REAL ANALYSIS – Semester 2018-1

Contents

1 Topological and metric spaces

1.1 Basic Definitions

Definition 1.1 (Topology)**.** Let S be a set. A subset $\mathcal{T}$ of the set $\mathfrak{P}(S)$ of subsets of S is called a *topology* iff it has the following properties:

- $\emptyset \in \mathcal{T}$ and $S \in \mathcal{T}$.
- Let $\{U_i\}_{i\in I}$ be a family of elements in $\mathcal{T}$. Then $\bigcup_{i\in I} U_i \in \mathcal{T}$.
- Let $U, V \in \mathcal{T}$. Then $U \cap V \in \mathcal{T}$.

A set equipped with a topology is called a *topological space*. The elements of $\mathcal{T}$ are called the *open* sets in S. A complement of an open set in S is called a *closed* set.

Definition 1.2. Let S be a topological space and $x \in S$. Then a subset $U \subseteq S$ is called a *neighborhood* of x iff it contains an open set which in turn contains x. We denote the set of neighborhoods of x by $\mathcal{N}_x$.

Definition 1.3. Let S be a topological space and U a subset. The *closure* $\overline{U}$ of U is the smallest closed set containing U. The *interior* $\mathring{U}$ of U is the largest open set contained in U. U is called *dense* in S iff $\overline{U} = S$.

Definition 1.4 (base)**.** Let $\mathcal{T}$ be a topology. A subset $\mathcal{B}$ of $\mathcal{T}$ is called a *base* of $\mathcal{T}$ iff the elements of $\mathcal{T}$ are precisely the unions of elements of $\mathcal{B}$. It is called a *subbase* iff the elements of $\mathcal{T}$ are precisely the finite intersections of unions of elements of $\mathcal{B}$.

Proposition 1.5. *Let S be a set and $\mathcal{B}$ a subset of $\mathfrak{P}(S)$. $\mathcal{B}$ is the base of a topology on S iff it satisfies all of the following properties:*

- *$\emptyset \in \mathcal{B}$.*
- *For every $x \in S$ there is a set $U \in \mathcal{B}$ such that $x \in U$.*
- *Let $U, V \in \mathcal{B}$. Then there exits a family $\{W_\alpha\}_{\alpha\in A}$ of elements of $\mathcal{B}$ such that $U \cap V = \bigcup_{\alpha\in A} W_\alpha$.*

Proof. **Exercise.** □

Definition 1.6. Let S be a topological space and p a point in S. We call a family $\{U_\alpha\}_{\alpha\in A}$ of open neighborhoods of p a *neighborhood base* at p iff for any neighborhood V of p there exists $\alpha \in A$ such that $U_\alpha \subseteq V$.

Definition 1.7 (Continuity)**.** Let S, T be topological spaces. A map $f : S \to T$ is called *continuous at* $p \in S$ iff $f^{-1}(\mathcal{N}_{f(p)}) \subseteq \mathcal{N}_p$. f is called *continuous* iff it is continuous at every $p \in S$. We denote the space of continuous maps from S to T by $\mathrm{C}(S, T)$.

Proposition 1.8. *Let S, T be topological spaces and $f : S \to T$ a map. Then, f is continuous iff for every open set $U \in T$ the preimage $f^{-1}(U)$ in S is open.*

Proof. **Exercise.** □

Proposition 1.9. *Let S, T, U be topological spaces, $f \in \mathrm{C}(S, T)$ and $g \in \mathrm{C}(T, U)$. Then, the composition $g \circ f : S \to U$ is continuous.*

Proof. Immediate. □

Definition 1.10. Let S, T be topological spaces. A bijection $f : S \to T$ is called a *homeomorphism* iff f and f^{-1} are both continuous. If such a homeomorphism exists S and T are called *homeomorphic*.

Definition 1.11. Let $\mathcal{T}_1$, $\mathcal{T}_2$ be topologies on the set S. Then, $\mathcal{T}_1$ is called *finer* than $\mathcal{T}_2$ and $\mathcal{T}_2$ is called *coarser* than $\mathcal{T}_1$ iff all open sets of $\mathcal{T}_2$ are also open sets of $\mathcal{T}_1$.

Definition 1.12 (Induced Topology)**.** Let S be a topological space and U a subset. Consider the topology given on U by the intersection of each open set on S with U. This is called the *induced topology* on U.

Definition 1.13 (Product Topology)**.** Let S be the cartesian product $S = \prod_{\alpha \in I} S_\alpha$ of a family of topological spaces. Consider subsets of S of the form $\prod_{\alpha \in I} U_\alpha$ where finitely many U_α are open sets in S_α and the others coincide with the whole space $U_\alpha = S_\alpha$. These subsets form the base of a topology on S which is called the *product topology*.

Exercise 1. Show that alternatively, the product topology can be characterized as the coarsest topology on $S = \prod_{\alpha \in I} S_\alpha$ such that all projections $S \twoheadrightarrow S_\alpha$ are continuous.

Proposition 1.14. *Let S, T, X be topological spaces and $f \in \mathrm{C}(S \times T, X)$, where $S \times T$ carries the product topology. Then the map $f_x : T \to X$ defined by $f_x(y) = f(x, y)$ is continuous for every $x \in S$.*

Proof. Fix $x \in S$. Let U be an open set in X. We want to show that $W := f_x^{-1}(U)$ is open. We do this by finding for any $y \in W$ an open neighborhood of y contained in W. If W is empty we are done, hence assume that this is not so. Pick $y \in W$. Then $(x, y) \in f^{-1}(U)$ with $f^{-1}(U)$ open by continuity of f. Since $S \times T$ carries the product topology there must be open sets $V_x \subseteq S$ and $V_y \subseteq T$ with $x \in V_x$, $y \in V_y$ and $V_x \times V_y \subseteq f^{-1}(U)$. But clearly $V_y \subseteq W$ and we are done. □

Definition 1.15 (Quotient Topology)**.** Let S be a topological space and $\sim$ an equivalence relation on S. Then, the *quotient topology* on $S/\sim$ is the finest topology such that the quotient map $S \twoheadrightarrow S/\sim$ is continuous.

1.2 Some properties of topological spaces

In a topological space it is useful if two distinct points can be distinguished by the topology. A strong form of this distinguishability is the *Hausdorff property*.

Definition 1.16 (Hausdorff)**.** Let S be a topological space. Assume that given any two distinct points $x, y \in S$ we can find open sets $U, V \subset S$ such that $x \in U$ and $y \in V$ and $U \cap V = \emptyset$. Then, S is said to have the *Hausdorff property*. We also say that S is a *Hausdorff space*.

Definition 1.17. Let S be a topological space. S is called *first-countable* iff there exists a countable neighborhood base at each point of S. S is called *second-countable* iff the topology of S admits a countable base.

Definition 1.18 (open cover)**.** Let S be a topological space and $U \subseteq S$ a subset. A family of open sets $\{U_\alpha\}_{\alpha \in A}$ is called an *open cover* of U iff $U \subseteq \bigcup_{\alpha \in A} U_\alpha$.

Proposition 1.19. *Let S be a second-countable topological space and $U \subseteq S$ a subset. Then, every open cover of U contains a countable subcover.*

Proof. **Exercise.** □

Definition 1.20 (compact)**.** Let S be a topological space and $U \subseteq S$ a subset. U is called *compact* iff every open cover of U contains a finite subcover.

Proposition 1.21. *A closed subset of a compact space is compact. A compact subset of a Hausdorff space is closed.*

Proof. **Exercise.** □

Proposition 1.22. *The image of a compact set under a continuous map is compact.*

Proof. **Exercise.** □

Definition 1.23. Let S be a topological space. S is called *locally compact* iff every point of S possesses a compact neighborhood.

Exercise 2 (One-point compactification)**.** Let S be a locally compact Hausdorff space. Let $\tilde{S} := S \cup \{\infty\}$ to be the set S with an extra element ∞ adjoint. Define a subset U of $\tilde{S}$ to be open iff either U is an open subset of S or U is the complement of a compact subset of S. Show that this makes $\tilde{S}$ into a compact Hausdorff space.

1.3 Sequences and convergence

Definition 1.24 (Convergence of sequences)**.** Let $x := \{x_n\}_{n\in\mathbb{N}}$ be a sequence of points in a topological space S. We say that x has an *accumulation point (or limit point)* p iff for every neighborhood U of p we have $x_k \in U$ for infinitely many $k \in \mathbb{N}$. We say that x *converges* to a point p iff for any neighborhood U of p there is a number $n \in \mathbb{N}$ such that for all $k \geq n$: $x_k \in U$.

Proposition 1.25. *Let S, T be topological spaces and $f : S \to T$. If f is continuous, then for any $p \in S$ and sequence $\{x_n\}_{n\in\mathbb{N}}$ converging to p, the sequence $\{f(x_n)\}_{n\in\mathbb{N}}$ in T converges to $f(p)$. Conversely, if S is first countable and for any $p \in S$ and sequence $\{x_n\}_{n\in\mathbb{N}}$ converging to p, the sequence $\{f(x_n)\}_{n\in\mathbb{N}}$ in T converges to $f(p)$, then f is continuous.*

Proof. **Exercise.** □

Proposition 1.26. *Let S be Hausdorff space and $\{x_n\}_{n\in\mathbb{N}}$ a sequence in S which converges to a point $p \in S$. Then, $\{x_n\}_{n\in\mathbb{N}}$ does not converge to any other point in S.*

Proof. **Exercise.** □

Definition 1.27. Let S be a topological space and $U \subseteq S$ a subset. Consider the set B_U of sequences of elements of U. Then the set $\overline{U}^s$ consisting of the points to which some element of B_U converges is called the *sequential closure* of U.

Proposition 1.28. *Let S be a topological space and $U \subseteq S$ a subset. Let x be a sequence of points in U which has an accumulation point $p \in S$. Then, $p \in \overline{U}$.*

Proof. Suppose $p \notin \overline{U}$. Since $\overline{U}$ is closed $S \setminus \overline{U}$ is an open neighborhood of p. But $S \setminus \overline{U}$ does not contain any point of x, so p cannot be accumulation point of x. This is a contradiction. □

Corollary 1.29. *Let S be a topological space and U a subset. Then, $U \subseteq \overline{U}^s \subseteq \overline{U}$.*

Proof. Immediate. □

Proposition 1.30. *Let S be a first-countable topological space and U a subset. Then, $\overline{U}^s = \overline{U}$.*

Proof. **Exercise.** □

Definition 1.31. Let S be a topological space and $U \subseteq S$ a subset. U is said to be *limit point compact* iff every sequence in U has an accumulation point (limit point) in U. U is called *sequentially compact* iff every sequence of elements of U contains a subsequence converging to a point in U.

Proposition 1.32. *Let S be a first-countable topological space and $x = \{x_n\}_{n\in\mathbb{N}}$ a sequence in S with accumulation point p. Then, x has a subsequence that converges to p.*

Proof. By first-countability choose a countable neighborhood base $\{U_n\}_{n\in\mathbb{N}}$ at p. Now consider the family $\{W_n\}_{n\in\mathbb{N}}$ of open neighborhoods $W_n := \bigcap_{k=1}^n U_k$ at p. It is easy to see that this is again a countable neighborhood base at p. Moreover, it has the property that $W_n \subseteq W_m$ if $n \geq m$. Now, Choose $n_1 \in \mathbb{N}$ such that $x_{n_1} \in W_1$. Recursively, choose $n_{k+1} > n_k$ such that $x_{n_{k+1}} \in W_{k+1}$. This is possible since W_{k+1} contains infinitely many points of x. Let V be a neighborhood of p. There exists some $k \in \mathbb{N}$ such that $U_k \subseteq V$. By construction, then $W_m \subseteq W_k \subseteq U_k$ for all $m \geq k$ and hence $x_{n_m} \in V$ for all $m \geq k$. Thus, the subsequence $\{x_{n_m}\}_{m\in\mathbb{N}}$ converges to p. □

Proposition 1.33. *Sequential compactness implies limit point compactness. In a first-countable space the converse is also true.*

Proof. **Exercise.** □

Proposition 1.34. *A compact set is limit point compact.*

Proof. Consider a sequence x in a compact set S. Suppose x does not have an accumulation point. Then, for each point $p \in S$ we can choose an open neighborhood U_p which contains only finitely many points of x. However, by compactness, S is covered by finitely many of the sets U_p. But their union can only contain a finite number of points of x, a contradiction. □

1.4 Metric and pseudometric spaces

Definition 1.35. Let S be a set and $d : S \times S \to \mathbb{R}^+_0$ a map with the following properties:

- $d(x,y) = d(y,x) \quad \forall x,y \in S$. (symmetry)
- $d(x,z) \leq d(x,y) + d(y,z) \quad \forall x,y,z \in S$. (triangle inequality)
- $d(x,x) = 0 \quad \forall x \in S$.

Then d is called a *pseudometric* on S. S is also called a *pseudometric space.* Suppose d also satisfies

- $d(x,y) = 0 \implies x = y \quad \forall x,y \in S$. (definiteness)

Then d is called a *metric* on S and S is called a *metric space.*

Definition 1.36. Let S be a pseudometric space, $x \in S$ and $r > 0$. Then the set $B_r(x) := \{y \in S : d(x,y) < r\}$ is called the *open ball* of radius r centered around x in S. The set $\overline{B}_r(x) := \{y \in S : d(x,y) \leq r\}$ is called the *closed ball* of radius r centered around x in S.

Proposition 1.37. *Let S be a pseudometric space. Then, the open balls in S together with the empty set form the basis of a topology on S. This topology is first-countable and such that closed balls are closed. Moreover, the topology is Hausdorff iff S is metric.*

Proof. **Exercise.** □

Definition 1.38. A topological space is called *(pseudo)metrizable* iff there exists a (pseudo)metric such that the open balls given by the (pseudo)metric are a basis of its topology.

Proposition 1.39. *In a pseudometric space any open ball can be obtained as the countable union of closed balls. Similarly, any closed ball can be obtained as the countable intersection of open balls.*

Proof. **Exercise.** □

Proposition 1.40. *Let S be a set equipped with two pseudometrics d^1 and d^2. Then, the topology generated by d^2 is finer than the topology generated by d^1 iff for all $x \in S$ and $r_1 > 0$ there exists $r_2 > 0$ such that $B^2_{r_2}(x) \subseteq B^1_{r_1}(x)$. In particular, d^1 and d^2 generate the same topology iff the condition holds both ways.*

Proof. **Exercise.** □

Proposition 1.41 (epsilon-delta criterion)**.** *Let S, T be pseudometric spaces and $f : S \to T$ a map. Then, f is continuous at $x \in S$ iff for every $\epsilon > 0$ there exists $\delta > 0$ such that $f(B_\delta(x)) \subseteq B_\epsilon(f(x))$.*

Proof. **Exercise.** □

1.5 Elementary properties of pseudometric spaces

Proposition 1.42. *Let S be a pseudometric space and $x := \{x_n\}_{n\in\mathbb{N}}$ a sequence in S. Then x converges to $p \in S$ iff for any $\epsilon > 0$ there exists an $n_0 \in \mathbb{N}$ such that $d(x_n, p) < \epsilon$ for all $n \geq n_0$.*

Proof. Immediate. □

Definition 1.43. Let S be a pseudometric space and $x := \{x_n\}_{n\in\mathbb{N}}$ a sequence in S. Then x is called a *Cauchy sequence* iff for all $\epsilon > 0$ there exists an $n_0 \in \mathbb{N}$ such that $d(x_n, x_m) < \epsilon$ for all $n, m \geq n_0$.

Exercise 3. Give an example of a set S, a sequence x in S and two metrics d^1 and d^2 on S that generate the same topology, but such that x is Cauchy with respect to d^1, but not with respect to d^2.

Proposition 1.44. *Any converging sequence in a pseudometric space is a Cauchy sequence.*

Proof. **Exercise.** □

Proposition 1.45. *Suppose x is a Cauchy sequence in a pseudometric space. If p is accumulation point of x then x converges to p.*

Proof. **Exercise.** □

Definition 1.46. Let S be a pseudometric space and $U \subseteq S$ a subset. If every Cauchy sequence in U converges to a point in U, then U is called *complete*.

Proposition 1.47. *A complete subset of a metric space is closed. A closed subset of a complete pseudometric space is complete.*

Proof. **Exercise.** □

Exercise 4. Give an example of a complete subset of a pseudometric space that is not closed.

Definition 1.48 (Totally boundedness)**.** Let S be a pseudometric space. A subset $U \subseteq S$ is called *totally bounded* iff for any $r > 0$ the set U admits a cover by finitely many open balls of radius r.

Proposition 1.49. *A subset of a pseudometric space is compact iff it is complete and totally bounded.*

Proof. We first show that compactness implies totally boundedness and completeness. Let U be a compact subset. Then, for $r > 0$ cover U by open balls of radius r centered at every point of U. Since U is compact, finitely many balls will cover it. Hence, U is totally bounded. Now, consider a Cauchy sequence x in U. Since U is compact x must have an accumulation point $p \in U$ (Proposition 1.34) and hence (Proposition 1.45) converge to p. Thus, U is complete.

We proceed to show that completeness together with totally boundedness imply compactness. Let U be a complete and totally bounded subset. Assume U is not compact and choose a covering $\{U_\alpha\}_{\alpha \in A}$ of U that does not admit a finite subcover. On the other hand, U is totally bounded and admits a covering by finitely many open balls of radius $1/2$. Hence, there must be at least one such ball B_1 such that $C_1 := B_1 \cap U$ is not covered by finitely many U_α. Choose a point x_1 in C_1. Observe that C_1 itself is totally bounded. Inductively, cover C_n by finitely many open balls of radius $2^{-(n+1)}$. For at least one of those, call it B_{n+1}, $C_{n+1} := B_{n+1} \cap C_n$ is not covered by finitely many U_α. Choose a point x_{n+1} in C_{n+1}. This process yields a Cauchy sequence $x := \{x_k\}_{k \in \mathbb{N}}$. Since U is complete the sequence converges to a point $p \in U$. There must be $\alpha \in A$ such that $p \in U_\alpha$. Since U_α is open there exists $r > 0$ such that $B_r(p) \subseteq U_\alpha$. This implies, $C_n \subseteq U_\alpha$ for all $n \in \mathbb{N}$ such that $2^{-n+1} < r$. However, this is a contradiction to the C_n not being finitely covered. Hence, U must be compact. □

Proposition 1.50. *The notions of compactness, limit point compactness and sequential compactness are equivalent in a pseudometric space.*

Proof. **Exercise.** □

Proposition 1.51. *A totally bounded pseudometric space is second-countable.*

Proof. **Exercise.** □

Proposition 1.52. *Let S be equipped with a pseudometric d. Then $p \sim q \iff d(p,q) = 0$ for $p, q \in S$ defines an equivalence relation on S. The prescription $\tilde{d}([p],[q]) := d(p,q)$ for $p, q \in S$ is well defined and yields a metric $\tilde{d}$ on the quotient space $S/\sim$. The topology induced by this metric on $S/\sim$ is the quotient topology with respect to that induced by d on S. Moreover, $S/\sim$ is complete iff S is complete.*

Proof. **Exercise.** □

1.6 Completion of metric spaces

Often it is desirable to work with a complete metric space when one is only given a non-complete metric space. To this end one can construct the *completion* of a metric space. This is detailed in the following exercise.

Exercise 5. Let S be a metric space.

- Let $x := \{x_n\}_{n\in\mathbb{N}}$ and $y := \{y_n\}_{n\in\mathbb{N}}$ be Cauchy sequences in S. Show that the limit $\lim_{n\to\infty} d(x_n, y_n)$ exists.

- Let T be the set of Cauchy sequences in S. Define the function $\tilde{d} : T \times T \to \mathbb{R}_0^+$ by $\tilde{d}(x,y) := \lim_{n\to\infty} d(x_n, y_n)$. Show that $\tilde{d}$ defines a pseudometric on T.

- Show that T is complete.

