Arnowitt-Deser-Misner Formalism

A Hamiltonian Formalism of Einstein Gravity

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Quantum Gravity Seminar

Outline

1. Motivation
2. Foliation of Space – Time
   - Gauss-Codazzi Equations
   - First glance at the constraints
   - ADM-Lagrangian
3. Constraint Hamiltonian Systems
   - Singular Systems
   - Legendre Transformation
   - Dirac-Bergman algorithm
   - First and Second Class
4. ADM Formalism
   - Constraints
5. Conclusions
6. References

Why do we want a Hamiltonian Formulation of GR

- We want Quantum Gravity and know how to do cannonical quantization
- Central for Wheeler-DeWitt and Geometrodynamics, LQG,...
- Deep insights into nature of constraint- and gauge systems.
- Numerical GR needs a description in terms of foliations in order to describe the dynamical evolution of events.

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2 Foliation of Space−Time

Requirements

- Hamiltonian formalism requires a time coordinate, since otherwise $p = \partial L / \partial \dot{q}$ cannot be defined.
- Must cast GR in a form where it exhibits a distinguished time.
  (does that not break $\mathrm{Diff}(\mathcal{M})$? $\Rightarrow$ No, to the contrary!)

Definition

A foliation of $\mathcal{M}$ is a diffeomorphism $X : \mathbb{R} \times \sigma \to \mathcal{M}$.

Facts

- space− time is a 4-dim globally hyperbolic manifold, and as such admits a foliation (topological: Geroch '70 and metrical: Bernal and Sanchez '03-'06)
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- EH-action is invariant under $\mathrm{Diff}(\mathcal{M})$ fixing a coordinate system breaks $\mathrm{Diff}(\mathcal{M})$
- specification of $X(\sigma)$ breaks $\mathrm{Diff}(\mathcal{M})$ but if we keep X generall $\mathrm{Diff}(\mathcal{M})$ is preserved.
- Define the pulled back action to be equal to the EH-action:

$$S_{\mathrm{ADM}}[X^* g] := S_{\mathrm{EH}}[g].$$

$\Rightarrow$ freedom of choice of the foliation is "equivalent" to $\mathrm{Diff}(\mathcal{M})$

$$S_{\mathrm{EH}}[\phi^* g] \equiv S_{\mathrm{ADM}}[X^* \circ \phi^* g] = S_{\mathrm{ADM}}[(\phi \circ X)^* g] = S_{\mathrm{ADM}}[X'^* g]$$

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Let $(\mathcal{M}, g, \nabla)$ be a 4-dim Lorentzian space– time and (σ, h, D) an embedded 3-dim Riemannian submanifold with the embedding $X : R \times \sigma \to \mathcal{M}$

$$\begin{aligned} t^\mu := \frac{\partial X^\mu(t,x)}{\partial t} &= N(X) n^\mu(X) + N^\mu(X) \\ T_p\mathcal{M} \quad &= \quad N_p\Sigma_\tau \quad \oplus \quad T_p\Sigma_\tau \end{aligned}$$

- The fuctions N and N^μ are called lapse function and shift vector respectively
- t^μ is interpreted as describing the "flow of time".

Metrics

$$\begin{array}{ccc} (X^*g) \text{ on } R\times\sigma & \text{and} & g_{\mu\nu} \text{ on } \mathcal{M} \\ h_{ab} \text{ on } \sigma & \text{and} & h_{\mu\nu} \text{ on } \Sigma_t := X_t(\sigma) \end{array}$$

We have the relations

$$h_{ab} := g_{\mu\nu} X^\mu_{,a} X^\nu_{,b} \qquad \text{and} \qquad h_{\mu\nu} := g_{\mu\nu} - s n_\mu n_\nu$$

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Dynamic ADM variables

After the foliation, what variables encode the 10 DOF of $g_{\mu\nu}$?

$$\begin{aligned} \mathrm{d}s^2 &= g_{\mu\nu}\mathrm{d}X^\mu \otimes \mathrm{d}X^\nu = g_{\mu\nu}[\dot{X}^\mu \mathrm{d}t + X^\mu_{,a}\mathrm{d}x^a] \otimes [\dot{X}^\nu \mathrm{d}t + X^\nu_{,b}\mathrm{d}x^b] \\ &= g_{\mu\nu}[(Nn^\mu + X^\mu_{,a}N^a)\mathrm{d}t + X^\mu_{,a}\mathrm{d}x^a] \otimes [(Nn^\nu + X^\nu_{,b}N^b)\mathrm{d}t + X^\nu_{,b}\mathrm{d}x^b] \\ &= (sN^2 + h_{ab})[\mathrm{d}t \otimes \mathrm{d}t] + h_{ab}N^b[\mathrm{d}x^a \otimes \mathrm{d}t + \mathrm{d}t \otimes \mathrm{d}x^a] + h_{ab}[\mathrm{d}x^a \otimes \mathrm{d}x^b] \end{aligned}$$

We choose (h_{ab}, N^a, N) as ADM variables since we can fully reconstruct $g_{\mu\nu}$ from (h_{ab}, N^a, N).

