $\tilde{\pi}_\omega$ preserves the *-structure because,

$$\langle \tilde{\pi}_\omega(a^\star)[b], [c]\rangle_\omega = [a^\star b, c]_\omega = \omega(c^\star a^\star b) = [b, ac]_\omega = \langle [b], \tilde{\pi}_\omega(a)[c]\rangle_\omega.$$

Since $\tilde{\pi}_\omega(a)$ is continuous it extends to a continuous operator $\pi_\omega(a) : H_\omega \to H_\omega$ on the completion H_ω of A/I_ω, with the same properties. In particular, π_ω is a homomorphism of unital *-algebras.

Due to the bound $\|\tilde{\pi}_\omega(a)\| \leq \|a\|$ and hence $\|\pi_\omega(a)\| \leq \|a\|$ (or due to Proposition 7.30.1) we find $\|\pi_\omega\| \leq 1$. Observe also that $\omega(e) = 1$ By Proposition 7.25 and hence $\|\pi_\omega(a)\|^2 \geq [ae, ae]_\omega / [e, e]_\omega = \omega(a^\star a)$. In particular, $\|\pi_\omega\| \geq \|\pi_\omega(e)\| \geq 1$. Thus, $\|\pi_\omega\| = 1$. □

The construction leading to the Hilbert spaces H_ω and this representation is called the *GNS-construction* (Gelfand-Naimark-Segal).

Definition 7.32. Let A be a unital C*-algebra, H a Hilbert space and $\phi : A \to \mathrm{CL}(H, H)$ a representation. A vector $\psi \in H$ is called a *cyclic vector* iff $\{\phi(a)\psi : a \in A\}$ is dense in H. The representation is then called a *cyclic representation*.

Proposition 7.33. *Let A be a unital C*-algebra and ω a state on A. Then, there is a cyclic vector $\psi \in H_\omega$ with the property $\omega(a) = \langle \pi_\omega(a)\psi, \psi\rangle_\omega$ for all $a \in A$.*

Proof. **Exercise.** □

A deficiency of the representation of Theorem 7.31 is that it is neither faithful nor full in general. Lack of faithfulness can be remedied. The idea is that we take the direct sum of the representations π_ω for all normalized states ω.

Proposition 7.34. *Let $\{H_\alpha\}_{\alpha \in I}$ be a family of Hilbert spaces. Consider collections ψ of elements $\psi_\alpha \in H_\alpha$ with $\alpha \in I$ such that $\sup_{J \subseteq I} \sum_{\alpha \in J} \|\psi_\alpha\|^2 < \infty$ where J ranges over all finite subsets of I. Then, the set H of such collections ψ is naturally a Hilbert space and we have isometric embeddings $H_\alpha \to H$ for all $\alpha \in I$.*

Proof. **Exercise.** □

Definition 7.35. The Hilbert space H constructed in the preceding Proposition is called the *direct sum* of the Hilbert spaces H_α and is denoted $\bigoplus_{\alpha \in I} H_\alpha$.

Proposition 7.36. *Let A be a unital C*-algebra, $\{H_\alpha\}_{\alpha\in I}$ a family of Hilbert spaces and $\phi_\alpha : A \to \mathrm{CL}(H_\alpha, H_\alpha)$ a representation for each $\alpha \in I$. Then, there exists a representation $\phi : A \to \mathrm{CL}(H, H)$ such that $\|\phi(a)\| = \sup_{\alpha\in I} \|\phi_\alpha(a)\|$ for all $a \in A$, where $H := \bigoplus_{\alpha\in I} H_\alpha$.*

Proof. **Exercise.** □

We are now ready to put everything together.

Theorem 7.37 (Gelfand-Naimark)**.** *Let A be a unital C*-algebra. Then, there exists a Hilbert space H and a faithful representation $\pi : A \to \mathrm{CL}(H, H)$.*

Proof. **Exercise.** □

This result concludes our characterization of the structure of C*-algebras: Each C*-algebra arises as a C*-subalgebra of the algebra of continuous operators on some Hilbert space.

Exercise 49. Let A be a unital C*-algebra, H_1, H_2 Hilbert spaces, $\phi_1 : A \to \mathrm{CL}(H_1, H_1)$ and $\phi_2 : A \to \mathrm{CL}(H_2, H_2)$ cyclic representations. Suppose that $\langle \phi_1(a)\psi_1, \psi_1\rangle_1 = \langle \phi_2(a)\psi_2, \psi_2\rangle_2$ for all $a \in A$, where ψ_1, ψ_2 are the cyclic vectors in H_1 and H_2 respectively. Show that there exists a unitary operator (i.e., an invertible linear isometry) $W : H_1 \to H_2$ such that $\phi(a) = W^\star\psi(a)W$ for all $a \in A$.