- Define $\overline{S}$ as the metric quotient $T/\sim$ as in Proposition 1.52. Then, $\overline{S}$ is complete.

- Show that there is a natural isometric embedding (i.e., a map that preserves the metric) $i_S : S \to \overline{S}$. Furthermore, show that this is a bijection iff S is complete.

Definition 1.53. The metric space $\overline{S}$ constructed above is called the *completion* of the metric space S.

Proposition 1.54 (Universal property of completion)**.** *Let S be a metric space, T a complete metric space and $f : S \to T$ an isometric map. Then, there is a unique isometric map $\overline{f} : \overline{S} \to T$ such that $f = \overline{f} \circ i_S$. Furthermore, the closure of $f(S)$ in T is equal to $\overline{f}(\overline{S})$.*

Proof. **Exercise.** □

1.7 Norms and seminorms

In the following $\mathbb{K}$ will denote a field which can be either $\mathbb{R}$ or $\mathbb{C}$.

Definition 1.55. Let V be a vector space over $\mathbb{K}$. Then a map $V \to \mathbb{R}_0^+ : x \mapsto \|x\|$ is called a *seminorm* iff it satisfies the following properties:

1. $\|\lambda x\| = |\lambda| \|x\|$ for all $\lambda \in \mathbb{K}, x \in V$.
2. For all $x, y \in V : \|x + y\| \leq \|x\| + \|y\|$. (triangle inequality)

A seminorm is called a *norm* iff it satisfies in addition the following property:

3. $\|x\| = 0 \implies x = 0$.

Proposition 1.56. *Let V be a seminormed vector space over $\mathbb{K}$. Then, $d(v, w) := \|v - w\|$ defines a pseudometric on V. Moreover, d is a metric iff the seminorm is a norm.*

Proof. **Exercise.** □

Remark 1.57. Since a seminormed space is a pseudometric space all the concepts developed for pseudometric spaces apply. In particular the notions of convergence, Cauchy sequence and completeness apply to seminormed spaces.

Exercise 6. Show that the operations of addition and multiplication are continuous in a seminormed space.

Definition 1.58. A complete normed vector space is called a *Banach space*.

Exercise 7. Show that $\mathbb{R}^n$ with norm given by $\|x\| = \sqrt{x_1^2 + \cdots + x_n^2}$ is a Banach space. Show that $\|x\| = |x_1| + \cdots + |x_n|$ is another norm that also makes $\mathbb{R}^n$ into a Banach space.

Exercise 8. Let S be a set and $\mathrm{F_b}(S, \mathbb{K})$ the set of bounded maps $S \to \mathbb{K}$.

1. $\mathrm{F_b}(S, \mathbb{K})$ is a vector space over $\mathbb{K}$.
2. The *supremum norm* on it is a norm defined by
$$\|f\|_{\mathrm{sup}} := \sup_{p \in S} |f(p)|.$$
3. $\mathrm{F_b}(S, \mathbb{K})$ with the supremum norm is a Banach space.

Exercise 9. Let $n \in \mathbb{N}$ and S be a set with n elements. Show that $\mathrm{F_b}(S, \mathbb{R})$ is isomorphic to $\mathbb{R}^n$ as a vector space and that the supremum norm yields in this way yet another norm on $\mathbb{R}^n$, different from the ones of Exercise 7, that also make it into a Banach space.

Exercise 10. Let S be a topological space and $\mathrm{C_b}(S, \mathbb{K})$ the set of bounded continuous maps $S \to \mathbb{K}$.

1. $\mathrm{C_b}(S, \mathbb{K})$ is a vector space over $\mathbb{K}$.
2. $\mathrm{C_b}(S, \mathbb{K})$ with the supremum norm is a Banach space.

Proposition 1.59. *Let V be a vector space with a seminorm $\|\cdot\|_V$. Consider the subset $A := \{v \in V : \|v\|_V = 0\}$. Then, A is a vector subspace. Moreover $v \sim w \iff v - w \in A$ defines an equivalence relation and $W := V/\sim$ is a vector space. The seminorm $\|\cdot\|_V$ induces a norm on W via $\|[v]\|_W := \|v\|_V$ for $v \in V$. Also, V is complete with respect to the seminorm $\|\cdot\|_V$ iff W is complete with respect to the norm $\|\cdot\|_W$.*

Proof. **Exercise.** □

Proposition 1.60. *Let V, W be seminormed vector spaces. Then, a linear map $\alpha : V \to W$ is continuous iff there exists a constant $c \geq 0$ such that*

$$\|\alpha(v)\|_W \leq c\|v\|_V \quad \forall v \in V.$$

Proof. **Exercise.** □

2 Measures

The basic idea behind integration theory via measures may be roughly described as follows: Given a space (set) we want to associate "sizes" to "pieces" of the space. To do this we first have to make precise what we mean by a "piece", i.e., what subsets we admit as "pieces". This is the purpose of the concept of a σ-algebra and a measurable space. Given that we know what a piece is, we want to assign a number to it, its "size", in such a way that sizes add up appropriately when we join pieces. This is provided by the concept of a measure. Then, we can declare the integral for the characteristic function on a piece to be the size of the piece. Approximating more arbitrary functions by linear combinations of characteristic functions for pieces then yields a general notion of integral.

2.1 σ-Algebras and Measurable Spaces

Definition 2.1 (Boolean Algebra)**.** Let A be a set equipped with three operations: $\wedge : A \times A \to A$, $\vee : A \times A \to A$ and $\neg : A \to A$ and two special elements $0, 1 \in A$. Suppose these satisfy the following properties:

- $(x \wedge y) \wedge z = x \wedge (y \wedge z)$ and $(x \vee y) \vee z = x \vee (y \vee z) \quad \forall x, y, z \in A$. (associativity)
- $x \wedge y = y \wedge x$ and $x \vee y = y \vee x \quad \forall x, y \in A$. (commutativity)
- $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ and $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z) \quad \forall x, y, z \in A$. (distributivity)
- $x \wedge (x \vee y) = x$ and $x \vee (x \wedge y) = x \quad \forall x, y \in A$. (absorption)
- $x \wedge \neg x = 0$ and $x \vee \neg x = 1 \quad \forall x \in A$. (complement)

Then, A is called a *Boolean algebra*.

Proposition 2.2. *Let A be a Boolean algebra. Then, the following properties hold:*

$$x \wedge x = x,\ x \vee x = x,\ x \wedge 0 = 0,\ x \wedge 1 = x,\ x \vee 0 = x,\ x \vee 1 = 1 \quad \forall x \in A.$$

Proof. **Exercise.** □

Exercise 11. Show that the set with two elements $0, 1$ forms a Boolean algebra. This is important in logic, where 0 stands for "false" and 1 for "true".

Exercise 12. Let S be a set. Show that the set $\mathfrak{P}(S)$ of subsets of S forms a Boolean algebra, where $\vee = \cup$ is the union, $\wedge = \cap$ is the intersection and $\neg$ is the complement of sets.

Definition 2.3 (Algebra of sets)**.** Let S be a set. A subset $\mathcal{M}$ of the set $\mathfrak{P}(S)$ of subsets of S is called an *algebra* of sets iff it is a Boolean subalgebra of $\mathfrak{P}(S)$.

Proposition 2.4. *Let S be a set and $\mathcal{M}$ a subset of the set $\mathfrak{P}(S)$ of subsets of S. Then $\mathcal{M}$ is an algebra of sets iff it contains the empty set and is closed under complements, finite unions, and finite intersections.*

Proof. Immediate. □

Exercise 13. Show that the above proposition remains true if we erase either the requirement for closedness under finite unions or the requirement for closedness under finite intersections.

Definition 2.5. Let S be a set and $\mathcal{M}$ an algebra of subsets of S. We call $\mathcal{M}$ a *σ-algebra* of sets iff it is closed under countable unions and countable intersections.

Exercise 14. Show that the above definition remains unchanged if we remove either the requirement for closedness under countable unions or closedness under countable intersections.

Definition 2.6. Let S be a set and $\mathcal{B}$ a subset of the set $\mathfrak{P}(S)$ of subsets of S. Then, the smallest σ-algebra $\mathcal{M}$ on S containing $\mathcal{B}$ is called the σ-algebra *generated* by $\mathcal{B}$.

Exercise 15. Justify the above definition by showing that the smallest σ-algebra in the sense of the definition always exists.

Definition 2.7. Let S be a set and $\mathcal{B}$ a subset of $\mathfrak{P}(S)$. Then, $\mathcal{B}$ is called *monotone* iff it satisfies the following properties:

- Let $\{A_n\}_{n\in\mathbb{N}}$ be a sequence of elements of $\mathcal{B}$ such that $A_n \subseteq A_{n+1}$. Then, $\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{B}$.
- Let $\{A_n\}_{n\in\mathbb{N}}$ be a sequence of elements of $\mathcal{B}$ such that $A_n \supseteq A_{n+1}$. Then, $\bigcap_{n\in\mathbb{N}} A_n \in \mathcal{B}$.

Proposition 2.8. *1. A σ-algebra is monotone. 2. An algebra that is monotone is a σ-algebra.*

Proof. **Exercise.** □

Proposition 2.9 (Monotone Class Theorem)**.** *Let S be a set and $\mathcal{N}$ an algebra of subsets of S. Then, the smallest set $\mathcal{M}$ of subsets of S which contains $\mathcal{N}$ and is monotone is the σ-algebra generated by $\mathcal{N}$.*

Proof. For each $A \in \mathcal{M}$ and consider

$$\mathcal{M}_A := \{B \in \mathcal{M} : A \cap B \in \mathcal{M}, A \cap \neg B \in \mathcal{M}, \neg A \cap B \in \mathcal{M}\}.$$

It is easy to see that $\mathcal{M}_A$ is monotone. [**Exercise.**Show this!] Furthermore, if $A \in \mathcal{N}$, then $\mathcal{N} \subseteq \mathcal{M}_A$ since $\mathcal{N}$ is an algebra. So in this case $\mathcal{M} \subseteq \mathcal{M}_A$ by minimality of $\mathcal{M}$ and consequently $\mathcal{M} = \mathcal{M}_A$. Thus, for $B \in \mathcal{M}$ we have $B \in \mathcal{M}_A$ and hence $A \in \mathcal{M}_B$ if $A \in \mathcal{N}$. So, $\mathcal{N} \subseteq \mathcal{M}_B$ and by minimality we conclude $\mathcal{M} = \mathcal{M}_B$ for any $B \in \mathcal{M}$. But this means that $\mathcal{M}$ is an algebra. Thus, by Proposition 2.8.2, $\mathcal{M}$ is a σ-algebra. Furthermore, by minimality and Proposition 2.8.1, it is the σ-algebra generated by $\mathcal{N}$. □

Definition 2.10. Let S be a set and $\mathcal{M}$ a σ-algebra of subsets of S. Then, we call the pair $(S, \mathcal{M})$ a *measurable space* and the elements of $\mathcal{M}$ *measurable sets*.

Definition 2.11. Let S be a measurable space and U a subset of S. Then, the σ-algebra on S intersected with U is called the *induced* σ-algebra on U.

Definition 2.12. Let S be a topological space. Then, the σ-algebra generated by the topology of S is called the algebra of *Borel* sets. Its elements are called *Borel measurable*.

2.2 Measurable Functions

As we see the concept of a measurable space is very similar to the concept of a topological space. Both are based on a set of subsets closed under certain operations. We can push this analogy further and consider the analog of a continuous function: a measurable function.

Definition 2.13. Let S, T be measurable spaces. Then a map $f : S \to T$ is called *measurable* iff the preimage of every measurable set of T is a measurable set of S. If either T or S or T and S are topological spaces instead we call f measurable iff it is measurable with respect to the generated σ-algebra(s) of Borel sets.

Proposition 2.14. *Let S, T, U be measurable spaces, $f : S \to T$ and $g : T \to U$ measurable. Then, $g \circ f : S \to U$ is measurable.*

Proof. Immediate. □

Proposition 2.15. *Let S be a measurable space, T a topological space and $f : S \to T$. Then, f is measurable iff the preimage of every open set is measurable. Also, f is measurable iff the preimage of every closed set is measurable.*

Proof. **Exercise.** □

Corollary 2.16. *Let S and T be topological spaces and $f : S \to T$ a continuous map. Then, f is measurable.*

Proposition 2.17. *Let S be a measurable space, T and U topological spaces, $f : S \to T \times U$. Denote by $f_T : S \to T$ and $f_U : S \to U$ the component functions. If the product $f : S \to T \times U$ is measurable, then both f_T and f_U are measurable. Conversely, if T and U are second-countable and f_T and f_U are measurable, then f is measurable.*

Proof. First suppose that f is measurable. Then, $f_T = p_T \circ f$, where p_T is the projection $T \times U \to T$. Since p_T is continuous, it is measurable by Corollary 2.16 and the composition f_T is measurable by Proposition 2.14. In the same way it follows that f_U is measurable.

Conversely, suppose now that f_T and f_U are measurable. If $V \subseteq T$ and $W \subseteq U$ are open sets, then $f_T^{-1}(V)$ and $f_U^{-1}(W)$ are measurable in S and so is their intersection $f^{-1}(V \times W) = f_T^{-1}(V) \cap f_U^{-1}(W)$. Since T and U are second-countable, every open set in $T \times U$ can be written as a countable union of products of open sets in T and U [**Exercise.**Show this!]. But the preimage of such a countable union in S under f^{-1} can be written as a countable union of preimages. Since these are measurable, their countable union is also measurable. It follows then from Proposition 2.15 that f is measurable. □

In the following $\mathbb{K}$ denotes either the field of real numbers $\mathbb{R}$ or the field of complex numbers $\mathbb{C}$.

Proposition 2.18. *Let S be a measurable space, $f, g : S \to \mathbb{K}$ measurable and $\lambda \in \mathbb{K}$. Then:*

1. *$|f| : x \mapsto |f(x)|$ is measurable.*
2. *$f + g : x \mapsto f(x) + g(x)$ is measurable.*
3. *$\lambda f : x \mapsto \lambda f(x)$ is measurable.*

4. *$fg : x \mapsto f(x)g(x)$ is measurable.*
5. *If $\mathbb{K} = \mathbb{R}$ then $\sup(f, g)$ and $\inf(f, g)$ are measurable.*

Proof. **Exercise.** $\square$

This shows in particular that measurable functions with values in $\mathbb{R}$ or $\mathbb{C}$ form an algebra. Another important property of the set of measurable maps is its closedness under pointwise limits. This can be formulated for the more general case when the values are taken in a metric space.

Theorem 2.19 (adapted from S. Lang)**.** *Let S be a measurable space and T a metric space. Suppose $\{f_n\}_{n\in\mathbb{N}}$ is a sequence of measurable functions $f_n : S \to T$ which converges pointwise to the function $f : S \to T$. Then, f is measurable.*

Proof. Let U be an open set in T. Suppose $x \in f^{-1}(U)$. Since $\{f_n(x)\}_{n\in\mathbb{N}}$ converges to $f(x)$ there exists $m \in \mathbb{N}$ such that $x \in f_n^{-1}(U)$ for all $n > m$. In particular, $x \in \bigcup_{n=k}^{\infty} f_n^{-1}(U)$ for any $k \in \mathbb{N}$ and so also $x \in \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} f_n^{-1}(U)$. Since this is true for any $x \in f^{-1}(U)$ we get

$$f^{-1}(U) \subseteq \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} f_n^{-1}(U).$$

Consider now for all $l \in \mathbb{N}$ the open sets

$$U_l := \{x \in U : \min_{y \notin U} d(x, y) > 1/l\}.$$

Then, $U = \bigcup_{l=1}^{\infty} U_l$ and applying the above reasoning to each U_l we get,

$$f^{-1}(U) \subseteq \bigcup_{l=1}^{\infty} \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} f_n^{-1}(U_l).$$

Suppose now that $x \notin f^{-1}(U)$ and fix $l \in \mathbb{N}$. Since $B_{1/l}(f(x)) \cap U_l = \emptyset$ there exists $m \in \mathbb{N}$ such that $x \notin f_n^{-1}(U_l)$ for all $n > m$. In particular, $x \notin \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} f_n^{-1}(U_l)$. Since this is true for any $l \in \mathbb{N}$ we get $x \notin \bigcup_{l=1}^{\infty} \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} f_n^{-1}(U_l)$. Since this is true for any $x \notin f^{-1}(U)$ we get, combining with the above result,

$$f^{-1}(U) = \bigcup_{l=1}^{\infty} \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} f_n^{-1}(U_l).$$

Since f_n is measurable for all $n \in \mathbb{N}$ the right hand side is measurable. We have thus shown that preimages of open sets are measurable. By Proposition 2.15 this is sufficient for f to be measurable. $\square$

Definition 2.20. Let S be a measurable space. A map $f : S \to \mathbb{K}$ is called a *simple map* iff it is measurable and takes only finitely many values.

Proposition 2.21. *Let S be a measurable space and $f : S \to \mathbb{K}$ a map that takes only finitely many values. Then f is a simple map (i.e., is measurable) iff the preimage of each of the values of f is measurable.*

Proof. **Exercise.** □

Proposition 2.22. *The simple functions with values in $\mathbb{K}$ form a subalgebra of the algebra of measurable functions with values in $\mathbb{K}$.*

Proof. **Exercise.** □

Theorem 2.23 (adapted from S. Lang)**.** *Let S be a measurable space and $f : S \to \mathbb{K}$ measurable. Then, f is the pointwise limit of a sequence of simple maps. If, moreover, f takes values in $\mathbb{R}^+_0$, then the sequence can be chosen to increase monotonically.*

Proof. Consider first the case $\mathbb{K} = \mathbb{R}$. Fix $n \in \mathbb{N}$. For each $k \in \{1, \ldots, 2^{n+1}n\}$ define the interval $I_k := [-n + \frac{k-1}{2^n}, -n + \frac{k}{2^n})$. Also, define $I_0 := (-\infty, -n)$ and $I_{2^{n+1}n+1} := [n, \infty)$. Notice that $\mathbb{R}$ is the disjoint union of the measurable intervals I_k for $k \in \{0, \ldots, 2^{n+1}n + 1\}$. Now set $X_k := f^{-1}(I_k)$ for all $k \in \{0, \ldots, 2^{n+1}n + 1\}$. Since the intervals I_k are measurable so are the sets X_k. Define the function $f_n : S \to \mathbb{R}$ by $f_n(X_k) := -n + \frac{k-1}{2^n}$ for all $k \in \{1, \ldots, 2^{n+1}n+1\}$ and $f_n(X_0) := -n$. It is easy to see that $\{f_n\}_{n\in\mathbb{N}}$ is a sequence of simple functions that converge pointwise to f. [**Exercise.**Show this!] Moreover, if f takes values in $\mathbb{R}^+_0$ only, the sequence is monotonically increasing. [**Exercise.**Show this!] To treat the case $\mathbb{K} = \mathbb{C}$ we decompose f into its real and imaginary part. The sum of simple sequences for each part is again a simple sequence. □

2.3 Positive Measures

Definition 2.24. Let $\{a_n\}_{n\in\mathbb{N}}$ be a monotonously increasing sequence of real numbers. Then we say that $\lim_{n\to\infty} a_n = \infty$ iff for any $a \in \mathbb{R}$ there exists $m \in \mathbb{N}$ such that $a_n > a$ for all $n > m$.

Definition 2.25 (Positive Measure)**.** Let S be a set with an algebra $\mathcal{M}$ of subsets. Then, a map $\mu : \mathcal{M} \to [0, \infty]$ is called a *(positive) measure* iff it is countably additive, i.e., satisfies the following properties:

- $\mu(\emptyset) = 0$.

- Let $\{U_n\}_{n\in\mathbb{N}}$ be a sequence of elements of $\mathcal{M}$ such that $U_n \cap U_m = \emptyset$ if $n \neq m$ and such that $\bigcup_{n\in\mathbb{N}} U_n \in \mathcal{M}$. Then,

$$\mu\left(\bigcup_{n\in\mathbb{N}} U_n\right) = \sum_{n\in\mathbb{N}} \mu\left(U_n\right).$$

If $U \in \mathcal{M}$, then $\mu(U)$ is called its *measure*. Moreover, a measurable space S with σ-algebra $\mathcal{M}$ and positive measure $\mu : \mathcal{M} \to [0,\infty]$ is called a *measure space*.