Curvature of Submanifolds

We have two different notions of curvature for the submanifolds Σ_t

- Extrinsic curvature (2nd fundamental form)

$$K_{\mu\nu} := \nabla_\mu n_\nu = h^\alpha_\mu \nabla_\alpha n_\nu = \frac{1}{2}\mathcal{L}_n h_{\mu\nu}$$

  measures how much a vector tangent to Σ_τ will fail to be tangent if we parallel transport it using ∇.

- Riemannian curvature ${}^{(3)}R$ of $D_\mu f := h^\nu_\mu \nabla_\nu \tilde{f}$

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$$S_{\mathrm{EH}} = \frac{1}{2\kappa} \int_{\mathcal{M}} \mathrm{d}^4 x \sqrt{-g} \left[{}^{(4)}R \right]$$

with $\kappa = 8\pi G/c^4$ and $\Lambda = T_{\mu\nu} = 0$

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Gauss-Codazzi Equations

Gauss equation

$${}^{(3)}R_{\mu\nu\alpha\beta} = 2s[K_{\alpha\mu}K_{\nu\beta} - K_{\alpha\nu}K_{\mu\beta}] + h^{\rho}_{\mu}h^{\sigma}_{\nu}h^{\lambda}_{\alpha}h^{\gamma}_{\beta}\,{}^{(4)}R_{\rho\sigma\lambda\gamma}$$

Codazzi Equation

$$\begin{aligned} D_\mu K_{\nu\lambda} - D_\nu K_{\mu\lambda} &= h^{\rho}_{\mu}h^{\sigma}_{\nu}h^{\lambda}_{\alpha}\,{}^{(4)}R_{\rho\sigma\lambda\gamma}n^\gamma \\ {}^{(4)}R &= {}^{(3)}R - s[K_{\mu\nu}K^{\mu\nu} - K^2] + 2s\nabla_\mu[n^\nu\nabla_\nu n^\mu - n^\mu\nabla_\nu n^\nu] \end{aligned}$$

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$$S_{\mathrm{EH}} = \frac{1}{2\kappa} \int_{\mathcal{M}} \mathrm{d}^4 x \sqrt{-g} \left[{}^{(4)}R \right]$$

with $\kappa = 8\pi G/c^4$ and $\Lambda = T_{\mu\nu} = 0$

- For foliation we need to reformulate this in terms of ADM variables (h_{ab}, N^a, N).

Gauss-Codazzi Equations

Gauss equation

$${}^{(3)}R_{\mu\nu\alpha\beta} = 2s[K_{\alpha\mu}K_{\nu\beta} - K_{\alpha\nu}K_{\mu\beta}] + h^\rho_\mu h^\sigma_\nu h^\lambda_\alpha h^\gamma_\beta \, {}^{(4)}R_{\rho\sigma\lambda\gamma}$$

Codazzi Equation

$$\begin{aligned} D_\mu K_{\nu\lambda} - D_\nu K_{\mu\lambda} &= h^\rho_\mu h^\sigma_\nu h^\lambda_\alpha \, {}^{(4)}R_{\rho\sigma\lambda\gamma} n^\gamma \\ {}^{(4)}R &= {}^{(3)}R - s[K_{\mu\nu}K^{\mu\nu} - K^2] + 2s\nabla_\mu[n^\nu \nabla_\nu n^\mu - n^\mu \nabla_\nu n^\nu] \end{aligned}$$

