Introduction to Braided Quantum Field Theory

Robert Oeckl*

Department of Applied Mathematics and Mathematical Physics,
University of Cambridge, Cambridge CB3 0WA, UK

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Indications from various areas of physics point to the possibility that space-time at small scales might not have the structure of a manifold. Noncommutative geometry provides an attractive framework for a perhaps more accurate description of nature. It encompasses the generalisation of spaces to noncommutative spaces and of symmetry groups to quantum groups. This motivates efforts to extend quantum field theory to noncommutative spaces and quantum group symmetries. One also expects that divergences of conventional theories might be regularised in this way.

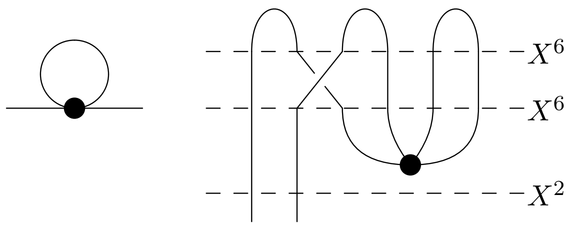
We present here an overview of a generalisation of quantum field theory to noncommutative spaces and quantum group symmetries. For a detailed account and references see [1], to which the present article might serve as an informal introduction. We start out by formulating (perturbative) quantum field theory in a purely algebraic language. The basic objects we consider are the space of fields and its group of symmetries. Feynman diagrams are viewed as built from representations and intertwiners of representations of this group. The next step is the generalisation to the noncommutative setting with quantum group symmetries. Here, the representation theory of quantum groups becomes relevant. Feynman diagrams are generalised to braided Feynman diagrams, which have non-trivial over- and under-crossings.

Finally, we outline the construction of ϕ^4-theory on the quantum 2-sphere with quantum $SU(2)$ symmetry. This was the first application of braided quantum field theory [1]. Regularisation of the basic divergence is achieved by the deformation. One should expect the same to happen for more complicated models in higher dimensions, e.g. quantum field theories on quantum deformations of Minkowski space. However, even for quantum field theories on noncommutative spaces with ordinary group symmetries, new insight can be gained by the methods presented here. This was shown

*email: r.oeckl@damtp.cam.ac.uk

(a) (b)

Figure 1: Propagator (a) and vertex (b).



(a) (b)

Figure 2: Ordinary tadpole diagram (a). Braided tadpole diagram (b).

recently for the noncommutative spaces appearing in string theory, where a duality exchanging noncommutativity and non-trivial statistics emerges [2].

1 Algebraic formulation of quantum field theory

Let us consider a quantum field theory with a space-time manifold M and (classical) fields $\phi_1(x), \dots, \phi_n(x)$. The indices may be spinor indices or distinguish different fields. For simplicity, we work purely over the real numbers. The fields span the vector space of $\mathbb{R}^n$ valued functions on M which we denote by X.[1]

We view a function of two variables as a sum of tensor products of functions in one variable. Thus, we consider the propagator $P_{ij}(x,y) := \langle 0|\phi_i(x)\phi_j(y)|0\rangle$ as an element of $X \otimes X$. Dually, the (polarised) free action is a bilinear map $S_0 : X \otimes X \to \mathbb{R}$ and P is characterised by the requirement

$$(S_0 \otimes \mathrm{id})(\mathrm{id} \otimes P) = \mathrm{id} \tag{1}$$

as maps $X \to X$. Correspondingly, an n-point function is an element of the n-fold tensor product $X^n := X \otimes \cdots \otimes X$. Further, we consider the interaction terms in the Lagrangian. Such a term is just a multi-linear form in the fields, i.e., a linear map $X \otimes \cdots \otimes X \to \mathbb{R}$. It gives rise to a vertex in

[1]Note that we leave out functional analytic details which do not affect our algebraic treatment.

perturbation theory, where the number of legs corresponds to the number of tensor factors.

We write Feynman diagrams in a way that directly reflects their algebraic structure. We call such Feynman diagrams “braided Feynman diagrams”, although the full justification for this name will only emerge in the next section. As an ordinary Feynman diagram, a braided one consists of propagators represented by lines, and vertices represented by dots that join lines together. However, the rules for drawing a braided Feynman diagram are more restrictive. First, one draws one arch (Figure 1.a) for every propagator appearing in the diagram at the top. Then, one draws all the appearing vertices (Figure 1.b) next to each other at the bottom. Finally, one joins the propagators with the vertices letting the external legs end on the bottom line to the left (say) of the vertices.