We shall mostly be interested in the case where $\mathcal{M}$ actually is a σ-algebra. However, it will turn out convenient to keep the definition more general when we consider constructing measures.

Proposition 2.26. *Let S be a set, $\mathcal{M}$ an algebra of subsets of S and $\mu : \mathcal{M} \to [0,\infty]$ a measure. Then, the following properties hold:*

1. *Let $A, B \in \mathcal{M}$ and $A \subseteq B$. Then, $\mu(A) \le \mu(B)$.*
2. *Let $\{A_n\}_{n\in\mathbb{N}}$ be a sequence of elements of $\mathcal{M}$ such that $\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{M}$. Then,*
$$\mu\left(\bigcup_{n\in\mathbb{N}} A_n\right) \le \sum_{n\in\mathbb{N}} \mu(A_n).$$
3. *Let $\{A_n\}_{n\in\mathbb{N}}$ be a sequence of elements of $\mathcal{M}$ such that $A_n \subseteq A_{n+1}$ for all $n \in \mathbb{N}$ and $\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{M}$. Then,*
$$\mu\left(\bigcup_{n\in\mathbb{N}} A_n\right) = \lim_{n\to\infty} \mu(A_n).$$
4. *Let $\{A_n\}_{n\in\mathbb{N}}$ be a sequence of elements of $\mathcal{M}$ such that $A_n \supseteq A_{n+1}$ for all $n \in \mathbb{N}$ and $\bigcap_{n\in\mathbb{N}} A_n \in \mathcal{M}$. If furthermore, $\mu(A_n) < \infty$ for some $n \in \mathbb{N}$ then,*
$$\mu\left(\bigcap_{n\in\mathbb{N}} A_n\right) = \lim_{n\to\infty} \mu(A_n).$$

Proof. **Exercise.** □

Exercise 16. Check whether the following examples are measures.

- Let S be a set and consider the σ-algebra of all subsets of S. If $A \subseteq S$ is finite define $\mu(A)$ to be its number of elements. If $A \subseteq S$ is infinite define $\mu(A) = \infty$. μ is called the *counting measure*.

- Let S be a set and consider the σ-algebra of all subsets of S. If $A \subseteq S$ is finite define $\mu(A) = 0$. If $A \subseteq S$ is infinite define $\mu(A) = \infty$.

- Let S be a set and consider the σ-algebra of all subsets of S. If $A \subseteq S$ is countable define $\mu(A) = 0$. If $A \subseteq S$ is not countable define $\mu(A) = \infty$.

- Let S be a set and consider the σ-algebra of all subsets of S. Let $x \in S$. For $A \subseteq S$ define $\mu(A) = 1$ if $x \in A$ and $\mu(A) = 0$ otherwise. μ is called the *Dirac measure* with respect to x.

Definition 2.27. Let S be a measure space and $A \subseteq S$ a measurable subset. We say that A is *σ-finite* iff it is equal to some countable union of measurable sets with finite measure. We say that a measure is *finite* respectively *σ-finite* iff the measure space is finite respectively σ-finite with respect to the measure.

Exercise 17. Which of the examples of measures above are σ-finite?

Definition 2.28. Let $(S, \mathcal{M}, \mu)$ be a measure space. If every subset of any set of measure 0 is measurable, then we call $(S, \mathcal{M}, \mu)$ *complete*.

Proposition 2.29. *Let $(S, \mathcal{M}, \mu)$ be a measure space. Then, there exists a unique complete measure space $(S, \mathcal{M}^*, \mu^*)$ such that $\mathcal{M}^*$ is a σ-algebra containing $\mathcal{M}$ and $\mu^*|_{\mathcal{M}} = \mu$ and $\mathcal{M}^*$ is smallest with these properties. $(S, \mathcal{M}^*, \mu^*)$ is called the* completion *of $(S, \mathcal{M}, \mu)$. Moreover, the element of $\mathcal{M}^*$ are precisely the sets of the form $A \cup N$, where $A \in \mathcal{M}$ and N is a subset of a set of measure 0 in $\mathcal{M}$.*

Proof. **Exercise.** □

Proposition 2.30. *Let $(S, \mathcal{M}, \mu)$ be a measure space and $f : S \to \mathbb{K}$ measurable with respect to $\mathcal{M}^*$. Then, there exists a function $g : S \to \mathbb{K}$ such that g is measurable with respect to $\mathcal{M}$ and g does not differ from f outside of a subset $N \in \mathcal{M}$ of measure 0.*

Proof. By Theorem 2.23 there exists a sequence $\{f_n\}_{n\in\mathbb{N}}$ of simple maps with respect to $\mathcal{M}^*$ that converges pointwise to f. For each f_n we can find a set $N_n \in \mathcal{M}$ of measure 0 such that the function $k_n : S \to \mathbb{K}$ defined by $k_n(p) = f_n(p)$ if $p \in S \setminus N_n$ and $k_n(p) = 0$ otherwise, is simple with respect to

$\mathcal{M}$. (**Exercise.**Show this!) The set $N := \bigcup_{n=1}^{\infty} N_n \in \mathcal{M}$ has measure zero. Moreover, $g_n : S \to \mathbb{K}$ defined by $g_n(p) = f_n(p)$ if $p \in S \setminus N$ and $g_n(p) = 0$ otherwise, is simple with respect to $\mathcal{M}$. Moreover, the sequence $\{g_n\}_{n\in\mathbb{N}}$ converges pointwise to $g : S \to \mathbb{K}$ defined by $g(p) = f(p)$ if $p \in S \setminus N$ and $g(p) = 0$ otherwise. Thus, by Theorem 2.19, g is measurable with respect to $\mathcal{M}$. $\square$

2.4 Extension of Measures

We now turn to the question of how to construct measures. We will focus here on the method of extension. That is, we consider a measure that is merely defined on an algebra of subsets and extend it to a measure on a σ-algebra.

Definition 2.31. Let S be a set and $\mathcal{M}$ a σ-algebra of subsets of S. Then, a map $\lambda : \mathcal{M} \to [0, \infty]$ is called an *outer measure* on $\mathcal{M}$ iff it satisfies the following properties:

- $\lambda(\emptyset) = 0$.
- Let $A, B \in \mathcal{M}$ and $A \subseteq B$. Then, $\lambda(A) \leq \lambda(B)$. (monotonicity)
- Let $\{A_n\}_{n\in\mathbb{N}}$ be a sequence of elements of $\mathcal{M}$. Then,

$$\lambda\left(\bigcup_{n\in\mathbb{N}} A_n\right) \leq \sum_{n\in\mathbb{N}} \lambda\left(A_n\right). \quad \text{(countable subadditivity)}$$

Lemma 2.32. *Let S be a set, $\mathcal{N}$ an algebra of subsets of S and μ a measure on $\mathcal{N}$. On the σ-algebra $\mathfrak{P}(S)$ of all subsets of S define the function $\lambda : \mathfrak{P}(S) \to [0, \infty]$ given by*

$$\lambda(X) = \inf\left\{\sum_{n\in\mathbb{N}} \mu(A_n) : A_n \in \mathcal{N}\,\forall n \in \mathbb{N}\,\textit{and}\, X \subseteq \bigcup_{n\in\mathbb{N}} A_n\right\}.$$

Then, λ is an outer measure on $\mathfrak{P}(S)$. Moreover, it extends μ, i.e., $\lambda(A) = \mu(A)$ for all $A \in \mathcal{N}$.

Proof. **Exercise.** $\square$

Definition 2.33. Let S be a set and λ an outer measure on the σ-algebra $\mathfrak{P}(S)$ of all subsets of S. Then, $A \subseteq S$ is called λ-*measurable* iff $\lambda(X) = \lambda(X \cap A) + \lambda(X \cap \neg A)$ for all $X \subseteq S$.

Lemma 2.34. *Let S be a set and λ an outer measure on the σ-algebra $\mathfrak{P}(S)$ of all subsets of S. Let $\mathcal{M}$ be the set of subsets of S that are λ-measurable. Then, $\mathcal{M}$ is a σ-algebra and λ is a complete measure on $\mathcal{M}$.*

Proof. It is clear that $\mathcal{M}$ contains the empty set and S. Also, from the definition it is clear that a set is λ-measurable iff its complement is. Let now $A, B \in \mathcal{M}$. We proceed to show that $A \cap B \in \mathcal{M}$. Let $C \subseteq S$ be arbitrary. Since B is λ-measurable we have,

$$\lambda(C \cap A \cap B) + \lambda(C \cap A \cap \neg B) = \lambda(C \cap A).$$

Adding $\lambda(C \cap \neg A)$ we get,

$$\lambda(C \cap A \cap B) + \lambda(C \cap A \cap \neg B) + \lambda(C \cap \neg A) = \lambda(C),$$

since A is λ-measurable. The λ-measurability of $A \cap B$ follows if we can show,

$$\lambda(C \cap A \cap \neg B) + \lambda(C \cap \neg A) = \lambda(C \cap \neg(A \cap B)).$$

But this equation can be rewritten term-wise as,

$$\lambda(C \cap \neg(A \cap B) \cap A) + \lambda(C \cap \neq (A \cap B) \cap \neg A) = \lambda(C \cap \neg(A \cap B)),$$

which follows from the λ-measurability of A. Thus $\mathcal{M}$ is an algebra.

Now consider a sequence $\{A_n\}_{n\in\mathbb{N}}$ of disjoint λ-measurable sets and let $A = \bigcup_{n\in N} A_n$ be their union. Let $C \subseteq S$ be arbitrary. By iteration we find that for any $n \in \mathbb{N}$ we have,

$$\lambda(C \cap (A_1 \cup \cdots \cup A_n)) = \sum_{k=1}^{n} \lambda(C \cap A_k).$$

On the other hand $C \cap \neg A \subseteq C \cap \neg(A_1 \cup \cdots \cup A_n)$ and so $\lambda(C \cap \neg A) \leq \lambda(C \cap \neg(A_1 \cup \cdots \cup A_n))$ since λ is an outer measure. With λ-measurability of $A_1 \cup \cdots \cup A_n$ we get thus,

$$\begin{aligned}\lambda(C) &= \lambda(C \cap (A_1 \cup \cdots \cup A_n)) + \lambda(C \cap \neg(A_1 \cup \cdots \cup A_n)) \\ &\geq \sum_{k=1}^{n} \lambda(C \cap A_k) + \lambda(C \cap \neg A).\end{aligned}$$

But this is true for any $n \in \mathbb{N}$, so,

$$\lambda(C) \geq \sum_{k=1}^{\infty} \lambda(C \cap A_k) + \lambda(C \cap \neg A) \geq \lambda(C \cap A) + \lambda(C \cap \neg A).$$

as λ is an outer measure. The same property of λ gives us directly the converse inequality,

$$\lambda(C) \leq \lambda(C \cap A) + \lambda(C \cap \neg A).$$

Thus, A is λ-measurable and consequently $\mathcal{M}$ is a σ-algebra. What is more, setting $C = A$ we see that λ is countably additive on $\mathcal{M}$, i.e., defines a positive measure on it.

Now let $A \in \mathcal{M}$ with $\lambda(A) = 0$ and $B \subseteq A$. Also let $C \subseteq S$ be arbitrary. Then we have,

$$\lambda(C \cap B) \leq \lambda(C \cap A) \leq \lambda(A) = 0.$$

We also get,

$$\lambda(C) = \lambda(C \cap A) + \lambda(C \cap \neg A) = \lambda(C \cap \neg A) \leq \lambda(C \cap \neg B) \leq \lambda(C).$$

In particular,

$$\lambda(C \cap B) = 0 \quad \text{and} \quad \lambda(C \cap \neg B) = \lambda(C).$$

Thus, B is λ-measurable. This shows completeness of $(S, \mathcal{M}, \lambda)$. $\square$

Theorem 2.35 (Hahn)**.** *Let S be a set, $\mathcal{N}$ an algebra of subsets of S and μ a measure on $\mathcal{N}$. Then, μ can be extended to a σ-algebra $\mathcal{M}$ containing $\mathcal{N}$ such that μ is a complete measure on $\mathcal{M}$ and for all $X \in \mathcal{M}$ we have*

$$\mu(X) = \inf \left\{ \sum_{n \in \mathbb{N}} \mu(A_n) : A_n \in \mathcal{N}\, \forall n \in \mathbb{N} \,\mathit{and}\, X \subseteq \bigcup_{n \in \mathbb{N}} A_n \right\}.$$

Proof. **Exercise.** $\square$

Proposition 2.36 (Uniqueness of Extension)**.** *Let S be a measurable space with σ-algebra $\mathcal{M}$ and measures μ_1, μ_2. Suppose there is an algebra $\mathcal{N} \subseteq \mathcal{M}$ generating $\mathcal{M}$ and such that $\mu(A) := \mu_1(A) = \mu_2(A)$ for all $A \in \mathcal{N}$. Furthermore, assume that μ is σ-finite with respect to $\mathcal{N}$. Then, $\mu_1 = \mu_2$ also on $\mathcal{M}$.*

Proof. Let $\{X_n\}_{n \in \mathbb{N}}$ be a sequence of elements of $\mathcal{N}$ such that $S = \bigcup_{n \in \mathbb{N}} X_n$ and $X_n \subseteq X_{n+1}$ and $\mu(X_n) < \infty$ for all $n \in \mathbb{N}$. (By σ-finiteness, there is a sequence $\{Y_k\}_{k \in \mathbb{N}}$ with $S = \bigcup_{k \in \mathbb{N}} Y_k$ and $\mu(Y_k) < \infty$ for all $k \in \mathbb{N}$. Now set $X_n := \bigcup_{k=1}^{n} Y_k$.) Define the finite measures $\mu_{1,n}(A) := \mu_1(A \cap X_n)$ and $\mu_{2,n}(A) := \mu_2(A \cap X_n)$ on $\mathcal{M}$ for all $n \in \mathbb{N}$. Now, let $\mathcal{B}_n$ be the subsets of

$\mathcal{M}$ where $\mu_{1,n}$ and $\mu_{2,n}$ agree. By construction, $\mathcal{N} \subseteq \mathcal{B}_n$ for all $n \in \mathbb{N}$. We show that the $\mathcal{B}_n$ are monotone.

Fix $n \in \mathbb{N}$. Let $\{A_k\}_{k\in\mathbb{N}}$ be a sequence of elements of $\mathcal{B}_n$ such that $A_k \subseteq A_{k+1}$ for all $k \in \mathbb{N}$ and set $A := \bigcup_{k\in\mathbb{N}} A_k$. Then, using Proposition 2.26,

$$\mu_{1,n}(A) = \lim_{k\to\infty} \mu_{1,n}(A_k) = \lim_{k\to\infty} \mu_{2,n}(A_k) = \mu_{2,n}(A).$$

So, $A \in \mathcal{B}_n$. Now, let $\{A_k\}_{k\in\mathbb{N}}$ be a sequence of elements of $\mathcal{B}_n$ such that $A_k \supseteq A_{k+1}$ for all $k \in \mathbb{N}$ and set $A := \bigcap_{k\in\mathbb{N}} A_k$. Again using Proposition 2.26 we get (note that the finiteness of the measure is essential here),

$$\mu_{1,n}(A) = \lim_{k\to\infty} \mu_{1,n}(A_k) = \lim_{k\to\infty} \mu_{2,n}(A_k) = \mu_{2,n}(A).$$

So, $A \in \mathcal{B}_n$. Hence, $\mathcal{B}_n$ is monotone and by Proposition 2.9 we must have $\mathcal{M} \subseteq \mathcal{B}_n$ and hence $\mathcal{M} = \mathcal{B}_n$.

Thus, $\mu_{1,n} = \mu_{2,n}$ for all $n \in \mathbb{N}$. But then, $\mu_1 = \lim_{n\to\infty} \mu_{1,n} = \lim_{n\to\infty} \mu_{2,n} = \mu_2$. This completes the proof. □

Proposition 2.37. *Let $(S, \mathcal{M}, \mu)$ be a measure space. Let $\mathcal{N}$ be an algebra of subsets of S that generates $\mathcal{M}$. Denote the completion of $\mathcal{M}$ with respect to μ by $\mathcal{M}^*$. Then, for any $X \in \mathcal{M}^*$ with finite measure and any $\epsilon > 0$ there exists $A \in \mathcal{N}$ such that*

$$\mu((X \setminus A) \cup (A \setminus X)) < \epsilon.$$

Proof. Let $X \in \mathcal{M}^*$. By Hahn's Theorem 2.35 there exists a sequence $\{A_n\}_{n\in\mathbb{N}}$ of disjoint elements of $\mathcal{N}$ such that $X \subseteq \bigcup_{n\in\mathbb{N}} A_n$ and

$$\sum_{n=1}^{\infty} \mu(A_n) < \mu(X) + \epsilon/2.$$

Now fix $k \in \mathbb{N}$ such that

$$\sum_{n=k+1}^{\infty} \mu(A_n) < \epsilon/2.$$

Set $A := \bigcup_{n=1}^{k} A_n$. Then, on the one hand,

$$\mu(A \setminus X) \le \mu\left(\left(\bigcup_{n=1}^{\infty} A_n\right) \setminus X\right) < \epsilon/2,$$

while on the other hand,

$$\mu(X \setminus A) \le \mu\left(\left(\bigcup_{n=1}^{\infty} A_n\right) \setminus A\right) = \mu\left(\bigcup_{n=k+1}^{\infty} A_n\right) < \epsilon/2.$$

This implies the statement. □

2.5 The Lebesgue Measure

In the following we are going to construct the Lebesgue measure. This is the unique (as we shall see) measure on the real numbers assigning to an interval its length. The construction proceeds in various stages.

Lemma 2.38. *The finite unions of intervals of the type $[a, b)$, $(-\infty, a)$, and $[a, \infty)$ together with $\emptyset$ form an algebra $\mathcal{N}$ of subsets of the real numbers.*

Proof. **Exercise.** □

Lemma 2.39. *The prescription $\mu([a, b)) = b - a$ determines uniquely a finitely additive function $\mu : \mathcal{N} \to [0, \infty]$ on the algebra $\mathcal{N}$ considered above.*

Proof. **Exercise.** □

Lemma 2.40. *The function $\mu : \mathcal{N} \to [0, \infty]$ defined above is countably additive and thus a measure.*

Proof. Let $\{A_n\}_{n\in\mathbb{N}}$ be a sequence of pairwise disjoint elements of $\mathcal{N}$ such that $A := \bigcup_{n\in\mathbb{N}} \in \mathcal{N}$. We wish to show that

$$\mu(A) = \sum_{n\in\mathbb{N}} \mu(A_n).$$

By finite additivity we have $\mu(A) \geq \mu(\bigcup_{n=1}^m A_n) = \sum_{n=1}^m \mu(A_n)$ for all $m \in \mathbb{N}$ and hence

$$\mu(A) \geq \sum_{n\in\mathbb{N}} \mu(A_n).$$

It remains to show the opposite inequality.