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Constraints

The vacuum Einstein equations $G_{\mu\nu} = 0$ yield

$$0 = G_{\mu\nu} n^\nu h^\mu_\alpha = R_{\mu\nu} n^\nu h^\mu_\alpha \quad (1)$$

$$0 = G_{\mu\nu} n^\mu n^\nu = R_{\mu\nu} n^\mu n^\nu + \frac{R}{2} \quad (2)$$

(1)+(Codazzi) gives the spacial diffeomorphism constraint

$$D_\mu K^\mu_\nu - D_\nu K = 0$$

(2)+(Gauss) gives the Hamiltonian constraint

$$K^2 - K_{\mu\nu} K^{\mu\nu} + {}^{(3)}R = 0$$

Initial Value Problem in GR

- $(\Sigma, h^{\mu\nu}, K^{\mu\nu})$ are initial data in GR
- Diffeomorphism and Hamiltonian constraint are inital value constraints that any choice of $(\Sigma, h^{\mu\nu}, K^{\mu\nu})$ will have to satisfy
- If constraints hold on Σ_0 and Einstein equations are satisfied everywhere, then the constraints hold on all later hypersurfaces Σ_T.

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ADM-Lagrangian

- For the volume element one finds $\sqrt{-g} = N\sqrt{h}$
- After inserting this and the Codazzi equation into S_{EH}, we pull it back to $R \times \sigma$ and get

$$S_{\mathrm{ADM}} = \frac{1}{2\kappa}\int_R \mathrm{d}t \int_\sigma \mathrm{d}^3x N\sqrt{h}\left[s(K^2 - K_{ab}K^{ab}) + {}^{(3)}R\right]$$

- ${}^{(3)}R = {}^{(3)}R(h, \partial_a h)$ and $K_{ab} = \frac{1}{2N}[\dot{h}_{ab} - (\mathcal{L}_{N^a}h)_{ab}] = K_{ab}(h, \dot{h}, N^a)$.
- $\Rightarrow S_{\mathrm{ADM}}$ does not depend on $\dot{N}, \dot{N}^a$

Conjugate Momenta

In order to perform the Legendre transformation we need the conjugate momenta

$$\begin{aligned} P^{ab} &:= \frac{\delta S_{\mathrm{ADM}}}{\delta \dot{h}_{ab}} = \frac{s}{2\kappa}\sqrt{h}[h^{ab}K - K^{ab}] \\ \Pi &:= \frac{\delta S_{\mathrm{ADM}}}{\delta \dot{N}} = 0, \qquad \Pi_a := \frac{\delta S_{\mathrm{ADM}}}{\delta \dot{N}^a} = 0 \end{aligned}$$

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Singular Systems

Euler-Lagrange-Equations for a system with N DOF

$$0 = -\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}^i} + \frac{\partial L}{\partial q^i} = -\frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j}\ddot{q}^j - \frac{\partial^2 L}{\partial \dot{q}^i \partial q^j}\dot{q}^j + \frac{\partial L}{\partial q^i} = -\underbrace{W_{ij}(q,\dot{q})}_{\text{Hessian}}\ddot{q}^j + V_i$$

$$\ddot{q}^j = (W^{-1})^{ij} V_i$$

- If $\det(W) = 0$ accelerations are not uniquely determined by $(q, \dot{q}) \Leftrightarrow$ Singular System. $\Rightarrow$ Different time evolutions will stem from the same initial conditions (Dirac's definition of gauge equivalence)
- Generalized Bianchi identities: Gauge theory $\overset{\not\Leftarrow}{\Rightarrow}$ Singular System
- Only if $W_{ij} = (\partial p_i)/(\partial \dot{q}^j)$ is invertible, can this relation be solved for all velocities in terms of phase space variables $\dot{q} = \dot{q}(q, p)$. In the other case not all momenta are independent.

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Primary constraint surface

Let $\mathrm{rank(W)} = \mathrm{R} < \mathrm{N} \Rightarrow \exists\, M = (N - R)$ null-eigenvectors

$$Y^i_m(q,\dot{q})W_{ij}(q,\dot{q}) = 0 \quad \Rightarrow \ \phi_m(q,\dot{q}) := Y^i_m V_i = 0$$

- The func. independent $\phi_k = 0$ with $k \in K \leq M$ are called Lagrange constraints.
- The ϕ_k define a constraint $2N - K$ dimensional primary constraint surface Γ_p.
- Call $F(q,p)$ weakly zero $F \approx 0$, if $F|_{\Gamma p} = 0$.