Consider for example the tadpole diagram (Figure 2.a). The corresponding braided diagram would look like Figure 2.b. To read off the algebraic structure we view the diagram as a map, to be read from top to bottom. A strand represents a field, i.e., an element of X. Horizontally parallel strands correspond to elements of tensor products $X \otimes \cdots \otimes X$ (indicated in Figure 2.b for various horizontal cuts). Since the propagator is an element of $X \otimes X$, two strands originate in it. Vertices on the other hand take in a number of fields to yield a real number, so strands join in them and end. The external legs then determine the tensor product of X’s, in which the resulting object lives. Denoting the vertex by $V : X^4 \to \mathbb{R}$ the diagram in Figure 2.b yields the expression

$$(\mathrm{id} \otimes \mathrm{id} \otimes V)(\mathrm{id} \otimes \tau \otimes \mathrm{id} \otimes \mathrm{id} \otimes \mathrm{id})(P \otimes P \otimes P) \in X \otimes X,$$

where $\tau : X \otimes X \to X \otimes X$ denotes the flip map corresponding to the crossing in the diagram.

We arrive at a completely algebraic description of perturbative quantum field theory. We can plug in an arbitrary vector space X (the space of fields), a scalar product S_0 (the free action) on it, and multi-linear functions $X^k \to \mathbb{R}$ (the interaction terms or vertices). To obtain the perturbation expansion we write down the braided Feynman diagrams determined by our data: For a given set of vertices and number of external legs we link the vertices and external legs (drawn at the bottom) in all possible ways with the propagators (drawn at the top) whereby the two legs of each propagator are treated as identical. This is the combinatorics of Wick’s theorem. Then we remove the vacuum diagrams corresponding to normalising the partition function.

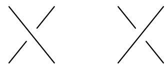


Figure 3: Over-crossing (left) and under-crossing (right).

2 Generalisation to quantum group symmetry

Although the algebraic setting already permits to consider fields on non-commutative spaces, it does not yet permit quantum group symmetries in a natural way. For this we need a genuine generalisation.

Suppose that G is a group of transformations of the fields that leaves the propagator and the vertices invariant. (E.g., for a field theory on Minkowski space one could take the Poincaré group.) Then, all the elements of perturbative quantum field theory in the algebraic formulation are representations of G or intertwiners of representations of G. Consequently, the composed objects, the Feynman diagrams and n-point functions are as well.

When generalising G to a quantum group[2] we have to take into account crucial differences between the representation theory of ordinary groups and quantum groups. Suppose that V and W are representations of G. Then the tensor product $V \otimes W$ is a representation of G in a natural way. This holds both for groups and quantum groups. However, while the map $\tau : V \otimes W \to W \otimes V$ which exchanges the components is an intertwiner if G is a group, this is not in general the case if G is a quantum group. Instead, for each pair V, W there is an intertwiner $\psi : V \otimes W \to W \otimes V$ (called the "braiding") which is non-trivial in general. It is generally drawn as an over-crossing, while its inverse is depicted as an under-crossing (Figure 3).

This has severe consequences. Consider for example the tadpole diagram (Figure 2.b) of the previous section. It contains a crossing of lines which we took to correspond to the flip map τ. Now, if τ is no longer an intertwiner, the diagram as a whole would not be covariant any more. To restore covariance, we could use the braiding ψ instead. However, we could also use its inverse ψ^{-1}. We only get a unique answer if $\psi^2 = \mathrm{id}$. In that case the braiding is called "symmetric". However, most of the interesting quantum groups, especially the ones which are not "close to commutative", have a non-symmetric braiding.

In order to deal with this situation properly, one has to go back to the foundations. It turns out that the concept of Gaussian integration which is fundamental to perturbation theory in the path integral approach can be generalised to braided spaces (i.e., spaces that are representations of a quantum group) in a natural way. The combinatorics of the perturbation

[2] To be precise, we take the terms quantum group and representation to mean either quasitriangular Hopf algebra and module or coquasitriangular Hopf algebra and comodule.

expansion is then governed by the braided generalisation of Wick's theorem:

The free n-point function $\mathcal{Z}_n$ is given in terms of the propagator P as

$$\mathcal{Z}_{2n} = [2n-1]_\psi!!\, P^n, \quad \mathcal{Z}_{2n-1} = 0, \quad \forall n \in \mathbb{N}. \tag{2}$$

$[n]_\psi$ is a braided integer, i.e., a map $X^n \to X^n$ defined as

$$[n]_\psi := \mathrm{id}^n + \psi^{-1} \otimes \mathrm{id}^{n-2} + \cdots + \psi^{-1}_{1,n-1}.$$

The double factorial is a map $X^{2n} \to X^{2n}$ composed of braided integers via

$$[2n-1]_\psi!! := (\mathrm{id} \otimes [2n-1]_\psi) \cdots (\mathrm{id}^{2n-3} \otimes [3]_\psi)(\mathrm{id}^{2n-1} \otimes [1]_\psi).$$

In terms of braided Feynman diagrams this means the following: As before, we draw the vertices and external legs at the bottom and the propagators at the top (the latter represented by the term P^n in (2)). Now, the double factorial of braided integers determines in which ways we have to connect the top with the bottom. The vacuum diagrams are cancelled in the usual way.