Assume at first that A is a finite interval $[a, b)$. Then, A is the disjoint union of a sequence of intervals $\{I_k\}_{k\in\mathbb{N}}$ with $I_k = [a_k, b_k)$ in such a way that each A_n is the finite union of some I_k. (We also allow the degenerate case $a_k = b_k$ in which case $I_k = \emptyset$.) Fix now $\epsilon > 0$ (with $\epsilon < b - a$) and define $I'_k := (a_k - 2^{-(k+1)}\epsilon, b_k)$ for all $k \in \mathbb{N}$. Then, the open sets $\{I'_k\}_{k\in\mathbb{N}}$ cover the compact interval $[a, b - \epsilon/2]$. Thus, there is a finite set of indices $I \subset \mathbb{N}$ such that $[a, b - \epsilon/2] \subset \bigcup_{k\in I} I'_k$. Then clearly also $[a, b - \epsilon/2) \subset \bigcup_{k\in I} I''_k$, where $I''_k := [a_k - 2^{-(k+1)}\epsilon, b_k)$. By finite additivity of μ we get

$$\begin{aligned}\mu([a, b - \epsilon/2)) \leq \mu\left(\bigcup_{k\in I} I''_k\right) &\leq \sum_{k\in I} \mu\left(I''_k\right) \\ &= \sum_{k\in I} \left(\mu(I_k) + \epsilon 2^{-(k+1)}\right) \leq \epsilon/2 + \sum_{k\in I} \mu(I_k).\end{aligned}$$

But since $\mu(A) = \mu([a, b-\epsilon/2)) + \epsilon/2$, we find $\mu(A) \le \epsilon + \sum_{k\in I} \mu(I_k)$. Thus, there exists $m \in \mathbb{N}$ such that $\mu(A) \le \epsilon + \sum_{n=1}^{m} \mu(A_n)$. But since ϵ was arbitrary we can conclude $\mu(A) \le \sum_{n\in\mathbb{N}} \mu(A_n)$ and hence equality.

Exercise. Complete the proof. □

Proposition 2.41. *Consider the real numbers with its σ-algebra $\mathcal{B}$ of Borel sets. Then, the prescription $\mu([a, b)) := b - a$ uniquely extends to a measure $\mu : \mathcal{B} \to [0, \infty]$.*

Proof. By Lemmas 2.38, 2.39 and 2.40 the prescription uniquely defines a measure μ on the algebra $\mathcal{N}$ of unions of intervals of the type $[a, b)$, $(-\infty, a)$, and $[a, \infty)$. By Theorem 2.35 μ extends to a σ-algebra $\mathcal{M}$ containing $\mathcal{N}$. But the σ-algebra generated by $\mathcal{N}$ is the σ-algebra $\mathcal{B}$ of Borel sets. (**Exercise.**Show this!) So, in particular, we get a measure on $\mathcal{B}$. By Proposition 2.36 this is unique since μ is σ-finite on $\mathcal{N}$. (**Exercise.**Show this latter statement!) □

Definition 2.42. The measure defined in the preceding Proposition is called the *Lebesgue measure* on $\mathbb{R}$.

Exercise 18. Consider the real numbers with the Lebesgue measure. Determine $\mu(\mathbb{Q})$ and $\mu(\mathbb{R} \setminus \mathbb{Q})$.

Exercise 19. The Cantor set C is a subset of the interval $[0, 1]$. It can be described for example as

$$C = \bigcap_{n=0}^{\infty} \bigcup_{k=0}^{(3^n-1)/2} \left[\frac{2k}{3^n}, \frac{2k+1}{3^n}\right].$$

Show that $\mu(C) = 0$.

Proposition 2.43. *The Lebesgue measure is translation invariant, i.e., $\mu(A + c) = \mu(A)$ for any measurable A and $c \in \mathbb{R}$.*

Proof. Straightforward. □

Exercise 20. Consider the following equivalence relation on $\mathbb{R}$: Let $x \sim y$ iff $x - y \in \mathbb{Q}$. Now choose (using the axiom of choice) one representative out of each equivalence class, such that this representative lies in $[0, 1]$. Call the set obtained in this way A.

1. Show that $(A+r)\cap(A+s) = \emptyset$ if r and s are distinct rational numbers. Supposing that A is Lebesgue measurable, conclude that $\mu(A) = 0$.

2. Show that $\mathbb{R} = \bigcup_{q\in\mathbb{Q}}(A+q)$. Supposing that A is Lebesgue measurable, conclude that $\mu(A) > 0$.

We obtain a contradiction showing that A is not Lebesgue measurable.

We can define the Lebesgue measure more generally for $\mathbb{R}^n$. The intervals of the type $[a,b)$ are replaced by products of such intervals. Otherwise the construction proceeds in parallel.

Proposition 2.44. *Consider $\mathbb{R}^n$ with its σ-algebra $\mathcal{B}$ of Borel sets. Then, the prescription $\mu([a_1,b_1)\times\cdots\times[a_n,b_n)) = (b_1-a_1)\cdots(b_n-a_n)$ uniquely extends to a measure $\mu : \mathcal{B} \to [0,\infty]$.*

Exercise 21. Sketch the proof by explaining the changes with respect to the one-dimensional case.

3 The integral

3.1 The integral of positive functions

Let X be a set and H a vector space of functions on S with values in $\mathbb{R}$. Denote by H^+ the subset of functions with values in $[0, \infty)$.

Definition 3.1. We say that a linear map $F : H \to \mathbb{R}$ is *positive* iff $f \in H$ with $f \geq 0$ implies $F(f) \geq 0$.

Proposition 3.2. *A linear map $F : H \to \mathbb{R}$ is positive iff for all $f, g \in X$ with $f \geq g$ we have $F(f) \geq F(g)$.*

Definition 3.3. We say that a map $F : H^+ \to [0, \infty]$ is *semilinear* iff

- $F(\lambda f) = \lambda F(f)$ for $\lambda \in [0, \infty)$ and $f \in H$, and,
- $F(f + g) = F(f) + F(g)$ for $f, g \in H$.

Proposition 3.4. *Consider a semilinear map $F : H^+ \to [0, \infty]$. Set $K^+ := F^{-1}([0, \infty))$ and $K := K^+ - K^+$ as a subset of H. Then, K is a vector subspace of H. Also, there is a unique positive linear map $F' : K \to \mathbb{R}$ such that $F'|_{K^+} = F|_{K^+}$.*

Proof. **Exercise.** □

Proposition 3.5. *Suppose that $f, g \in H$ implies* $\sup(f, g) \in H$ *and (equivalently)* $\inf(f, g) \in H$*. Then, $f, g \in K$ implies* $\sup(f, g) \in K$ *and* $\inf(f, g) \in K$.

Proof. Note that the equivalence between the conditions follows from $\inf(f, g) = -\sup(-f, -g)$. Moreover, note that the condition that $f, g \in H$ implies $\sup(f, g) \in H$ is equivalent to the apparently weaker condition that $f \in H$ implies $\sup(f, 0) \in H$. This is because $\sup(f, g) = \sup(f - g, 0) + g$. Now let $f \in K$. By definition there exist $f_1, f_2 \in K^+$ such that $f = f_1 - f_2$. But, as is easy to see $\sup(f, 0) \leq f_1$. So $\sup(f, 0) \in K^+ \subset K$. This completes the proof. □

Let $(X, \mathcal{M})$ be a measurable space. We denote the vector space of measurable functions on X with values in $\mathbb{K}$ by $\mathcal{L}(X, \mathbb{K})$ and the subspace of simple functions by $\mathcal{S}(X, \mathbb{K})$. In the case $\mathbb{K} = \mathbb{R}$ we also simply write $\mathcal{L}(X)$ and $\mathcal{S}(X)$. We denote the subsets of functions with values in $[0, \infty)$ by $\mathcal{L}^+(X)$ and $\mathcal{S}^+(X)$ respectively. We call such functions *positive*. Note that in this sense 0 is a positive function.

Let $(X, \mathcal{M}, \mu)$ be a measure space. We define in the following the *μ-integral*, or simply *integral*, which associates to certain measurable functions $f : X \to \mathbb{K}$ a value, denoted

$$f \mapsto \int_X f \, \mathrm{d}\mu.$$

When it is clear with respect to which measure the integral is taken, the symbol $\mathrm{d}\mu$ may be omitted. When the integral is taken with respect to the whole measure space and it is clear which measure space this is, the subscript indicating the set over which is integrated may be omitted.

Definition 3.6. The *integral* for positive simple functions is the map $\mathcal{S}^+(X) \to [0, \infty]$ defined as follows. Given $f \in \mathcal{S}^+(X)$ let $f(X) = \{a_1, \ldots, a_n\}$ and $X_i := f^{-1}(a_i)$. Then,

$$\int_X f \, \mathrm{d}\mu := \sum_{i=1}^n a_i \mu(X_i).$$

Proposition 3.7. *The integrals $\mathcal{S}^+(Y) \to [0, \infty]$ for $Y \in \mathcal{M}$ are the unique collection of maps with the following properties:*

1. *They are semilinear maps.*
2. *$\int_Y 1 \, \mathrm{d}\mu = \mu(Y)$, where 1 is the constant function with value 1.*
3. *Let $Y_1, Y_2 \in \mathcal{M}$ such that $Y_1 \cap Y_2 = \emptyset$ and $Y = Y_1 \cup Y_2$. Then,*

$$\int_Y f \, \mathrm{d}\mu = \int_{Y_1} f \, \mathrm{d}\mu + \int_{Y_2} f \, \mathrm{d}\mu.$$

Proof. We first show unicity. Thus, we suppose that we are given an integral with the described properties. Let $f \in \mathcal{S}^+(X)$ and set $f(X) = \{a_1, \ldots, a_n\}$ and $X_i := f^{-1}(a_i)$. Iterating property (3), then applying semilinearity (1), then property (2), we recover the previous definition of the integral,

$$\int_X f \, \mathrm{d}\mu = \sum_{i=1}^n \int_{X_i} a_i \cdot 1 \, \mathrm{d}\mu = \sum_{i=1}^n a_i \int_{X_i} 1 \, \mathrm{d}\mu = \sum_{i=1}^n a_i \mu(X_i).$$

We turn to the proof of the different properties for the defined integral. Property (2) is immediate from the definition. We proceed to demonstrate property (3). Thus, let $f \in \mathcal{S}^+(Y)$ and set $f(Y) = \{a_1, \ldots, a_n\}$ and $X_i := f^{-1}(a_i)$. Then, the restrictions $f|_{Y_1}$ and $f|_{Y_2}$ take values in subsets of $\{a_1, \ldots, a_n\}$ and we have $X_i \cap Y_j = f|_{Y_j}^{-1}(a_i)$. Thus, we get from the definition of the integral,

$$\int_{Y_j} f \, \mathrm{d}\mu = \sum_{i=1}^{n} a_i \mu(X_i \cap Y_j).$$

Strictly speaking we should only sum over the values a_i actually occuring in Y_j. However, the summands for the additional values vanish since for these $X_i \cap Y_j = \emptyset$ and thus $\mu(X_i \cap Y_j) = 0$, so including them does not modify the sum. We then have,

$$\int_{Y_1} f \, \mathrm{d}\mu + \int_{Y_2} f \, \mathrm{d}\mu = \sum_{i=1}^{n} a_i \left(\mu(X_i \cap Y_1) + \mu(X_i \cap Y_2)\right)$$
$$= \sum_{i=1}^{n} a_i \mu((X_i \cap Y_1) \cup (X_i \cap Y_2)) = \sum_{i=1}^{n} a_i \mu(X_i) = \int_{Y} f \, \mathrm{d}\mu.$$

We proceed to demonstrate property (1). We start with the multiplicative property of a semilinear map. Let $f \in \mathcal{S}^+(X)$, $\lambda \in [0, \infty)$ and $g = \lambda f$. If $\lambda = 0$ we have immediately $\int_X g \, \mathrm{d}\mu = 0$, as required. Suppose thus $\lambda \neq 0$. Set $f(X) = \{a_1, \ldots, a_n\}$ and $X_i := f^{-1}(a_i)$. Then $g(X) = \{\lambda a_1, \ldots, \lambda a_n\}$. Note that the values λa_i are all distinct. By definition we then have, as required,

$$\int_X g \, \mathrm{d}\mu = \sum_{i=1}^{n} \lambda a_i \mu(X_i) = \lambda \sum_{i=1}^{n} a_i \mu(X_i) = \lambda \int_X f \, \mathrm{d}\mu.$$

It remains to show additivity of the integral. Thus, let $f, g \in \mathcal{S}^+(X)$. Set $f(X) = \{a_1, \ldots, a_n\}$ and $X_i := f^{-1}(a_i)$ as well as $g(X) = \{b_1, \ldots, b_m\}$ and $Y_j := g^{-1}(b_j)$. Define $Z_{ij} = X_i \cap Y_j$. Note that the Z_{ij} form a disjoint partition of X. Moreover the function $f + g$ takes the constant value $a_i + b_j$ on Z_{ij}. Using property (3) and the definition of the integral we get,

$$\int_X (f + g) \, \mathrm{d}\mu = \sum_{i,j} \int_{Z_{ij}} (f + g) \, \mathrm{d}\mu = \sum_{i,j} (a_i + b_j) \mu(Z_{ij})$$
$$= \sum_{i,j} a_i \mu(Z_{ij}) + \sum_{i,j} b_j \mu(Z_{ij}) = \sum_{i=1}^{n} a_i \mu(X_i) + \sum_{j=1}^{m} b_j \mu(Y_j)$$
$$= \int_X f \, \mathrm{d}\mu + \int_X g \, \mathrm{d}\mu.$$

This completes the proof. $\square$

Definition 3.8. The *μ-integral* for positive measurable functions is the map $\mathcal{L}^+(X) \to [0,\infty]$ defined as follows. Given $f \in \mathcal{L}^+(X)$,

$$\int_X f \,\mathrm{d}\mu := \sup\left\{\int_X g \,\mathrm{d}\mu : 0 \le g \le f \text{ and } g \in \mathcal{S}^+(X)\right\}$$

The coincidence of this definition with the previous one in the case of simple maps is implied by the following result.

Proposition 3.9. *The integrals $\mathcal{L}^+(Y) \to [0,\infty]$ for $Y \in \mathcal{M}$ are a collection of maps with the following properties:*

1. *They coincide with Definition 3.6 for simple maps.*
2. *They are multiplicative, i.e. $\int_Y \lambda f \mathrm{d}\mu = \lambda \int_Y f \mathrm{d}\mu$ for $\lambda \in [0,\infty)$.*
3. *They are positive, i.e. $\int_Y f \,\mathrm{d}\mu \le \int_Y g \,\mathrm{d}\mu$ if $f \le g$.*
4. *Let $Y_1, Y_2 \in \mathcal{M}$ such that $Y_1 \cap Y_2 = \emptyset$ and $Y = Y_1 \cup Y_2$. Then,*

$$\int_Y f \,\mathrm{d}\mu = \int_{Y_1} f \,\mathrm{d}\mu + \int_{Y_2} f \,\mathrm{d}\mu.$$

Proof. **Exercise.** □

Theorem 3.10 (Monotone Convergence Theorem)**.** *Let $\{f_n\}_{n\in\mathbb{N}}$ be an increasing sequence of positive measurable functions on X that converges pointwise to a function $f : X \to [0,\infty)$. Then, f is measurable and*

$$\lim_{n\to\infty} \int_X f_n \,\mathrm{d}\mu = \int_X f \,\mathrm{d}\mu.$$

Proof. The measurability of f follows from Theorem 2.19. We denote,

$$b := \lim_{n\to\infty} \int_X f_n \,\mathrm{d}\mu.$$

Since $f_n \le f$ for all $n \in \mathbb{N}$ positivity of the integral implies,

$$b \le \int_X f \,\mathrm{d}\mu.$$

Let $g \in \mathcal{S}^+(X)$ such that $0 \le g \le f$ and choose $0 < c < 1$. Define $E_n \in \mathcal{M}$ as,

$$E_n := \{x \in X : f_n(x) \ge cg(x)\}$$

Then $\{E_n\}_{n\in\mathbb{N}}$ is an increasing sequence of measurable subsets of X with $X = \bigcup_{n\in N} E_n$. Moreover, for any $n \in \mathbb{N}$,

$$\int_X f_n \,\mathrm{d}\mu \geq \int_{E_n} f_n \,\mathrm{d}\mu \geq c \int_{E_n} g \,\mathrm{d}\mu.$$

The limit $n \to \infty$ exits on both sides. On the left hand side this is b. To see the limit on the right hand side set $g(X) = \{a_1, \ldots, a_m\}$ and $X_i := g^{-1}(\{a_i\})$. Then, we have

$$\int_{E_n} g \,\mathrm{d}\mu = \sum_{i=1}^{m} a_i \mu(E_n \cap X_i).$$

But $\mu(E_n \cap X_i) \to \mu(X_i)$ as $n \to \infty$ by Proposition 2.26.4. We obtain,

$$b \geq c \int_X g \,\mathrm{d}\mu.$$

But c was arbitrary, so the inequality is valid without c. On the other hand, by definition of the integral of f as a supremum of intgerals of simple functions g we obtain,

$$b \geq \int_X f \,\mathrm{d}\mu.$$

Combining both inequalities yields the desired equality. □

Proposition 3.11. *The integral $\mathcal{L}^+(Y) \to [0,\infty]$ is a semilinear map.*

Proof. It remains to show additivity. **Exercise.**Hint: Use approximability from below by simple functions (Theorem 2.23) and apply the Monotone Convergence Theorem 3.10. □

Lemma 3.12 (Fatou's Lemma)**.** *Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of positive measurable functions on X such that $f := \liminf_{n\to\infty} f_n$ takes only finite values. Then, f is measurable and*

$$\int f \leq \liminf_{n\to\infty} \int f_n.$$

Proof. Define $g_n := \inf_{k\geq n} f_k$. Note that g_n is the limit $m \to \infty$ of the decreasing sequence of measurable functions $\{h_{n,m}\}_{m\geq n}$ defined as $h_{n,m} := \inf\{f_k : n \leq k \leq m\}$, so it is measurable. The sequence $\{g_n\}_{n\in\mathbb{N}}$ is increasing. Moreover, $f = \lim_{n\to\infty} g_n$. So, by the Monotone Convergence Theorem 3.10 we have,

$$\lim_{n\to\infty} \int g_n = \int f.$$

On the other hand, by definition of g_n and positivity of the integral we have,

$$\int g_n \leq \int f_k \quad \forall k \geq n.$$

This implies,

$$\int g_n \leq \inf_{k \geq n} \int f_k.$$

Taking the limit yields,

$$\lim_{n\to\infty} \int g_n \leq \liminf_{n\to\infty} \int f_k.$$

This completes the proof. $\square$

3.2 Integrable functions

Let $\mathcal{L}^{1+}(X, \mu)$ denote the subset of $\mathcal{L}^+(X)$ such that the integral is finite,

$$\mathcal{L}^{1+}(X, \mu) := \left\{ f \in \mathcal{L}^+(X) : \int_X f \, \mathrm{d}\mu < \infty \right\}.$$

Define now $\mathcal{L}^1(X, \mu) := \mathcal{L}^{1+}(X, \mu) - \mathcal{L}^{1+}(X, \mu)$. By Proposition 3.4 $\mathcal{L}^1(X, \mu)$ is a vector space and we obtain a uniquely defined positive linear map

$$\int_X : \mathcal{L}^1(X, \mu) \to \mathbb{R}.$$

We call $\mathcal{L}^1(X, \mu)$ the space of *integrable* functions. Note also that given $f, g \in \mathcal{L}^1(X, \mu, \mathbb{R})$, $\sup(f, g)$ and $\inf(f, g)$ are measurable by Proposition 2.18 and integrable by Proposition 3.5.

We may now extend the notion of integral to functions that take values in the complex numbers rather than the real numbers. A further extension to functions taking values in Banach spaces over $\mathbb{R}$ or $\mathbb{C}$ is also straightforward, but we shall not consider this here.

We define the complex vector space $\mathcal{L}^1(X, \mu, \mathbb{C}) := \mathcal{L}^1(X, \mu) + \mathrm{i}\mathcal{L}^1(X, \mu)$ of *integrable* complex valued functions. The integral is extended from $\mathbb{R}$ to $\mathbb{C}$ by complex linearity. For $f = f_R + \mathrm{i} f_I$ with $f_R, f_I \in \mathcal{L}^1(X, \mu)$ we define,

$$\int_X f \, \mathrm{d}\mu := \int_X f_R \, \mathrm{d}\mu + \mathrm{i} \int_X f_I \, \mathrm{d}\mu.$$

We also write $\mathcal{L}^1(X, \mu, \mathbb{R}) := \mathcal{L}^1(X, \mu)$ and $\mathcal{L}^1(X, \mu, \mathbb{K})$ if we want to make statements valid for both cases $\mathbb{K} = \mathbb{R}$ and $\mathbb{K} = \mathbb{C}$.