Theorems for primary constraints

- **Theorem 1** If $F(q,p)|_{\Gamma p} = 0$, then $F = f^k \phi_k$ for some $f^k \in C^\infty$
- **Theorem 2** If $\lambda_i \delta q^i + \mu^i \delta p_i = 0$, then

$$\lambda_i \approx u^k \frac{\partial \phi_k}{\partial q^i}, \quad \text{and} \quad \mu_i \approx u^k \frac{\partial \phi_k}{\partial p^i}.$$

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Canonical Hamiltonian

Define the canonical Hamiltonian through the Legendre Transformation

$$H_c := \dot{q}^i p_i - L$$

The following shows that H_c is a function of p and q only.

$$\begin{aligned} \delta H_c &= \dot{q}^i \delta p_i + \delta \dot{q}^i p_i - \delta \dot{q}^i \frac{\partial L}{\partial \dot{q}^i} - \delta q^i \frac{\partial L}{\partial q^i} = \dot{q}^i \delta p_i - \delta q^i \frac{\partial L}{\partial q^i} \\ &= \delta \dot{q}^i \frac{\partial H_c}{\partial q_i} + \delta p_i \frac{\partial H_c}{\partial p_i} \end{aligned}$$

and thus

$$\left(\frac{\partial H_c}{\partial q^i} + \frac{\partial L}{\partial q^i} \right) \delta q^i + \left(\frac{\partial H_c}{\partial p^i} - \dot{q}^i \right) \delta p^i = 0$$

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$$\begin{aligned} \delta H_c &= \dot{q}^i \delta p_i + \delta \dot{q}^i p_i - \delta \dot{q}^i \frac{\partial L}{\partial \dot{q}^i} - \delta q^i \frac{\partial L}{\partial q^i} = \dot{q}^i \delta p_i - \delta q^i \frac{\partial L}{\partial q^i} \\ &= \delta \dot{q}^i \frac{\partial H_c}{\partial q_i} + \delta p_i \frac{\partial H_c}{\partial p_i} \end{aligned}$$

and thus

$$\left(\frac{\partial H_c}{\partial q^i} + \frac{\partial L}{\partial q^i} \right) \delta q^i + \left(\frac{\partial H_c}{\partial p^i} - \dot{q}^i \right) \delta p^i = 0$$

with Theorem 2 it follows that

$$\begin{aligned} \dot{q}^i &\approx \frac{\partial H_c}{\partial p_i} + u^k \frac{\partial \phi_k}{\partial p_i} \\ -\dot{p}^i &= -\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{q}^i} = -\frac{\partial L}{\partial q^i} \approx \frac{\partial H_c}{\partial q_i} + u^k \frac{\partial \phi_k}{\partial q_i} \end{aligned}$$

Canonical Hamiltonian

Define the canonical Hamiltonian through the Legendre Transformation

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Primary Hamiltonian

This motivates the definition of the primary Hamiltonian H_p

$$H_p := H_c + u^k \phi_k$$

For any phase space function $F(q,p)$ the time evolution then follows from

$$\dot{F} \approx \{F, H_p\}$$

Consistency conditions

We must enforce consistency conditions that ensure that the EOM preserve the constraints.

$$\dot{\phi}_m \approx \{\phi_m, H_c\} + \{\phi_m, \phi_n\}u^n =: h_m + C_{mn}u^n \approx 0$$

Distinguish two cases

1.) $\det C \not\approx 0$

u is uniquely fixed to be $u^n \approx C^{nm} h_m \Rightarrow$ evolution preserves Γ_p

2.) $\det C \approx 0$

u is not fixed and $\dot{\phi}_m \approx 0$ leads to a certain number R of secondary constraints

$$\phi_r \approx 0, \quad r \in \{M+1, \ldots M+R\}$$

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Dirac-Bergman algorithm

- The primary and secondary constraints $\phi_j,\ j = 1, \ldots, M + R$ define the hypersurface $\Gamma_1 \subseteq \Gamma_p$
- We have to check the consistency for the primary and secondary constranits on Γ_1: $\dot{\phi}_j \approx 0$.
- This might lead to tertiary constraints and $\Gamma_2 \subseteq \Gamma_1$.
- This procedure terminates after a finite number of iterations on $\Gamma \subseteq \cdots \subseteq \Gamma_1 \subseteq \Gamma_p$ with $\phi_j \approx 0, \quad j = 1, \ldots, M + K$.
- Note that the primary constraints are merely consequences of the definition of the momenta, whereas we used the EOM to arrive at the secondary constraints.