An interesting consequence of the braided generalisation is that symmetry factors of ordinary Feynman diagrams get resolved into different braided Feynman diagrams. For example, the tadpole diagram (Figure 2.a) carries ordinarily a factor of 12 which is resolved into 12 different braided Feynman diagrams. One of them is in fact precisely the diagram (Figure 2.b).

We arrive at a generalisation of perturbative quantum field theory. It allows us to start with a vector space X (space of fields), a scalar product S_0 (free action) on X, multi-linear maps $V : X^k \to \mathbb{R}$ (vertices) and a quantum group G (symmetries) acting on X and leaving S_0 and V invariant. The full treatment shows that the propagator is determined by

$$(S_0 \otimes \mathrm{id})(\mathrm{id} \otimes \psi^{-1})(\mathrm{id} \otimes P) = \mathrm{id}, \tag{3}$$

which is equivalent to (1) in the group case since then $\tau P = P$. Braided Wick's Theorem then yields the G-invariant perturbative n-point functions expressed in terms of braided Feynman diagrams.

3 ϕ^4-Theory on the quantum 2-sphere

Let us illustrate the theory with an example. We consider real massive scalar ϕ^4-theory on the (standard) quantum 2-sphere with quantum $SU(2)$-symmetry. Thus, $G = SU_q(2)$ and $X = S^2_q$ as a homogeneous space[3] under G and q is real. We choose a basis $\{t^{(l)}_{0\,n}\}$ (l the integer spin and $n \in \{-l, \ldots, l-1, l\}$) of spherical harmonics on S^2_q, induced by the Peter-Weyl decomposition of $SU_q(2)$.

[3]The symbols for the quantum spaces denote the deformed function algebras.

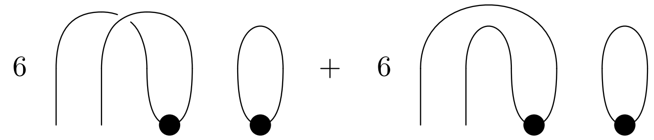


Figure 4: Simplified tadpole diagram contribution.

On the ordinary 2-sphere the (polarised) free action is given by

$$S_0(\phi \otimes \phi') = \int_{S^2} \phi(m^2 - \Delta)\phi'.$$

We view the Laplace operator as a Casimir operator of the Lie algebra $\mathfrak{su}_2$. The quantum Casimir provides a natural q-deformed generalisation and the operator $L := m^2 - \Delta$ has the eigenvalue

$$\mathsf{L}_l = [l]_q[l+1]_q + m^2.$$

on the spin-l component of S^2_q. Here the q-integers are defined as

$$[n]_q := \sum_{k=0}^{n-1} q^{n-2k-1} = \frac{q^n - q^{-n}}{q - q^{-1}}.$$

Applying (3) yields the propagator

$$P = \sum_{l,n} [2l+1]_q \, \mathsf{L}_l^{-1} \, q^{-2l(l+1)} \, (-q)^n \, t^{(l)}_{0\,n} \otimes t^{(l)}_{0\,-n}.$$

In the undeformed case this can be rewritten in the usual way

$$P|_{q=1}(x,y) = (m^2 - \Delta_x)^{-1}\delta(x,y).$$

Consider now the tadpole diagram, which is the only basic divergent diagram in the undeformed theory. As mentioned in the previous section, it corresponds to 12 different diagrams in the braided case. However, due to certain identities and a factorisation property that hold in the case at hand, we only obtain two different diagrams, each with a multiplicity of 6 (Figure 4). The factorisation breaks the ϕ^4-interaction into ϕ^2-terms. The disconnected loop (which is a loop with a ϕ^2-vertex) comes out as

$$\delta_{\text{loop}} := \sum_l \frac{[2l+1]_q}{[l]_q[l+1]_q + m^2} \, q^{-2l(l+1)},$$

while the whole diagram is

$$6\,\delta_{\text{loop}} \sum_{l,n} [2l+1]_q \, \mathsf{L}_l^{-2} \, q^{-4l(l+1)} (1 + q^{-2l(l+1)}) \, (-q)^n \, t^{(l)}_{0\,n} \otimes t^{(l)}_{0\,-n}.$$

δ_{loop} is a divergent sum at $q = 1$. This is the usual divergence of the ordinary tadpole diagram. However, at $q > 1$ the term converges and the diagram becomes finite and well defined. The divergence has been converted to a divergence in the deformation parameter q.

Thus, we have obtained a regularisation that neither truly breaks the symmetry (as conventional quantum field theoretic schemes do) nor resorts to zero-dimensional approximations (as lattice methods or some other non-commutative methods do).

Acknowledgements

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References

[1] R. Oeckl, *Braided Quantum Field Theory*, Commun. Math. Phys. **217** (2001) 451–473, `hep-th/9906225`.

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