Proposition 3.13. *We summarize basic properties of the integral.*

1. *The integral is a positive linear function.*

2. $f, g \in \mathcal{L}^1(X, \mu, \mathbb{R})$ *implies* $\sup(f, g), \inf(f, g) \in \mathcal{L}^1(X, \mu, \mathbb{R})$.

3. *Let* $Y_1, Y_2 \in \mathcal{M}$ *such that* $Y_1 \cap Y_2 = \emptyset$ *and* $Y = Y_1 \cup Y_2$. *Then, for* $f \in \mathcal{L}^1(Y, \mu, \mathbb{K})$,

$$\int_Y f \, \mathrm{d}\mu = \int_{Y_1} f \, \mathrm{d}\mu + \int_{Y_2} f \, \mathrm{d}\mu.$$

Proof. It remains to demonstrate the validity of (3). This follows from linearity upon decomposing f into a linear combination of positive integrable functions and Proposition 3.9.(3). □

Theorem 3.14. *Let* $f \in \mathcal{L}(X, \mathbb{K})$. *Then* f *is integrable iff* $|f| \in \mathcal{L}^+(X)$ *is integrable.*

Proof. Without loss of generality take $\mathbb{K} = \mathbb{C}$. Suppose that $f \in \mathcal{L}(X, \mathbb{C})$. Let $f_{\mathrm{R}} = \Re(f)$ and $f_{\mathrm{I}} = \Im(f)$. Define $f_{\mathrm{R}}^+ = \sup(f_{\mathrm{R}}, 0)$ and $f_{\mathrm{R}}^- = \sup(-f_{\mathrm{R}}, 0)$. Similarly, $f_{\mathrm{I}}^+ = \sup(f_{\mathrm{I}}, 0)$ and $f_{\mathrm{I}}^- = \sup(-f_{\mathrm{I}}, 0)$. Note that all these component functions are positive and $f = f_{\mathrm{R}}^+ - f_{\mathrm{R}}^- + \mathrm{i} f_{\mathrm{I}}^+ - \mathrm{i} f_{\mathrm{I}}^-$. Now suppose that f is integrable. Then, by definition both f_{R} and f_{I} are integrable. Moreover, f_{R}^+, f_{R}^-, f_{I}^+, f_{I}^- are all integrable and so is their sum. The inequality

$$|f| \le |f_{\mathrm{R}}| + |f_{\mathrm{I}}| = f_{\mathrm{R}}^+ + f_{\mathrm{R}}^- + f_{\mathrm{I}}^+ + f_{\mathrm{I}}^-$$

implies the integrability of $|f|$. Conversely, suppose that $|f|$ is integrable. But all of f_{R}^+, f_{R}^-, f_{I}^+, f_{I}^- are smaller or equal to $|f|$, so they are all integrable. So is thus their linear combination f. □

Theorem 3.15. *For* $f \in \mathcal{L}^1(X, \mu, \mathbb{K})$, $\left|\int f\right| \le \int |f|$.

Proof. Let $c \in \mathbb{K}$ with $|c| = 1$ such that $|\int f| = c \int f$. Then,

$$\left|\int f\right| = c \int f = \int cf = \int \Re(cf) \le \int |cf| = \int |f|.$$

□

Proposition 3.16. *Let* f *be an integrable map. Then,* f *vanishes outside a* σ*-finite set.*

Proof. **Exercise.** □

Proposition 3.17. *The space $\mathcal{L}^1(X, \mu, \mathbb{K})$ carries a* seminorm *given by*

$$\|f\|_1 := \int_X |f| \, \mathrm{d}\mu.$$

Proof. **Exercise.** □

The fact that we only have a seminorm and not necessarily a norm comes from the inability of the integral to "see" sets of measure zero.

Proposition 3.18. *Let $f \in \mathcal{L}^1(X, \mu, \mathbb{K})$. Then, $\|f\|_1 = 0$ iff f vanishes outside a set of measure zero.*

Proof. **Exercise.** □

We also say "almost everywhere" to mean "outside a set of measure zero".

Theorem 3.19 (Dominated Convergence Theorem)**.** *Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of integrable functions that converges pointwise to a function f. Also assume that there exists a positive integrable function g with $|f_n| \le g$ for all $n \in \mathbb{N}$. Then, f is integrable, $\{f_n\}_{n\in\mathbb{N}}$ converges to f in the $\|\cdot\|_1$-seminorm and $\{\int f_n\}_{n\in\mathbb{N}}$ converges to $\int f$.*

Proof. f is measurable by Theorem 2.19. We have $|f| \le g$ from pointwise convergence. By positivity of the integral and integrability of g this implies the integrability of f. Define a sequence of positive integrable functions via $h_n := 2g - |f - f_n|$. Note that $\{h_n\}_{n\in\mathbb{N}}$ converges pointwise to $2g$. We apply Fatou's Lemma 3.12 to this sequence. This yields,

$$\int 2g \le \liminf_{n\to\infty} \int h_n.$$

We may substract the constant $\int 2g$ on both sides and multiply the inequality by -1 to get,

$$0 \ge \limsup_{n\to\infty} \int |f - f_n|.$$

Since the integrals on the right hand side are bounded from below by 0, the limes superior is actually a proper limes and the inequality is an equality,

$$0 = \lim_{n\to\infty} \int |f - f_n|.$$

This is precisely the convergence of $\{f_n\}_{n\in\mathbb{N}}$ to f in the $\|\cdot\|_1$-seminorm. The convergence of the integral itself follows with Theorem 3.15. □

Theorem 3.20. *Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of integrable functions that converges pointwise to a function f. Also assume there is a constant $c \geq 0$ such that $\|f_n\|_1 \leq c$ for all $n \in \mathbb{N}$. Then, f is integrable.*

Proof. f is measurable by Theorem 2.19. We consider the sequence of absolute value functions $\{|f_n|\}_{n\in\mathbb{N}}$ and apply Fatou's Lemma 3.12. This yields,

$$\int |f| = \int \liminf_{n\to\infty} |f_n| \leq \liminf_{n\to\infty} \int |f_n| \leq c.$$

Thus, $|f|$ is integrable and so is f by Theorem 3.14. □

Theorem 3.21. *Let $\{f_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathcal{L}^1(X,\mu,\mathbb{K})$ with respect to the seminorm $\|\cdot\|_1$. Then, the sequence converges to some $f \in \mathcal{L}^1(x,\mu,\mathbb{K})$ in the seminorm $\|\cdot\|_1$. In particular, $\mathcal{L}^1(X,\mu,\mathbb{K})$ is complete. Furthermore, there exists a subsequence which converges pointwise almost everywhere to f and for any $\epsilon > 0$ converges uniformly to f outside of a set of measure less than ϵ.*

Proof. Since $\{f_n\}_{n\in\mathbb{N}}$ is Cauchy, there exists a subsequence $\{f_{n_k}\}_{k\in\mathbb{N}}$ such that

$$\|f_{n_l} - f_{n_k}\|_1 < 2^{-2k} \quad \forall k \in \mathbb{N} \quad \text{and} \quad \forall l \geq k.$$

Define

$$Y_k := \{x \in X : |f_{n_{k+1}}(x) - f_{n_k}(x)| \geq 2^{-k}\} \quad \forall k \in \mathbb{N}.$$

Then,

$$2^{-k}\mu(Y_k) \leq \int_{Y_k} |f_{n_{k+1}} - f_{n_k}| \leq \int_X |f_{n_{k+1}} - f_{n_k}| \leq 2^{-2k} \quad \forall k \in \mathbb{N}.$$

This implies, $\mu(Y_k) \leq 2^{-k}$ for all $k \in \mathbb{N}$. Define now $Z_j := \bigcup_{k=j}^{\infty} Y_k$ for all $j \in \mathbb{N}$. Then, $\mu(Z_j) \leq 2^{1-j}$ for all $j \in \mathbb{N}$.

Fix $\epsilon > 0$ and choose $j \in \mathbb{N}$ such that $2^{1-j} < \epsilon$. Let $x \in X \setminus Z_j$. Then, for $k \geq j$ we have

$$|f_{n_{k+1}}(x) - f_{n_k}(x)| < 2^{-k}.$$

Thus, the sum $\sum_{k=1}^{\infty} f_{n_{k+1}}(x) - f_{n_k}(x)$ converges absolutely. In particular, the limit

$$f(x) := \lim_{l\to\infty} f_{n_l}(x) = f_{n_1}(x) + \sum_{l=1}^{\infty} f_{n_{l+1}}(x) - f_{n_l}(x)$$

exists. For all $k \geq j$ we have the estimate,

$$|f(x) - f_{n_k}(x)| = \left|\sum_{l=k}^{\infty} f_{n_{l+1}}(x) - f_{n_l}(x)\right| \leq \sum_{l=k}^{\infty} \left|f_{n_{l+1}}(x) - f_{n_l}(x)\right| \leq 2^{1-k}$$

Thus, $\{f_{n_k}\}_{k\in\mathbb{N}}$ converges to f uniformly outside of Z_j, where $\mu(Z_j) < \epsilon$.

Repeating the argument for arbitrarily small ϵ we find that f is defined on $X \setminus Z$, where $Z := \bigcap_{j=1}^{\infty} Z_j$. Furthermore, $\{f_{n_k}\}_{k\in\mathbb{N}}$ converges to f pointwise on $X \setminus Z$. Note that $\mu(Z) = 0$. By Theorem 2.19, f is measurable on $X \setminus Z$. We extend f to a measurable function on all of X by declaring $f(x) = 0$ if $x \in Z$.

Note that the Cauchy property implies that the sequence $\{\|f_{n_k}\|\}_{k\in\mathbb{N}}$ is bounded. Thus we can apply Theorem 3.20 and conclude that f is integrable. It remains to show that $\{f_n\}_{n\in\mathbb{N}}$ converges to f in the seminorm $\|\cdot\|_1$. Let $\epsilon > 0$. By the Cauchy property there exists $m \in \mathbb{N}$ such that $\|f_{n_k} - f_{n_l}\|_1 < \epsilon$ for all $k, l \geq m$. Let $k \geq m$ be arbitrary. We apply Fatou's Lemma 3.12 to the sequence $\{|f_{n_k} - f_{n_l}|\}_{l\in\mathbb{N}}$. This yields,

$$\|f_{n_k} - f\|_1 = \int_X |f_{n_k} - f| = \int_X \liminf_{l\to\infty} |f_{n_k} - f_{n_l}| \leq \liminf_{l\to\infty} \int_X |f_{n_k} - f_{n_l}| \leq \epsilon.$$

Thus, the sequence $\{f_{n_k}\}_{k\in\mathbb{N}}$ converges to f in the seminorm $\|\cdot\|_1$. But since this is a subsequence of the Cauchy sequence $\{f_n\}_{n\in\mathbb{N}}$ the latter must also converge to f. □

Theorem 3.22 (Averaging Theorem)**.** *Let X be a measure space with σ-finite measure μ. Let $S \subseteq \mathbb{K}$ be a closed subset and $f \in \mathcal{L}^1(X, \mu, \mathbb{K})$. If for every measurable set A of finite and positive measure we have*

$$\frac{1}{\mu(A)} \int_A f \mathrm{d}\mu \in S,$$

then $f(x) \in S$ for almost all $x \in X$.

Proof. Let $C := \{x \in X : f(x) \notin S\}$. We need to show that $\mu(C) = 0$. Assume the contrary, i.e., $\mu(C) > 0$. Write $\mathbb{K} \setminus S = \bigcup_{n\in\mathbb{N}} B_n$ as a countable union of closed balls $\{B_n\}_{n\in\mathbb{N}}$. (Use second countability of $\mathbb{K}$ and recall Proposition 1.39.) Their preimages are measurable and cover C. There is at least one closed ball B_n such that $\mu(f^{-1}(B_n)) > 0$. Say this closed ball has center x and radius r. Furthermore, there is a measurable subset

$D \subseteq f^{-1}(B_n)$ such that $0 < \mu(D) < \infty$. Then,

$$\begin{aligned}\left|\frac{1}{\mu(D)}\int_D f\,\mathrm{d}\mu - x\right| &= \frac{1}{\mu(D)}\left|\int_D (f-x)\,\mathrm{d}\mu\right| \\ &\le \frac{1}{\mu(D)}\int_D |f-x|\,\mathrm{d}\mu \le \frac{1}{\mu(D)}\int_D r\,\mathrm{d}\mu = r.\end{aligned}$$

In particular, $\frac{1}{\mu(D)}\int_D f\,\mathrm{d}\mu \in B_n$. But $B_n \cap S = \emptyset$, so we get a contradiction with the assumptions. □

Exercise 22. 1. Explain where in the above proof σ-finiteness was used. 2. Extend the proof to the case where μ is not σ-finite by replacing $f(x) \in S$ with $f(x) \in S \cup \{0\}$ in the statement of the Theorem.

Finally, we return to the simple functions.

Proposition 3.23. *The space of integrable simple functions $\mathcal{S}^1(X,\mu,\mathbb{K})$ is precisely the space of simple functions that vanish outside of a set of finite measure.*

Proof. **Exercise.** □

Lemma 3.24. *Let $f \in \mathcal{L}^1(X,\mu,\mathbb{K})$ and $\epsilon > 0$. Then there exists $g \in \mathcal{S}^1(X,\mu,\mathbb{K})$ such that $\|f-g\|_1 < \epsilon$. In particular, $\mathcal{S}^1(X,\mu,\mathbb{K})$ is a dense subspace of $\mathcal{L}^1(X,\mu,\mathbb{K})$.*

Proof. **Exercise.** □

Lemma 3.25. *Let $(X,\mathcal{M},\mu)$ be a measure space and $\mathcal{N}$ an algebra of subsets of X that generates the σ-algebra $\mathcal{M}$. Let $f \in \mathcal{S}^1(X,\mu,\mathbb{K})$ and $\epsilon > 0$. Then, there exists $g \in \mathcal{S}^1(X,\mu,\mathbb{K})$ such that g is measurable with respect to $\mathcal{N}$ (i.e., $g^{-1}(\{p\}) \subseteq \mathcal{N}$ for all $p \in \mathbb{K}$) and such that $\|f-g\|_1 < \epsilon$.*

Proof. **Exercise.**Hint: Use Proposition 2.37. □

3.3 Exercises

Exercise 23 (Lang)**.** Consider the interval $[0,1]$ with the Lebesgue measure μ. Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of continuous functions $f_n : [0,1] \to [0,1]$ which converges pointwise to 0 everywhere. Show that

$$\lim_{n\to\infty}\int_0^1 f_n\,\mathrm{d}\mu = 0.$$

Exercise 24 (Lang)**.** Let X, Y be measurable spaces and $f : X \to Y$ a measurable map. Denote the σ-algebra on X by $\mathcal{M}$ and the σ-algebra on Y by $\mathcal{N}$. Let μ be a positive measure on $\mathcal{M}$. Define a function $\nu : \mathcal{N} \to [0, \infty]$ as follows: $\nu(N) := \mu(f^{-1}(N))$. Show that ν is a positive measure on $\mathcal{N}$. Moreover show that if $g \in \mathcal{L}^1(Y, \nu)$, then $g \circ f \in \mathcal{L}^1(X, \mu)$ and

$$\int_X g \circ f \, \mathrm{d}\mu = \int_Y g \, \mathrm{d}\nu.$$

Exercise 25 (Lang, extended)**.** Let X be a measure space with finite measure μ and $f \in \mathcal{L}^1(X, \mu)$. Show that the limit

$$\lim_{n\to\infty} \int_X |f|^{1/n} \, \mathrm{d}\mu$$

exists and compute it. Give an example where the limit does not exist if $\mu(X) = \infty$.

Exercise 26 (Fundamental Theorem of Differentiation and Integration)**.** Let $f : \mathbb{R} \to \mathbb{R}$ be continuously differentiable and $a, b \in \mathbb{R}$ with $a \le b$. Then,

$$\int_a^b f' \, \mathrm{d}\mu = f(b) - f(a),$$

where μ is the Lebesgue measure. [Hint: Note that f' is integrable on $[a, b]$. Consider the map $g : \mathbb{R} \to \mathbb{R}$ given by $g(y) := \int_a^y f' \, \mathrm{d}\mu$. Show that g is continuously differentiable and that $g' = f'$. Apply the fact that a function with vanishing derivative is constant to the difference $f - g$ to conclude the proof.]

Exercise 27 (Partial Integration)**.** Let $f, g : \mathbb{R} \to \mathbb{R}$ be continuously differentiable and $a, b \in \mathbb{R}$ with $a \le b$. Show that,

$$\int_a^b f g' \, \mathrm{d}\mu = f g|_a^b - \int_a^b f' g \, \mathrm{d}\mu,$$

where $\mathrm{d}\mu$ is the Lebesgue measure.

Exercise 28 (adapted from Lang)**.** Equip the space $[0, \infty]$ with the topology of the one-point compactification by adding the point ∞ to the interval $[0, \infty)$ with its usual topology. (Recall Exercise 2).

1. Let X be a measurable space and $f : X \to [0, \infty]$. Let $Y := f^{-1}([0, \infty))$. Show that f is a measurable function iff Y is a measurable set and $f|_Y : Y \to [0, \infty)$ is a measurable function.

2. Let X be a measure space with σ-finite measure μ. Show that $f : X \to [0, \infty]$ is measurable iff there exists an increasing sequence $\{f_n\}_{n\in\mathbb{N}}$ of integrable simple functions $f_n : X \to [0, \infty)$ which converges pointwise to f.