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First and second class constraints

A function $F(q,p)$ is called first class, if $\{F,\phi_j\} \approx 0$ for all (primary and secondary) constraints ϕ_j. Otherwise it is called second class.

$$\begin{aligned}\dot{F} \approx \{F, H_p\} &= \{F, H_c\} + \{F, \phi_k\} u^k \\ &= \{F, H_c\} + \{F, \gamma_a\} u^a + \{F, \chi_b\} u^b\end{aligned}$$

where γ_a are FCC: $\{\gamma_a, \gamma_b\} = \{\gamma_a, \chi_b\} = 0$,
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The consistence condition $0 \approx \dot{\chi} \approx \{\chi_a, H_c\} + \Delta_{ab} u^b$ leads to $u^b \approx -\Delta^{ba}\{\chi_a, H_c\}$

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Flows of the constraints

- Flows generated by the SCC lead off the constraint surface
- Flows generated by the FCC stay on the constraint surface and are identified with gauge transformation
- Dirac conjecture: all FCC generate gauge transformations (exotic counterexamples)

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canonical ADM-Hamiltonian

We deduced the canonical variables h^{ab}, N^a, N and their canonical momenta

$$P^{ab} = \frac{s}{2\kappa}\sqrt{h}[h^{ab}K - K^{ab}], \quad \Pi = 0, \quad \Pi_a = 0$$

$$K^2 = \frac{\kappa^2}{\det(h)} P^2, \qquad K_{ab}K^{ab} = \frac{\kappa^2}{\det(h)}[4P_{ab}P^{ab} - P^2]$$

The canonical ADM-Hamiltonian is obtained via a Legendre transformation

$$\begin{aligned} H^c_{\mathrm{ADM}} &:= \int dx^3 [P^{ab}\dot{h}_{ab} - \mathcal{L}_{\mathrm{ADM}}] \\ &= \int dx^3 (L_N h)_{ab} P^{ab} + \frac{N}{2\kappa}\left(\frac{-4s\kappa^2}{\sqrt{h}}\left[P_{ab}P^{ab} - \frac{P^2}{2}\right] - \sqrt{h}R\right) \\ &=: \int dx^3 NH + N^a H_a \end{aligned}$$

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Secondary Constraints

The primary Hamiltonian is

$$H^p_{\mathrm{ADM}} \quad := \quad H^c_{\mathrm{ADM}} + u^a \Pi_a + u\Pi$$

The consistancy of the primary constraints Π_a, Π must be ensured

$$\{\Pi_a, H^p_{\mathrm{ADM}}\} = H_a, \qquad \{\Pi, H^p_{\mathrm{ADM}}\} = H$$

- Following the Dirac-Bergman algorithm, we must impose H_a (the spatial Diffeomorphism constraint) and H (the Hamiltonian constraint) as secondary constraints.
- N_a and N can be treated as Lagrange multipliers, and are thus arbitrary
- The Hamilton is a linear combination of constaints and thus vanishes on the physical phase space. (No true Hamiltonian)
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Geometrical Interpretation of Constraints

We want to know what the Hamiltonian flow of the physical phase space variables P_{ab} and q_{ab} with respect to the constraints is.

$$\begin{aligned}
\{H(N), P^{ab}\} &= [0]_{\Gamma, G_{\mu\nu}=0} + L_{Nn} P^{ab} \\
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\end{aligned}$$

- spacial Diffeomorphism constraint generates diffeomorphisms that preserve Σ_t.
- Hamiltonian constraint generates diffeomorphisms of $\mathcal{M}$ orthogonal to Σ_t.
- However, their algebra is not a Lie algebra (Bergman-Komar "group")

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- Hamiltonian constraint generates diffeomorphisms of $\mathcal{M}$ orthogonal to Σ_t.
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The ADM-EOM are

$$\dot{P}^{ab} = \{P^{ab}, H^p_{\mathrm{ADM}}\}, \qquad \dot{q}^{ab} = \{q^{ab}, H^p_{\mathrm{ADM}}\}$$

Time evolution well defined

Further topics

- Matter can be coupled and leads to a variation of the constraints, e.g. $H_a \to H_a + \sqrt{h} J_a$ (J_a Poynting vector)
- For fermions the vierbein is decomposed, rather then the 4-metric
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5 Conclusions

- Gravity can be cast into Hamiltonian form, the physical variables are the 3-metric and its momentum
- Lapse function and shift vector turn out to be Lagrange multipliers
- Diffeomorphism invariance is preserved
- Gauge theories are singular and thus constraint Hamiltonian systems
- Consistency of the primary constraints yield spacial Diffeomorphism- and Hamiltonian constraint that generate Diffeomorphisms
- Let the Canonical Quantization begin...

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6 References

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