3. (X and μ as above.) Let $f : X \to [0, \infty]$ measurable. We define its integral by extension of Definition 3.8. For each measurable subset $A \subseteq X$ define
$$\mu_f(A) := \int_A f \, \mathrm{d}\mu.$$
Show that μ_f is a positive measure. Let $g : X \to [0, \infty]$ measurable and show that,
$$\int_X g \, \mathrm{d}\mu_f = \int_X fg \, \mathrm{d}\mu.$$

4 The spaces $\mathcal{L}^p$ and L^p

4.1 Elementary inequalities and seminorms

Lemma 4.1. *Let $a, b \geq 0$ and $p \geq 1$. Then,*

$$\left(\frac{a+b}{2}\right)^p \leq \frac{a^p + b^p}{2}.$$

Let $a, b \geq 0$ and $p > 1$. Set q such that $1/p + 1/q = 1$. Then,

$$a^{1/p} b^{1/q} \leq \frac{a}{p} + \frac{b}{q}.$$

Proof. **Exercise.** □

Definition 4.2. Let X be a measure space with measure μ and $p > 0$.

$$\mathcal{L}^p(X, \mu, \mathbb{K}) := \{f : X \to \mathbb{K} \,\text{measurable} : |f|^p \,\text{integrable}\}.$$

Define also the function $\|\cdot\|_p : \mathcal{L}^p(X, \mu, \mathbb{K}) \to \mathbb{R}_0^+$ given by

$$\|f\|_p := \left(\int_X |f|^p\right)^{1/p}.$$

Proposition 4.3. *The set $\mathcal{L}^p(X, \mu, \mathbb{K})$ for $p \in (0, \infty)$ is a vector space. Also, $\|\cdot\|_p$ is multiplicative, i.e., $\|\lambda f\|_p = |\lambda| \|f\|_p$ for all $\lambda \in \mathbb{K}$ and $f \in \mathcal{L}^p$. Furthermore, if $p \leq 1$ the function $d_p : \mathcal{L}^p(X, \mu, \mathbb{K}) \times \mathcal{L}^p(X, \mu, \mathbb{K}) \to [0, \infty)$ given by $d_p(f, g) := \|f - g\|_p^p$ is a pseudometric.*

Proof. **Exercise.** □

Definition 4.4. Let X be a measure space with measure μ. We call a measurable function $f : X \to \mathbb{K}$ *essentially bounded* iff there exists a bounded measurable function $g : X \to \mathbb{K}$ such that $g = f$ almost everywhere. We denote the set of essentially bounded functions by $\mathcal{L}^\infty(X, \mu, \mathbb{K})$. Define also the function $\|\cdot\|_\infty : \mathcal{L}^\infty(X, \mu, \mathbb{K}) \to \mathbb{R}_0^+$ given by

$$\|f\|_\infty := \inf\{\|g\|_{\sup} : g = f \,\text{a.e. and}\, g \,\text{bounded measurable}\}.$$

Proposition 4.5. *The set $\mathcal{L}^\infty(X, \mu, \mathbb{K})$ is a vector space and $\|\cdot\|_\infty$ is a seminorm.*

Proof. **Exercise.** □

Proposition 4.6. *Let $f \in \mathcal{L}^p$ for $p \in (0, \infty)$. Then, f vanishes outside of a σ-finite set.*

Proof. By Proposition 3.16, $|f|^p$ vanishes outside a σ-finite set and hence so does f. □

Proposition 4.7. *Let $f \in \mathcal{L}^\infty$. Then, the set $\{x : |f(x)| > \|f\|_\infty\}$ has measure zero. Moreover, there exists $g \in \mathcal{L}^\infty$ bounded such that $g = f$ almost everywhere and $\|g\|_{sup} = \|g\|_\infty = \|f\|_\infty$.*

Proof. Fix $c > 0$ and consider the set $A_c := \{x : |f(x)| \geq \|f\|_\infty + c\}$. Since there exists a bounded measurable function g such that $g = f$ almost everywhere and $\|g\|_{\text{sup}} < \|f\|_\infty + c$ we must have $\mu(A_c) = 0$. Thus $\{A_{1/n}\}_{n \in \mathbb{N}}$ is an increasing sequence of sets of measure zero. So, their union $A := \bigcup_{n \in \mathbb{N}} A_n = \{x : |f(x)| > \|f\|_\infty\}$ must have measure zero. Define now

$$g(x) := \begin{cases} f(x) & \text{if } x \in X \setminus A \\ 0 & \text{if } x \in A \end{cases}.$$

Then, g is measurable, bounded, and $g = f$ almost everywhere. Moreover, $\|g\|_{\text{sup}} \leq \|f\|_\infty$. On, the other hand, since $g = f$ almost everywhere we must have $\|g\|_{\text{sup}} \geq \|f\|_\infty$ by the definition of $\|\cdot\|_\infty$. Also, $f - g = 0$ almost everywhere and hence $\|f - g\|_\infty \leq \|0\|_{\text{sup}}$, i.e., $\|f - g\|_\infty = 0$ and thus $\|f\|_\infty = \|g\|_\infty$. □

Proposition 4.8. *Let $f \in \mathcal{L}^p$ for $p \in (0, \infty]$. Then $\|f\|_p = 0$ iff $f = 0$ almost everywhere.*

Proof. If $p < \infty$ apply Proposition 3.18 to $|f|^p$. **Exercise.**Complete the proof for $p = \infty$. □

Theorem 4.9 (Hölder's inequality)**.** *Let $p \in [1, \infty]$ and q such that $1/p + 1/q = 1$. Given $f \in \mathcal{L}^p$ and $g \in \mathcal{L}^q$ we have $fg \in \mathcal{L}^1$ and,*

$$\|fg\|_1 \leq \|f\|_p \|g\|_q.$$

Proof. First observe that fg is measurable by Proposition 2.18 since f and g are measurable.

We start with the case $p = 1$ and $q = \infty$. (The case $q = 1$ and $p = \infty$ is analogous.) By Proposition 4.7 there is a bounded measurable function $h \in \mathcal{L}^\infty$ such that $h = g$ almost everywhere and $\|h\|_{\text{sup}} = \|g\|_\infty$. We have

$$|fh| \leq |f| \|h\|_{\text{sup}}.$$

Thus, the measurable function $|fh|$ is bounded from above by an integrable function and hence is integrable itself by positivity of the integral. By Theorem 3.14 fh itself is integrable. But $fh = fg$ almost everywhere and so fg is integrable. Moreover, integrating the above inequality over X we obtain,

$$\|fg\|_1 = \int_X |fg| = \int_X |fh| \le \|h\|_{\sup} \int_X |f| = \|f\|_1 \|g\|_\infty.$$

It remains to consider the case $p \in (1, \infty)$. If $\|f\|_p = 0$ or $\|g\|_q = 0$ then f or g vanishes almost everywhere by Proposition 4.8. Thus, fg vanishes almost everywhere and $\|fg\|_1 = 0$ by the same Proposition (and in particular $fg \in \mathcal{L}^1$). We thus assume now $\|f\|_p \neq 0$ and $\|g\|_q \neq 0$. Set

$$a := \frac{|f|^p}{\|f\|_p^p}, \quad \text{and} \quad b := \frac{|g|^q}{\|g\|_q^q}.$$

Using the second inequality of Lemma 4.1 we find,

$$\frac{|fg|}{\|f\|_p \|g\|_q} \le \frac{1}{p} \frac{|f|^p}{\|f\|_p^p} + \frac{1}{q} \frac{|g|^q}{\|g\|_q^q}.$$

This implies that $|fg|$ is bounded from above by an integrable function and is hence integrable by positivity of the integral. Moreover, integrating both sides of the inequality over X yields the inequality that is to be demonstrated. □

Proposition 4.10 (Minkowski's inequality)**.** *Let $p \in [1, \infty]$ and $f, g \in \mathcal{L}^p$. Then,*

$$\|f + g\|_p \le \|f\|_p + \|g\|_p.$$

In particular, $\| \cdot \|_p$ is a seminorm.

Proof. The case $p = 1$ is already implied by Proposition 3.17 while the case $p = \infty$ is implied by Proposition 4.5. We may thus assume $p \in (1, \infty)$. Set q such that $1/p + 1/q = 1$. We have,

$$|f + g|^p \le |f||f + g|^{p-1} + |g||f + g|^{p-1}.$$

Notice that $|f + g|^{p-1} \in \mathcal{L}^q$ so that the two summands on the right hand side are integrable by Theorem 4.9. Integrating on both sides and applying Hölder's inequality to both summands on the right hand side yields,

$$\|f + g\|_p^p \le \|f\|_p \||f + g|^{p-1}\|_q + \|g\|_p \||f + g|^{p-1}\|_q$$

Noticing that $\||f+g|^{p-1}\|_q = \|f+g\|_p^{p-1}$ we find,

$$\|f+g\|_p^p \leq (\|f\|_p + \|g\|_p)\|f+g\|_p^{p-1}.$$

Dividing by $\|f+g\|_p^{p-1}$ yields the desired inequality. This is nothing but the triangle inequality for $\|\cdot\|_p$. The other properties making this into a seminorm are immediately verified. □

4.2 Properties of $\mathcal{L}^p$ spaces

Theorem 4.11 (Dominated Convergence Theorem in $\mathcal{L}^p$)**.** *Let $p \in [1,\infty)$. Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of functions in $\mathcal{L}^p$ such that there exists a real valued function $g \in \mathcal{L}^p$ with $|f_n| \leq g$ for all $n \in \mathbb{N}$. Assume also that $\{f_n\}_{n\in\mathbb{N}}$ converges pointwise almost everywhere to a measurable function f. Then, $f \in \mathcal{L}^p$ and $\{f_n\}_{n\in\mathbb{N}}$ converges to f in the $\|\cdot\|_p$-seminorm.*

Proof. **Exercise.**Adapt the proof of Theorem 3.19. □

Theorem 4.12. *Let $p \in [1,\infty)$ and $\{f_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathcal{L}^p$. Then, the sequence converges to some $f \in \mathcal{L}^p$ in the $\|\cdot\|_p$-seminorm. That is, $\mathcal{L}^p$ is complete. Furthermore, there exists a subsequence which converges pointwise almost everywhere to f and for any $\epsilon > 0$ converges uniformly to f outside of a set of measure less than ϵ.*

Proof. Since $\{f_n\}_{n\in\mathbb{N}}$ is Cauchy, there exists a subsequence $\{f_{n_k}\}_{k\in\mathbb{N}}$ such that

$$\|f_{n_l} - f_{n_k}\|_p < 2^{-2k} \quad \forall k \in \mathbb{N} \quad \text{and} \quad \forall l \geq k.$$

Define

$$Y_k := \{x \in X : |f_{n_{k+1}}(x) - f_{n_k}(x)| \geq 2^{-k}\} \quad \forall k \in \mathbb{N}.$$

Then,

$$2^{-kp}\mu(Y_k) \leq \int_{Y_k} |f_{n_{k+1}} - f_{n_k}|^p \leq \int_X |f_{n_{k+1}} - f_{n_k}|^p < 2^{-2kp} \quad \forall k \in \mathbb{N}.$$

This implies, $\mu(Y_k) < 2^{-kp} \leq 2^{-k}$ for all $k \in \mathbb{N}$. Define now $Z_j := \bigcup_{k=j}^\infty Y_k$ for all $j \in \mathbb{N}$. Then, $\mu(Z_j) \leq 2^{1-j}$ for all $j \in \mathbb{N}$.

Fix $\epsilon > 0$ and choose $j \in \mathbb{N}$ such that $2^{1-j} < \epsilon$. Let $x \in X \setminus Z_j$. Then, for $k \geq j$ we have

$$|f_{n_{k+1}}(x) - f_{n_k}(x)| < 2^{-k}.$$

Thus, the sum $\sum_{k=1}^{\infty} f_{n_{k+1}}(x) - f_{n_k}(x)$ converges absolutely. In particular, the limit

$$f(x) := \lim_{l\to\infty} f_{n_l}(x) = f_{n_1}(x) + \sum_{l=1}^{\infty} f_{n_{l+1}}(x) - f_{n_l}(x)$$

exists. For all $k \ge j$ we have the estimate,

$$|f(x) - f_{n_k}(x)| = \left|\sum_{l=k}^{\infty} f_{n_{l+1}}(x) - f_{n_l}(x)\right| \le \sum_{l=k}^{\infty} \left|f_{n_{l+1}}(x) - f_{n_l}(x)\right| \le 2^{1-k}$$

Thus, $\{f_{n_k}\}_{k\in\mathbb{N}}$ converges to f uniformly outside of Z_j, where $\mu(Z_j) < \epsilon$.

Repeating the argument for arbitrarily small ϵ we find that f is defined on $X \setminus Z$, where $Z := \bigcap_{j=1}^{\infty} Z_j$. Furthermore, $\{f_{n_k}\}_{k\in\mathbb{N}}$ converges to f pointwise on $X \setminus Z$. Note that $\mu(Z) = 0$. By Theorem 2.19, f is measurable on $X \setminus Z$. We extend f to a measurable function on all of X by declaring $f(x) = 0$ if $x \in Z$.

For fixed $k \in \mathbb{N}$ consider the sequence $\{g_l\}_{l\in\mathbb{N}}$ of integrable functions given by

$$g_l := |f_{n_l} - f_{n_k}|^p.$$

Then $g := \liminf_{l\to\infty} g_l$ is equal to $|f - f_{n_k}|^p$ almost everywhere. We apply Fatou's Lemma 3.12 to obtain,

$$\int_X |f - f_{n_k}|^p \le \liminf_{l\to\infty} \int_X |f_{n_l} - f_{n_k}|^p \le 2^{-2kp}.$$

In particular, $f - f_{n_k} \in \mathcal{L}^p$ and so $f \in \mathcal{L}^p$ and

$$\|f - f_{n_k}\|_p \le 2^{-2k}.$$

So $\{f_{n_k}\}_{k\in\mathbb{N}}$ and therefore also $\{f_n\}_{n\in\mathbb{N}}$ converge to f in the $\|\cdot\|_p$-seminorm. □

Theorem 4.13. *Let $\{f_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathcal{L}^\infty$. Then, the sequence converges uniformly almost everywhere to a function $f \in \mathcal{L}^\infty$. Furthermore, the sequence converges to f in the $\mathcal{L}^\infty$-seminorm. In particular, $\mathcal{L}^\infty$ is complete.*

Proof. Define $Z_n := \{x \in X : |f_n(x)| > \|f_n\|_\infty\}$ for all $n \in \mathbb{N}$ and $Y_{n,m} := \{x \in X : |f_n(x) - f_m(x)| > \|f_n - f_m\|_\infty\}$. By Proposition 4.7 $\mu(Z_n) = 0$ for all $n \in \mathbb{N}$ and $\mu(Y_{n,m}) = 0$ for all $n, m \in \mathbb{N}$. Define

$$Z := \left(\bigcup_{n\in\mathbb{N}} Z_n\right) \cup \left(\bigcup_{n,m\in\mathbb{N}} Y_{n,m}\right).$$

Then, $\mu(Z) = 0$. So, $\{f_n(x)\}_{n\in\mathbb{N}}$ converges uniformly on $X \setminus Z$ to some measurable function f. We extend f to a measurable function on all of X by defining $f(x) = 0$ if $x \in Z$. **Exercise.** Complete the proof. $\square$

Proposition 4.14. *Let $p \in [1, \infty)$. Then, $\mathcal{S}^1 \subseteq \mathcal{L}^p$ is a dense subset.*

Proof. **Exercise.** $\square$

Proposition 4.15. *The simple maps form a dense subset of $\mathcal{L}^\infty$.*

Proof. Let $f \in \mathcal{L}^\infty$ and fix $\epsilon > 0$. The statement follows if we can show that there exists a simple map h such that $\|f-h\|_\infty < \epsilon$. By Proposition 4.7 there is a bounded map $g \in \mathcal{L}^\infty$ such that $g = f$ almost everywhere and $\|g\|_{\sup} = \|f\|_\infty$. Since g is bounded, its image $A \subset \mathbb{K}$ is bounded and thus contained in a compact set. This means that we can cover A by a finite number of open balls $\{B_k\}_{k\in\{1,\ldots,n\}}$ of radius ϵ. Denote the centers of the balls by $\{x_k\}_{k\in\{1,\ldots,n\}}$. Now take measurable subsets $C_k \subseteq B_k$ such that $C_i \cap C_j = \emptyset$ if $i \neq j$ while still covering A, i.e., $A \subseteq \bigcup_{k\in\{1,\ldots,n\}} C_k$. (**Exercise.** Explain how this can be done.) Define $D_k := g^{-1}(C_k)$. $\{D_k\}_{k\in\{1,\ldots,k\}}$ form a measurable partition of X. Now set $h(x) := x_k$ if $x \in D_k$. Then, h is simple and $\|f-h\|_\infty = \|g-h\|_\infty \leq \|g-h\|_{\sup} < \epsilon$. $\square$

Exercise 29. Analogues of the Monotone Convergence Theorem (Theorem 3.10) and the Dominated Convergence Theorem (Theorem 3.19 or 4.11) are not true in $\mathcal{L}^\infty$. Give a counterexample to both. More precisely, give a pointwise increasing sequence $\{f_n\}_{n\in\mathbb{N}}$ of real non-negative valued functions $f_n \in \mathcal{L}^\infty$ on some measure space X such that $\{f_n\}_{n\in\mathbb{N}}$ converges pointwise to some $f \in \mathcal{L}^\infty$, but $\{f_n\}_{n\in\mathbb{N}}$ does not converge to any function in the $\|\cdot\|_\infty$-seminorm.

We have seen already that the spaces $\mathcal{L}^p$ with $p \in [1, \infty]$ are vector spaces with a seminorm $\|\cdot\|_p$ and are complete with respect to this seminorm. In order to convert a vector space with a seminorm into a vector space with a norm, we may quotient by those elements whose seminorm is zero.

Definition 4.16. Let $p \in [1, \infty]$. Then the quotient space $\mathcal{L}^p/\sim$ in the sense of Proposition 1.59 is denoted by L^p. It is a Banach space.

Banach spaces have many useful properties that make it easy to work with them. So usually, one works with the spaces L^p instead of the spaces $\mathcal{L}^p$. Nevertheless one can still think of the these as "spaces of functions" even though they are spaces of equivalence classes. But (because of Proposition 4.8) two functions are in one equivalence class only if they are "essentially the same", i.e., equal almost everywhere.

Proposition 4.17. *Let $p, q \in (0, \infty]$ and set $r \in (0, \infty]$ such that $1/r = 1/p + 1/q$. Then, given $f \in \mathcal{L}^p$ and $g \in \mathcal{L}^q$ we have $fg \in \mathcal{L}^r$. Moreover, the following inequality holds,*

$$\|fg\|_r \leq \|f\|_p \|g\|_q.$$

Proof. **Exercise.**[Hint: For $f \in \mathcal{L}^p$ and $g \in \mathcal{L}^q$ apply Hölder's Theorem (Theorem 4.9) to $|f|^r$ and $|g|^r$, in the case $r < \infty$. Treat the case $r = \infty$ separately.] □

Proposition 4.18. *Let $0 < p \leq q < r \leq \infty$. Then, $\mathcal{L}^p \cap \mathcal{L}^r \subseteq \mathcal{L}^q$. Moreover, if $r < \infty$,*

$$\|f\|_q^{q(r-p)} \leq \|f\|_p^{p(r-q)} \|f\|_r^{r(q-p)} \quad \forall f \in \mathcal{L}^p \cap \mathcal{L}^r.$$

If $r = \infty$ we have,

$$\|f\|_q^q \leq \|f\|_p^p \|f\|_\infty^{q-p} \quad \forall f \in \mathcal{L}^p \cap \mathcal{L}^\infty.$$

If $p \geq 1$, then also $\mathrm{L}^p \cap \mathrm{L}^r \subseteq \mathrm{L}^q$.

Proof. **Exercise.** □

Proposition 4.19. *Let X be a measure space with finite measure μ. Let $0 < p \leq q \leq \infty$. Then, $\mathcal{L}^q(X, \mu) \subseteq \mathcal{L}^p(X, \mu)$. Moreover,*

$$\|f\|_p \leq \|f\|_q \, (\mu(X))^{1/p - 1/q} \quad \forall f \in \mathcal{L}^q(X, \mu).$$

If $p \geq 1$, then also $\mathrm{L}^q(X, \mu) \subseteq \mathrm{L}^p(X, \mu)$.

Proof. **Exercise.** □

Lemma 4.20. *Let X be a measure space with σ-finite measure μ and let $p \in (0, \infty)$. Then, there exists a function $w \in \mathcal{L}^p(X, \mu)$ such that $0 < w < 1$.*

Proof. Let $\{X_n\}_{n \in \mathbb{N}}$ be a sequence of disjoint sets of finite measure such that $X = \bigcup_{n \in \mathbb{N}} X_n$. Define

$$w(x) := \left(\frac{2^{-n}}{1 + \mu(X_n)} \right)^{1/p} \quad \text{if}\, x \in X_n.$$

This has the desired properties. **Exercise.**Show this. □

<u>Exercise</u> 30 (adapted from Lang)**.** Let X be a measure space with σ-finite measure μ and let $p \in [1, \infty)$. Let $T : \mathcal{L}^p \to \mathcal{L}^p$ be a bounded linear map. For each $g \in \mathcal{L}^\infty$ consider the bounded linear map $M_g : \mathcal{L}^p \to \mathcal{L}^p$ given by $f \mapsto gf$. Assume that T and M_g commute for all $g \in \mathcal{L}^\infty$, i.e., $T \circ M_g = M_g \circ T$. Show that $T = M_h$ for some $h \in \mathcal{L}^\infty$. [Hint: Use Lemma 4.20 to obtain a function $w \in \mathcal{L}^p \cap \mathcal{L}^\infty$ with $0 < w$. Then, for $f \in \mathcal{L}^p \cap \mathcal{L}^\infty$ we have

$$T(wf) = wT(f) = fT(w).$$

If we define $h := T(w)/w$ we thus have $T(f) = hf$. Prove that h is essentially bounded by contradiction: Assume it is not and consider sets of positive measure where $|h| > c$ for some constant c and evaluate T on the characteristic function of such sets. Finally, prove that $T(f) = hf$ for all $f \in \mathcal{L}^p$.]

4.3 Hilbert spaces and L^2

Definition 4.21. Let V be a complex vector space and $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$ a map. $\langle \cdot, \cdot \rangle$ is called a *sesquilinear* form iff it satisfies the following properties:

- $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$ and
  $\langle u, v + w \rangle = \langle u, v \rangle + \langle u, w \rangle$ for all $u, v, w \in V$.
- $\langle \lambda u, v \rangle = \lambda \langle u, v \rangle$ and $\langle u, \lambda v \rangle = \overline{\lambda} \langle u, v \rangle$ for all $\lambda \in \mathbb{C}$ and $v \in V$.

$\langle \cdot, \cdot \rangle$ is called *hermitian* iff it satisfies in addition the following property:

- $\langle u, v \rangle = \overline{\langle v, u \rangle}$ for all $u, v \in V$.

$\langle \cdot, \cdot \rangle$ is called *positive* iff it satisfies in addition the following property:

- $\langle v, v \rangle \geq 0$ for all $v \in V$.

$\langle \cdot, \cdot \rangle$ is called *definite* iff it satisfies in addition the following property:

- If $\langle v, v \rangle = 0$ then $v = 0$ for all $v \in V$.

Proposition 4.22 (from Lang)**.** *Let V be a complex vector space with a positive hermitian sesquilinear form $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$. If $v \in V$ is such that $\langle v, v \rangle = 0$, then $\langle v, w \rangle = \langle w, v \rangle = 0$ for all $w \in V$.*

Proof. Suppose $\langle v, v\rangle = 0$ for a fixed $v \in V$. Fix some $w \in V$. For any $t \in \mathbb{R}$ we have,

$$0 \le \langle tv + w, tv + w\rangle = 2t\,\Re(\langle v, w\rangle) + \langle w, w\rangle.$$

If $\Re(\langle v, w\rangle) \neq 0$ we could find $t \in \mathbb{R}$ such that the right hand side would be negative, a contradiction. Hence, we can conclude $\Re(\langle v, w\rangle) = 0$, for all $w \in V$. Thus, also $0 = \Re(\langle v, \mathrm{i}w\rangle) = \Re(-\mathrm{i}\langle v, w\rangle) = \Im(\langle v, w\rangle)$ for all $w \in V$. Hence, $\langle v, w\rangle = 0$ and $\langle w, v\rangle = \overline{\langle v, w\rangle} = 0$ for all $w \in V$. □

Theorem 4.23 (Schwarz Inequality)**.** *Let V be a complex vector space with a positive hermitian sesquilinear form $\langle\cdot,\cdot\rangle : V \times V \to \mathbb{C}$. Then, the following inequality is satisfied:*

$$|\langle v, w\rangle|^2 \le \langle v, v\rangle\langle w, w\rangle \quad \forall v, w \in V.$$

Proof. If $\langle v, v\rangle = 0$ then also $\langle v, w\rangle = 0$ by Proposition 4.22 and the inequality holds. Thus, we may assume $\alpha := \langle v, v\rangle \neq 0$ and we set $\beta := -\langle w, v\rangle$. By positivity we have,

$$0 \le \langle \beta v + \alpha w, \beta v + \alpha w\rangle.$$

Using sesquilinearity and hermiticity on the right hand side this yields,

$$0 \le |\langle v, v\rangle|^2\langle w, w\rangle - \langle v, v\rangle|\langle v, w\rangle|^2.$$

(**Exercise**.Show this.) Since $\langle v, v\rangle \neq 0$ we can divide by it and arrive at the required inequality. □

Proposition 4.24. *Let V be a complex vector space with a positive hermitian sesquilinear form $\langle\cdot,\cdot\rangle : V \times V \to \mathbb{C}$. Then, V carries a seminorm given by $\|v\| := \sqrt{\langle v, v\rangle}$. If $\langle\cdot,\cdot\rangle$ is also definite then $\|\cdot\|$ is a norm.*

Proof. **Exercise**.Hint: To prove the triangle inequality, show that $\|v + w\|^2 \le (\|v\| + \|w\|)^2$ can be derived from the Schwarz inequality (Theorem 4.23). □

Definition 4.25. A positive definite hermitian sesquilinear form is also called an *inner product* or a *scalar product.* A complex vector space equipped with such a form is called an *inner product space* or a *pre-Hilbert space.* It is called a *Hilbert space* iff it is complete with respect to the induced norm.

Proposition 4.26. *Consider the map $\langle\cdot,\cdot\rangle : \mathcal{L}^2 \times \mathcal{L}^2 \to \mathbb{C}$ given by*

$$\langle f, g\rangle := \int f\overline{g}.$$

Then, $\langle\cdot,\cdot\rangle$ is a positive hermitian sesquilinear form on $\mathcal{L}^2$. Moreover, the seminorm induced by it according to Proposition 4.24 is the $\|\cdot\|_2$-seminorm. Also, the map $\langle\cdot,\cdot\rangle : \mathrm{L}^2 \times \mathrm{L}^2 \to \mathbb{C}$ given by $\langle [f], [g]\rangle := \langle f, g\rangle$ defines a positive definite hermitian sesquilinear form on L^2. The norm induced by it is the $\|\cdot\|_2$-norm. This makes L^2 into a Hilbert space.

Proof. **Exercise.** □

The following Theorem about Hilbert spaces is fundamental, but we do not include the proof here, as we will only use it one single time.

Theorem 4.27. *Let H be a complex Hilbert space and $\alpha : H \to \mathbb{C}$ a bounded linear map. Then, there exists a unique element $w \in H$ such that*

$$\alpha(v) = \langle v, w\rangle \quad \forall v \in H.$$

5 Measures and integrals on product spaces

5.1 The Product of measures

Definition 5.1. Let S, T be sets and $\mathcal{M} \subseteq \mathfrak{P}(S)$, $\mathcal{N} \subseteq \mathfrak{P}(T)$ be algebras of subsets. For $(A, B) \in \mathcal{M} \times \mathcal{N}$ we view $A \times B$ as a subset of $S \times T$, called a *rectangle*. We denote the set of rectangles by $\mathcal{M} \times \mathcal{N} \subseteq \mathfrak{P}(S \times T)$. Then, $\mathcal{M} \square \mathcal{N} \subseteq \mathfrak{P}(S \times T)$ denotes the algebra generated by the set of rectangles. We also call this the *product algebra*. Similarly, $\mathcal{M} \boxtimes \mathcal{N}$ denotes the σ-algebra generated by $\mathcal{M} \square \mathcal{N}$ which we call the *product σ-algebra*.

Proposition 5.2. *$\mathcal{M} \square \mathcal{N}$ consists of the finite disjoint union of elements of $\mathcal{M} \times \mathcal{N}$.*

Proof. **Exercise**. □

Proposition 5.3. *Let $\mathcal{M}'$, $\mathcal{N}'$ be the σ-algebras generated by $\mathcal{M}$ and $\mathcal{N}$ respectively. Then,*

$$\mathcal{N} \boxtimes \mathcal{M} = \mathcal{N}' \boxtimes \mathcal{M}'.$$

Proof. **Exercise**. □

Lemma 5.4. *Let $(S, \mathcal{M})$, $(T, \mathcal{N})$ be measurable spaces. Let $U \in \mathcal{M} \boxtimes \mathcal{N}$ and $p \in S$. Set $U_p := \{q \in T : (p, q) \in U\} \subseteq T$. Then, $U_p \in \mathcal{N}$.*

Proof. Let $\mathcal{A}$ denote the set of subsets $V \subseteq S \times T$ such that $V \in \mathcal{M} \boxtimes \mathcal{N}$ and $V_p \in \mathcal{N}$. Let $(A, B) \in \mathcal{M} \times \mathcal{N}$. Then the rectangle $A \times B$ is in $\mathcal{A}$ since $(A \times B)_p = B$ if $p \in A$ and $(A \times B)_p = \emptyset$ otherwise. Thus, all rectangles are in $\mathcal{A}$. Moreover, $\mathcal{A}$ is an algebra: Clearly $\emptyset \in \mathcal{A}$. Also, if $V \in \mathcal{A}$, then $\neg V \in A$ since $(\neg V)_p = \neg(V_p)$. Similarly, for $A, B \in \mathcal{A}$ we have $(A \cap B)_p = A_p \cap B_p$. So, $\mathcal{M} \square \mathcal{N} \subseteq \mathcal{A}$. But $\mathcal{A}$ is even a σ-algebra: Let $(A_n)_{n \in \mathbb{N}}$ be a sequence of elements of $\mathcal{A}$. Then, $(\bigcup_{n \in \mathbb{N}} A_n)_p = \bigcup_{n \in \mathbb{N}} (A_n)_p$. Thus, $\mathcal{M} \boxtimes \mathcal{N} \subseteq \mathcal{A}$. But $\mathcal{A} \subseteq \mathcal{M} \boxtimes \mathcal{N}$ by construction. □

Lemma 5.5. *Let $(S, \mathcal{M})$, $(T, \mathcal{N})$, $(U, \mathcal{A})$ be measurable spaces and $f : S \times T \to U$ a measurable map, where $S \times T$ is equipped with the product σ-algebra $\mathcal{M} \boxtimes \mathcal{N}$. For $p \in S$ denote by $f_p : T \to U$ the map $f_p(q) := f(p, q)$. Then, f_p is measurable for all $p \in S$.*

Proof. Let $V \in \mathcal{A}$. Then, $f_p^{-1}(V) = (f^{-1}(V))_p$, using the notation of Lemma 5.4. But by that same Lemma, $(f^{-1}(V))_p \in \mathcal{N}$. □

Theorem 5.6. *Let $(S, \mathcal{M}, \mu)$ and $(T, \mathcal{N}, \nu)$ be measure spaces with σ-finite measures. Then, there exists a unique measure $\mu \boxtimes \nu$ on the measurable space $(S \times T, \mathcal{M} \boxtimes \mathcal{N})$ such that for sets of finite measure $A \in \mathcal{M}$ and $B \in \mathcal{N}$ we have*

$$(\mu \boxtimes \nu)(A \times B) = \mu(A)\nu(B).$$

Proof. At first we assume the measures to be finite. It is clear from Proposition 5.2 that $\mu \boxtimes \nu$, if it exists, is uniquely determined on $\mathcal{M} \square \mathcal{N}$ by additivity. A priori it is not clear, however, if $\mu \boxtimes \nu$ can be well defined even merely on $\mathcal{M} \square \mathcal{N}$, since a given element of $\mathcal{M} \square \mathcal{N}$ can be presented as a disjoint union of rectangles in different ways. For $U \in \mathcal{M} \square \mathcal{N}$ define $\alpha_U : S \to \mathbb{R}_0^+$ by $\alpha_U(p) := \nu(U_p)$. If $U = A \times B$ is a rectangle, we have $\alpha_U(p) = \chi_A(p)\nu(B)$ for $p \in S$. In particular, α_U is integrable on S and we have

$$\mu(A)\nu(B) = \int_S \alpha_U \, \mathrm{d}\mu.$$

For U a finite disjoint union of rectangles the function α_U is simply the sum of the corresponding functions for the individual rectangles and is thus integrable on S. In particular, we must have

$$(\mu \boxtimes \nu)(U) = \int_S \alpha_U \, \mathrm{d}\mu,$$

incidentally showing that $\mu \boxtimes \nu$ is well defined on $\mathcal{M} \square \mathcal{N}$.

We proceed to show that $\mu \boxtimes \nu$ is countably additive on $\mathcal{M} \square \mathcal{N}$. Let $\{U_n\}_{n \in \mathbb{N}}$ be an increasing sequence of elements of $\mathcal{M} \square \mathcal{N}$ such that $U := \bigcup_{n \in \mathbb{N}} U_n \in \mathcal{M} \square \mathcal{N}$. Then, $\{\alpha_{U_n}\}_{n \in \mathbb{N}}$ is an increasing sequence of integrable positive functions on S and we can apply the Monotone Convergence Theorem 3.10. Since α_{U_n} converges pointwise to α_U we must have

$$\lim_{n \to \infty} \int_S \alpha_{U_n} \, \mathrm{d}\mu = \int_S \alpha_U \, \mathrm{d}\mu.$$

That is, $\lim_{n \to \infty} (\mu \boxtimes \nu)(U_n) = (\mu \boxtimes \nu)(U)$, implying countable additivity. It is now guaranteed by Hahn's Theorem 2.35 and Proposition 2.36 that $\mu \boxtimes \nu$ extends to a measure on $\mathcal{M} \boxtimes \mathcal{N}$, and uniquely so.

It remains to consider the case of σ-finite measures. **Exercise.** $\square$

Exercise 31. Show whether the operation of taking the product measure is associative.

Exercise 32. Show that the Lebesgue measure on $\mathbb{R}^{n+m}$ is the completion of the product measure of the Lesbegue measures on $\mathbb{R}^n$ and $\mathbb{R}^m$.

In the following we denote the completion of a σ-algebra $\mathcal{A}$ with respect to a given measure by $\mathcal{A}^*$.

Lemma 5.7. *Let $(S, \mathcal{M}, \mu)$ and $(T, \mathcal{N}, \nu)$ be measure spaces with σ-finite complete measures. Let $Z \in (\mathcal{M} \boxtimes \mathcal{N})^*$ of measure 0. Then, for almost all $p \in S$ we have $\nu(Z_p) = 0$.*

Proof. We consider first the case that the measures are finite. For all $n \in \mathbb{N}$ define $Y_n := \{p \in S : \nu(Z_p) \geq 1/n\}$. Now fix $n \in \mathbb{N}$ and $j \in \mathbb{N}$. Since the algebra $\mathcal{N} \square \mathcal{M}$ generates the σ-algebra $\mathcal{N} \boxtimes \mathcal{M}$, Theorem 2.35, implies that there is a sequence of disjoint rectangles $\{A_{j,k} \times B_{j,k}\}_{k\in\mathbb{N}}$ such that $Z \subseteq R_j$ and $(\mu \boxtimes \nu)(R_j) < 1/(nj)$, where $R_j := \bigcup_{k=1}^{\infty}(A_{j,k} \times B_{j,k})$. Define now $X_j := \{p \in S : \nu((R_j)_p) \geq 1/n\}$. Obviously, $Y_n \subseteq X_j$. Moreover, X_j is measurable since $p \mapsto \nu((R_j)_p) = \sum_{k=1}^{\infty} \chi_{A_{j,k}}(p)\nu(B_{j,k})$ is measurable, being a pointwise limit of measurable functions (Theorem 2.19). We have then,

$$
\begin{aligned}
(\mu \boxtimes \nu)(R_j) &= \sum_{k=1}^{\infty} \mu(A_{j,k})\nu(B_{j,k}) = \sum_{k=1}^{\infty} \int_S \chi_{A_{j,k}}(p)\nu(B_{j,k}) \,\mathrm{d}\mu(p) \\
&= \int_S \sum_{k=1}^{\infty} \chi_{A_{j,k}}(p)\nu(B_{j,k}) \,\mathrm{d}\mu(p) = \int_S \nu((R_j)_p) \,\mathrm{d}\mu(p) \\
&\geq \int_{X_j} \nu((R_j)_p) \,\mathrm{d}\mu(p) \geq \int_{X_j} \frac{1}{n} \,\mathrm{d}\mu = \frac{1}{n}\mu(X_j)
\end{aligned}
$$

(**<u>Exercise</u>.**Justify the interchange of sum and integral!) Thus we get the estimate $\mu(X_j) < 1/j$. Repeating the construction for all $j \in \mathbb{N}$ set $X := \bigcap_{j=1}^{\infty} X_j$. We then have $Y_n \subseteq X$, but $\mu(X) = 0$. Thus, since μ is complete, Y_n is measurable and has measure 0. This in turn implies that $Y := \{p \in S : \nu(Z_p) > 0\} = \bigcup_{n=1}^{\infty} Y_n$ has measure 0 as required. **<u>Exercise</u>.**Complete the proof for the σ-finite case! □

5.2 Fubini's Theorem

Lemma 5.8. *Let $(S, \mathcal{M}, \mu)$ and $(T, \mathcal{N}, \nu)$ be measure spaces with σ-finite measures. Let $A \times B \subseteq S \times T$ be a rectangle such that $0 < (\mu \boxtimes \nu)(A \times B) < \infty$. Then, $0 < \mu(A) < \infty$ and $0 < \nu(B) < \infty$.*

Proof. **<u>Exercise</u>.** □

Lemma 5.9. *Let $(S, \mathcal{M}, \mu)$ and $(T, \mathcal{N}, \nu)$ be measure spaces with σ-finite complete measures. Let $\{(\lambda_1, A_1, B_1), \ldots, (\lambda_n, A_n, B_n)\}$ be triples of elements of $\mathbb{K}, \mathcal{M}, \mathcal{N}$ respectively and such that $0 \leq \mu(A_i) < \infty$ and $0 \leq$*

$\nu(B_i) < \infty$. Define $g : S \times T \to \mathbb{K}$ by

$$g(p,q) := \sum_{k=1}^{n} \lambda_k \chi_{A_k}(p) \chi_{B_k}(q).$$

Then, $g \in \mathcal{S}^1(S \times T, \mu \boxtimes \nu)$. Moreover, $g_p \in \mathcal{S}^1(T, \nu)$ for all $p \in S$ and

$$p \mapsto \int_T g_p \, \mathrm{d}\nu$$

defines a function in $\mathcal{S}^1(S, \mu)$ satisfying

$$\int_S \left(\int_T g_p \, \mathrm{d}\nu \right) \mathrm{d}\mu(p) = \int_{S \times T} g \, \mathrm{d}(\mu \boxtimes \nu).$$

Proof. **Exercise.** □

Theorem 5.10 (Fubini's Theorem, Part 1)**.** *Let $(S, \mathcal{M}, \mu)$ and $(T, \mathcal{N}, \nu)$ be measure spaces with σ-finite complete measures and $f \in \mathcal{L}^1(S \times T, (\mathcal{M} \boxtimes \mathcal{N})^*, \mu \boxtimes \nu)$. Then, $f_p \in \mathcal{L}^1(T, \mathcal{N}, \nu)$ for almost all $p \in S$ and*

$$p \mapsto \int_T f_p \, \mathrm{d}\nu$$

defines almost everywhere a function in $\mathcal{L}^1(S, \mathcal{M}, \mu)$ satisfying

$$\int_S \left(\int_T f_p \, \mathrm{d}\nu \right) \mathrm{d}\mu(p) = \int_{S \times T} f \, \mathrm{d}(\mu \boxtimes \nu).$$

Proof. By Lemmas 3.24 and 3.25 there is a sequence $\{f_n\}_{n \in \mathbb{N}}$ of integrable simple functions, measurable with respect to $\mathcal{M} \square \mathcal{N}$, that converges to f in the $\|\cdot\|_1$-seminorm. Each function f_n can be written as a linear combination of characteristic functions on elements of $\mathcal{M} \square \mathcal{N}$ with finite measure. By modifying f_n if necessary, but without affecting convergence of the sequence we can also arrange that the supports of the characteristic functions all have non-zero measure. Due to Theorem 3.21, by replacing $\{f_n\}_{n \in \mathbb{N}}$ with a subsequence if necessary, we can ensure moreover pointwise convergence to f, except on a set N of measure zero. Taking into account Lemma 5.8 we notice that the functions f_n satisfy the conditions of Lemma 5.9.

By Lemma 5.7, there exists a subset $X \subseteq S$ with measure 0 such that $\nu(N_p) = 0$ if $p \notin X$. Fix for the moment $p \in S \setminus X$. Then, $\{(f_n)_p\}_{n \in \mathbb{N}}$ converges to f_p pointwise outside N_p. Moreover, since the $(f_n)_p$ are measurable with respect to $(T, \mathcal{N})$ by construction, so is f_p outside of N_p due to Theorem 2.19. But, N_p has measure zero and $(T, \mathcal{N}, \nu)$ is complete by assumption, so f_p is measurable everywhere.

Since $\{f_n\}_{n\in\mathbb{N}}$ is Cauchy, we can restrict to a subsequence such that

$$\|f_l - f_k\|_1 < 2^{-2k} \quad \forall k \in \mathbb{N}, \forall l \geq k.$$

By applying Lemma 5.9 to $|f_l - f_k|$, we have for all $k \in \mathbb{N}$ and $l \geq k$,

$$\int_S \|(f_l)_p - (f_k)_p\|_{1,\nu} \, \mathrm{d}\mu(p) = \int_S \left(\int_T |(f_l)_p - (f_k)_p| \, \mathrm{d}\nu\right) \mathrm{d}\mu(p)$$
$$= \int_S \left(\int_T |f_l - f_k|_p \, \mathrm{d}\nu\right) \mathrm{d}\mu(p) = \int_{S\times T} |f_l - f_k| \, \mathrm{d}(\mu\boxtimes\nu) = \|f_l - f_k\|_1 < 2^{-2k}.$$

Now for $k \in \mathbb{N}$ set $Y_k \subseteq S$ to

$$Y_k := \left\{p \in S : \|(f_{k+1})_p - (f_k)_p\|_{1,\nu} \geq 2^{-k}\right\}.$$

Then, for all $k \in \mathbb{N}$,

$$2^{-k}\mu(Y_k) \leq \int_{Y_k} \|(f_{k+1})_p - (f_k)_p\|_{1,\nu} \mathrm{d}\mu(p)$$
$$\leq \int_S \|(f_{k+1})_p - (f_k)_p\|_{1,\nu} \mathrm{d}\mu(p) \leq 2^{-2k}.$$

This implies, $\mu(Y_k) \leq 2^{-k}$ for all $k \in \mathbb{N}$. Define now $Z_j := \bigcup_{k=j}^{\infty} Y_k$ for all $j \in \mathbb{N}$. Then, $\mu(Z_j) \leq 2^{1-j}$ for all $j \in \mathbb{N}$.

Fix $j \in \mathbb{N}$ and let $p \in S \setminus Z_j$. Then, for $k \geq j$ we have

$$\|(f_{k+1})_p - (f_k)_p\|_{1,\nu} < 2^{-k}.$$

This implies for $k \geq j$ and $l \geq k$,

$$\|(f_l)_p - (f_k)_p\|_{1,\nu} < 2^{1-k}.$$

In particular, $\{(f_n)_p\}_{n\in\mathbb{N}}$ is a Cauchy sequence with respect to the $\|\cdot\|_{1,\nu}$-seminorm. Since j was arbitrary, this remains true for $p \in S \setminus Z$, where $Z := \bigcap_{j=1}^{\infty} Z_j$. Note that $\mu(Z) = 0$. Now let $p \in S \setminus (X \cup Z)$. Since $\{(f_n)_p\}_{n\in\mathbb{N}}$ converges to f_p pointwise almost everywhere, and f_p is measurable, Theorem 3.21 then implies that f_p is integrable and that $\{(f_n)_p\}_{n\in\mathbb{N}}$ converges to f_p in the $\|\cdot\|_{1,\nu}$-seminorm.

Now define

$$h_n : p \mapsto \int_T (f_n)_p \, \mathrm{d}\nu$$

By Lemma 5.9 this is an integrable simple map and by the previous arguments it converges pointwise outside of $X \cup Z$ to

$$h : p \mapsto \int_T (f)_p \, \mathrm{d}\nu.$$

Thus, h is measurable in $S \setminus (X \cup Z)$ by Theorem 2.19 and can be extended to a measurable function on all of S, for example by setting $h(p) = 0$ if $p \in X \cup Z$. On the other hand, $\{h_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence with respect to the $\|\cdot\|_{1,\mu}$-seminorm since, for all $l, k \in \mathbb{N}$,

$$\begin{aligned}\|h_l - h_k\|_{1,\mu} &= \int_S |h_l - h_k| \mathrm{d}\mu = \int_S \left| \int_T ((f_l)_p - (f_k)_p) \, \mathrm{d}\nu \right| \mathrm{d}\mu(p) \\ &\le \int_S \left(\int_T |(f_l)_p - (f_k)_p| \, \mathrm{d}\nu \right) \mathrm{d}\mu(p) = \|f_l - f_k\|_1\end{aligned}$$

and $\{f_n\}_{n\in\mathbb{N}}$ is Cauchy. Thus, by Theorem 3.21, h is integrable and $\{h_n\}_{n\in\mathbb{N}}$ converges to h in the $\|\cdot\|_{1,\mu}$-seminorm. Then,

$$\begin{aligned}\int_{S\times T} f \, \mathrm{d}(\mu \boxtimes \nu) &= \lim_{n\to\infty} \int_{S\times T} f_n \, \mathrm{d}(\mu \boxtimes \nu) = \lim_{n\to\infty} \int_S \left(\int_T (f_n)_p \, \mathrm{d}\nu \right) \mathrm{d}\mu(p) \\ &= \lim_{n\to\infty} \int_S h_n \, \mathrm{d}\mu = \int_S h \, \mathrm{d}\mu = \int_S \left(\int_T f_p \, \mathrm{d}\nu \right) \mathrm{d}\mu(p).\end{aligned}$$

□

Lemma 5.11. *Let $(S, \mathcal{M}, \mu)$ and $(T, \mathcal{N}, \nu)$ be measure spaces with σ-finite complete measures and $f : S \times T \to \mathbb{K}$ measurable with respect to $(\mathcal{M} \boxtimes \mathcal{N})^*$. Then, for almost all $p \in S$, f_p is measurable with respect to $\mathcal{N}$.*

Proof. By Proposition 2.30, there is a function $g : S \times T \to \mathbb{K}$ that is measurable with respect to $\mathcal{M} \boxtimes \mathcal{N}$ and such that g coincides with f at least outside a set $N \in \mathcal{M} \boxtimes \mathcal{N}$ of measure 0. By Lemma 5.5, g_p is measurable for all $p \in S$. By Lemma 5.7, $\nu(N_p) = 0$ for all $p \in S \setminus Y$, where $Y \in \mathcal{M}$ is of measure 0. Let $p \in S \setminus Y$, then g_p coincides with f_p almost everywhere and since $(T, \mathcal{N}, \nu)$ is complete f_p must be measurable. □

Theorem 5.12 (Fubini's Theorem, Part 2)**.** *Let $(S, \mathcal{M}, \mu)$ and $(T, \mathcal{N}, \nu)$ be measure spaces with σ-finite complete measures and $f : S \times T \to \mathbb{K}$ be measurable with respect to $(\mathcal{M} \boxtimes \mathcal{N})^*$. Suppose that $f_p \in \mathcal{L}^1(T, \mathcal{N}, \nu)$ for almost all $p \in S$. Moreover suppose that the function*

$$p \mapsto \int_T |f_p| \, \mathrm{d}\nu$$

defined almost everywhere in this way is in $\mathcal{L}^1(S, \mathcal{M}, \mu)$*. Then,* $f \in \mathcal{L}^1(S \times T, (\mathcal{N} \boxtimes \mathcal{M})^*, \mu \boxtimes \nu)$*.*

Proof. Denote by $X \in \mathcal{M}$ a set of measure 0 such that $f_p \in \mathcal{L}^1(T, \mathcal{N}, \nu)$ for $p \in S \setminus X$. By Theorem 2.23 there exists a an increasing sequence $\{f_n\}_{n\in\mathbb{N}}$ of simple functions $f_n : S \times T \to \mathbb{R}_0^+$ with respect to $(\mathcal{M} \boxtimes \mathcal{N})^*$ that converges pointwise to $|f|$. Moreover, because of σ-finiteness the f_n can be chosen to have finite support. (**Exercise.**Explain!) In particular, this implies that each f_n is integrable. Applying Theorem 5.10 to f_n yields a set $N_n \in \mathcal{M}$ of measure 0 such that $(f_n)_p \in \mathcal{L}^1(T, \mathcal{N}, \nu)$ for all $p \in S \setminus N_n$. Moreover, it implies that $h_n : S \to \mathbb{R}_0^+$ defined by $h_n(p) := \int_T (f_n)_p \,\mathrm{d}\nu$ for $p \in S \setminus N_n$ and $h_n(p) = 0$ otherwise, is integrable. Also it implies,

$$\int_S h_n \,\mathrm{d}\mu = \int_{S\times T} f_n \,\mathrm{d}(\mu \boxtimes \nu)$$

Let $N := \bigcup_{n\in\mathbb{N}} N_n$. This has measure 0. Note that since $f_n \le |f|$ for all $n \in \mathbb{N}$ we also have $h_n(p) \le \int_T |f_p| \,\mathrm{d}\nu$ for all $p \in S \setminus \{N \cup X\}$. Putting things together we get for all $n \in \mathbb{N}$

$$\int_{S\times T} f_n \,\mathrm{d}(\mu \boxtimes \nu) = \int_S h_n \,\mathrm{d}\mu \le \int_S \left(\int_T |f_p| \,\mathrm{d}\nu \right) \mathrm{d}\mu$$

Thus, by the Monotone Convergence Theorem 3.10, $\{f_n\}_{n\in\mathbb{N}}$ converges pointwise almost everywhere to an integrable function. But $\{f_n\}_{n\in\mathbb{N}}$ converges pointwise to $|f|$, which is measurable, so $|f|$ must be integrable. Then, by Theorem 3.14, f is integrable. $\square$

6 Relations between measures

Proposition 6.1. *Let X be a measured space with σ-algebra $\mathcal{M}$. Let μ_1, μ_2 be positive measures on $\mathcal{M}$. Then, $\mu := \mu_1 + \mu_2$ is a positive measure on $(X, \mathcal{M})$. Moreover, $\mathcal{L}^1(\mu) = \mathcal{L}^1(\mu_1) \cap \mathcal{L}^1(\mu_2)$ and*

$$\int_A f \, \mathrm{d}\mu = \int_A f \, \mathrm{d}\mu_1 + \int_A f \, \mathrm{d}\mu_2 \quad \forall f \in \mathcal{L}^1(\mu), A \in \mathcal{M}.$$

Proof. **Exercise.** □

Definition 6.2 (Complex Measure)**.** Let X be a measured space with σ-algebra $\mathcal{M}$. Then, a map $\mu : \mathcal{M} \to \mathbb{C}$ is called a *complex measure* iff it is countably additive, i.e., satisfies the following property: If $\{A_n\}_{n \in \mathbb{N}}$ is a sequence of elements of $\mathcal{M}$ such that $A_n \cap A_m = \emptyset$ if $n \neq m$, then

$$\mu\left(\bigcup_{n \in \mathbb{N}} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n).$$

Remark 6.3. 1. The above definition implies $\mu(\emptyset) = 0$. 2. The convergence of the series in the definition is absolute since its limit must be invariant under reorderings. 3. In contrast to positive measures, a complex measure is always finite.

Exercise 33. Show that the complex measures on a given σ-algebra form a complex vector space.

Definition 6.4. Let X be a measured space with σ-algebra $\mathcal{M}$. Let μ be a positive measure on $(X, \mathcal{M})$ and ν a positive or complex measure on $(X, \mathcal{M})$. We say that ν is *absolutely continuous* with respect to μ, denoted $\nu \ll \mu$ iff $\mu(A) = 0$ implies $\nu(A) = 0$ for all $A \in \mathcal{M}$.

Definition 6.5. Let X be a measured space with σ-algebra $\mathcal{M}$. Let μ be a positive or complex measure on $(X, \mathcal{M})$. We say that μ is *concentrated* on $A \in \mathcal{M}$ iff $\mu(B) = \mu(B \cap A)$ for all $B \in \mathcal{M}$.

Definition 6.6. Let X be a measured space with σ-algebra $\mathcal{M}$. Let μ, ν be positive or complex measures on $(X, \mathcal{M})$. We say that μ and ν are *mutually singular*, denoted $\mu \perp \nu$, iff there exist disjoint sets $A, B \in \mathcal{M}$ such that μ is concentrated on A and ν is concentrated on B.

Proposition 6.7. *Let μ be a positive measure and ν, ν_1, ν_2 be positive or complex measures.*

1. *If μ is concentrated on A and $\nu \ll \mu$, then ν is concentrated on A.*
2. *If $\nu_1 \ll \mu$ and $\nu_2 \perp \mu$, then $\nu_1 \perp \nu_2$.*
3. *If $\nu \ll \mu$ and $\nu \perp \mu$, then $\nu = 0$.*
4. *If $\nu_1 \ll \mu$ and $\nu_2 \ll \mu$, then $\nu_1 + \nu_2 \ll \mu$.*
5. *If $\nu_1 \perp \nu$ and $\nu_2 \perp \nu$, then $\nu_1 + \nu_2 \perp \nu$.*

Proof. **Exercise.** □

Theorem 6.8. *Let X be a measure space with σ-algebra $\mathcal{M}$ and σ-finite measure μ. Let ν be a finite measure on $(X, \mathcal{M})$.*

1. *(Lebesgue) Then, there exists a unique decomposition*
$$\nu = \nu_a + \nu_s,$$
into finite measures such that $\nu_a \ll \mu$ and $\nu_s \perp \mu$.
2. *(Radon-Nikodym) There exists a unique $[h] \in \mathrm{L}^1(\mu)$ such that for all $A \in \mathcal{M}$,*
$$\nu_a(A) = \int_A h \, \mathrm{d}\mu.$$

Proof. We first show the uniqueness of the decomposition $\nu = \nu_a + \nu_s$ in (1.). Suppose there is another decomposition $\nu = \nu'_a + \nu'_s$. Note that all the measures involved here are finite and thus are also complex measures. In particular, we obtain the following equality of complex measures, $\nu_a - \nu'_a = \nu'_s - \nu_s$. However, by Proposition 6.7 the left hand side is absolutely continuous with respect to μ while the right hand side is singular with respect to μ. Again by Proposition 6.7, the equality of both sides implies that they must be zero, i.e., $\nu'_a = \nu_a$ and $\nu'_s = \nu_s$.

To show the uniqueness of $[h] \in \mathrm{L}^1(\mu)$ in (2.) we note that given another element $[h'] \in \mathrm{L}^1(\mu)$ with the same property, we would get $\int_A (h - h') \, \mathrm{d}\mu = 0$ for all measurable sets A. By the Averaging Theorem 3.22 $(h - h')(x) = 0$ almost everywhere and so $0 = [h - h'] = [h] - [h'] \in \mathrm{L}^1(\mu)$.

We proceed to construct the decomposition $\nu = \nu_a + \nu_s$ and the element $[h] \in \mathrm{L}^1(\mu)$. By Lemma 4.20, there is a function $w \in \mathcal{L}^1(\mu)$ with $0 < w < 1$. This yields the finite measure μ_w, given by
$$\mu_w(A) := \int_A w \, \mathrm{d}\mu \quad \forall A \in \mathcal{M}.$$

(Recall the last part of Exercise 28.) Define the finite measure $\varphi := \nu + \mu_w$. Note that $\mathcal{L}^1(\varphi) \subseteq \mathcal{L}^1(\nu)$ and $\mathcal{L}^1(\varphi) \subseteq \mathcal{L}^1(\mu_w)$ and we have (using Proposition 6.1),

$$\int_X f\,\mathrm{d}\varphi = \int_X f\,\mathrm{d}\nu + \int_X fw\,\mathrm{d}\mu \quad \forall f \in \mathcal{L}^1(\varphi). \tag{1}$$

In particular, we may deduce

$$\left|\int_X f \mathrm{d}\nu\right| \le \|f\|_{\nu,1} \le \|f\|_{\varphi,1} \quad \forall f \in \mathcal{L}^1(\varphi).$$

By Proposition 4.19 we have $\mathcal{L}^2(\varphi) \subseteq \mathcal{L}^1(\varphi)$ and even

$$\|f\|_{\varphi,1} \le \|f\|_{\varphi,2}\,(\varphi(X))^{1/2} \quad \forall f \in \mathcal{L}^2(\varphi).$$

Combining the inequalities we find

$$\left|\int_X f \mathrm{d}\nu\right| \le \|f\|_{\varphi,2}\,(\varphi(X))^{1/2} \quad \forall f \in \mathcal{L}^2(\varphi).$$

This means that the linear map $\alpha : \mathrm{L}^2(\varphi) \to \mathbb{K} \subseteq \mathbb{C}$ given by $[f] \mapsto \int_X [f]\mathrm{d}\nu$ is bounded. Since $\mathrm{L}^2(\varphi)$ is a Hilbert space, Theorem 4.27 implies that there is an element $g \in \mathcal{L}^2(\varphi)$ such that $\alpha([f]) = \langle [f], [\overline{g}]\rangle$ for all $f \in \mathcal{L}^2(\varphi)$. This implies,

$$\int_X f \mathrm{d}\nu = \int_X fg\,\mathrm{d}\varphi \quad \forall f \in \mathcal{L}^2(\varphi) \tag{2}$$

By inserting characteristic functions for f we obtain

$$\nu(A) = \int_A g\,\mathrm{d}\varphi \quad \forall A \in \mathcal{M}.$$

On the other hand we have $\nu(A) \le \varphi(A)$ for all measurable sets A and hence,

$$0 \le \frac{1}{\varphi(A)} \int_A g\,\mathrm{d}\varphi = \frac{\nu(A)}{\varphi(A)} \le 1 \quad \forall A \in \mathcal{M} : \varphi(A) > 0.$$

We can now apply the Averaging Theorem (Theorem 3.22) to conclude that $0 \le g \le 1$ almost everywhere. We modify g on a set of measure zero if necessary so that $0 \le g \le 1$ everywhere. In particular, if $f \in \mathcal{L}^2(\varphi)$ then $(1-g)f \in \mathcal{L}^2(\varphi)$ and $gf \in \mathcal{L}^2(\varphi)$. Combining (1) and (2) we find

$$\int_X (1-g) f\,\mathrm{d}\nu = \int_X fgw\,\mathrm{d}\mu \quad \forall f \in \mathcal{L}^2(\varphi).$$

Set $Z_a := \{x \in X : g(x) < 1\}$ and $Z_s := \{x \in X : g(x) = 1\}$ and define the measures $\nu_a(A) := \nu(A \cap Z_a)$ and $\nu_s := \nu(A \cap Z_s)$ for all $A \in \mathcal{M}$. Since X is the disjoint union of Z_a and Z_s we obviously have $\nu = \nu_a + \nu_s$. Taking f to be the characteristic function of Z_s we find that $\int_{Z_s} w \, \mathrm{d}\mu = 0$. Since $0 < w$, we conclude that $\mu(Z_s) = 0$. In particular, this implies that μ is supported on Z_a, while ν_s is supported on Z_s, so $\nu_s \perp \mu$.

Define now the sequence $\{f_n\}_{n\in\mathbb{N}}$ of functions $f_n := \sum_{k=1}^{n} g^{k-1}$. Since g is bounded, f_n is bounded. Multiplying with characteristic functions we find for measurable sets A,

$$\int_A (1 - g^n) \, \mathrm{d}\nu = \int_A (1 - g) f_n \, \mathrm{d}\nu = \int_A f_n g w \, \mathrm{d}\mu.$$

Note that $\{1 - g^n\}_{n\in\mathbb{N}}$ increases monotonically and converges pointwise to the characteristic function of Z_a. Thus, by the Monotone Convergence Theorem 3.10 the left hand side converges to $\nu(A \cap Z_a) = \nu_a(A)$.

The sequence $\{f_n g w\}_{n\in\mathbb{N}}$ is also increasing monotonically with its μ-integrals over A bounded by $\nu_a(A)$. So the Monotone Convergence Theorem 3.10 applies and the pointwise limit is a μ-integrable function h. We get

$$\nu_a(A) = \int_A h \, \mathrm{d}\mu,$$

showing existence in (2.) and also $\nu_a \ll \mu$, thus completing the existence proof for (1.). □

Remark 6.9. The function h appearing in the above Theorem is also called the *Radon-Nikodym derivative*, denoted as $h = \mathrm{d}\nu_a / \mathrm{d}\mu$.