Quantum Geometrodynamics

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Quantum Gravity Seminar

Contents

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- No background spacetime is assumed to define the theory
- Application to some cosmological models
- Semiclassical solutions

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- Known quantization procedure
- No background spacetime is assumed to define the theory
- Application to some cosmological models
- Semiclassical solutions

Disadvantages

- Lack of rigorous mathematical definition
- Difficulties in solving the dynamical equations of the theory
- Conceptual issues (problem of time)

2 types of canonical quantization

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  2. Quantize first and then solve the quantum constraints.
     $\longrightarrow$ Dirac's canonical quantization programme for constrained systems.

The programme of canonical quantization

The canonical procedure for constrained systems has been developed by Dirac and can be summarized in the following steps:

1. put the classical theory in a canonical form and identify the conjugate canonical variables that satisfy a Poisson algebra;
2. represent these classical quantities as operators acting on the space of wavefunctions and promote the Poisson bracket to the status of commutators;
3. write the constraints present in the theory, i.e. quantities vanishing identically at the classical level, as quantum operators and identify the physical states of the quantum theory with those states annihilated by the action of the constraints operators;
4. define an inner product in the space of the physical states, then complete it to obtain the physical Hilbert space $\mathcal{H}_{phys}$ of the quantum theory;
5. define a set of observables as those quantities that commute with all the constraints, and provide predictions in order to give a physical interpretation to the states $\mathcal{H}_{phys}$.

ADM formulation of GR (I)

Starting point of ADM formalism: 4d spacetime manifold M, equipped with metric $g_{\mu\nu}$, has topology $\Sigma \times \mathbb{R}$ where Σ is 3d submanifold, and can be foliated by a family of spacelike 3d hypersurfaces Σ_t, each diffeomorphic to Σ, indexed by the value of the time parameter t, realizing a 3+1 decomposition of the original 4d geometry. From such decomposition we derive three things:

1. a 3-metric q_{ab} induced on the 3d spacelike hypersurface Σ_t (embedded in a 4d spacetime), on which a coordinate system $\{x^a\}$ has been defined. q_{ab} captures the intrinsic geometry of Σ_t and plays the role of the configuration variable of the theory;
2. an extrinsic curvature tensor K_{ab}, obtained from the covariant derivate of the normal n^a to Σ_t and associated to the "velocity" $\dot{q}_{ab}$ of the configuration variable. K_{ab} is related to the momentum conjugate to the 3-metric. The extrinsic curvature encodes the information on how the hypersurface is embedded in the 4d spacetime;
3. the way in which coordinates defined on the spatial hypersurface evolve in time, i.e. specifying the correspondence between points (in space) on each surface (i.e. at different times). By the introduction of the lapse function N and shift vector N^a the spacetime interval between (t, x^a) and $(t + dt, x^a + dx^a)$ results to be

$$ds^2 = -N^2 dt^2 + q_{ab}\,(dx^a + N^a dt)\,(dx^b + N^b dt).$$

After the choice of the lapse function and the shift vector, the 4-metric can be reconstructed from the t dependence of 3-metric.

ADM formulation of GR (II)

The Einstein Lagrangian $\int R\sqrt{-g}d^3x$ is

$$L = \frac{1}{16\pi G}\int_{\Sigma_t} d^3x\sqrt{q}N\left(K^{ab}K_{ab} - K^2 + R\right) + \text{surface term},$$

$q \equiv \det q_{ab}$, $K = q^{ab}K_{ab}$, $K_{ab} = \frac{1}{2N}(\dot{q}_{ab} - N_{a,b} - N_{b,a})$, q^{ab} is the inverse of q_{ab} and R is the 3d intrinsic curvature.

The momentum π^{ab} conjugate to q_{ab} is

$$\pi^{ab} = \frac{\delta L}{\delta \dot{q}_{ab}} = \frac{c^3}{8\pi G}\sqrt{q}(K^{ab} - q^{ab}K).$$

The Poisson bracket satisfied by the canonical variables are

$$\{q_{ab}(t,x), \pi^{cd}(t,x')\} = \frac{1}{2}(\delta_a^c\ \delta_b^d + \delta_a^d\ \delta_b^c)\ \delta(x - x').$$

The momenta conjugate to N and N_a are

$$\pi = \frac{\delta L}{\delta \dot{N}} = 0, \qquad \pi^a = \frac{\delta L}{\delta \dot{N}_a} = 0.$$

These eqs are *primary constraints*. They express the fact that the Einstein Lagrangian is independent of the "velocities" $\dot{N}$ and $\dot{N}_a$. They do not involve the dynamical equations.

ADM formulation of GR (III)

The Einstein-Hilbert action for pure gravity takes the form

$$S_{EH} = \int \mathrm{d}t \int_{\Sigma_t} \mathrm{d}^3x \left(\pi^{ab}\dot{q}_{ab} - N\mathcal{H} - N_a\mathcal{H}^a \right) + \text{boundary terms},$$

where the lapse and the shift functions represent Lagrange multipliers and

$$\mathcal{H} = \frac{8\pi G}{c^3\sqrt{q}}(\pi_{ab}\pi^{ab} - \frac{1}{2}\pi^2) - \frac{c^3}{8\pi G}\sqrt{q}R \approx 0, \qquad \mathcal{H}^a = -2\nabla_b \pi^{ab} \approx 0.$$

$\mathcal{H}$ and $\mathcal{H}^a$ are the Hamiltonian and the diffeomorphism (or momentum) constraints respectively (∇_b is the covariant differentiation on Σ_t). These constraints are obtained by requiring the preservation of the primary constraints. $\mathcal{H}$ and $\mathcal{H}_a$ are therefore four *secondary or dynamical constraints*. They do not lead to further constraints.
The theory is completely defined by these four constraints.
All the Poisson brackets among the constraints are equal to linear combinations of the constraints, and the constraints are therefore of first class and there are no second class constraints.
The Hamiltonian results to be

$$H = \int_{\Sigma_t} d^3x \left(N\mathcal{H} + N_a\mathcal{H}^a \right),$$

and therefore vanishes weakly, $H \approx 0$, on the constraint surface.

- Poisson brackets become commutators and classical quantities become operators

$$\begin{aligned}
\left[\hat{q}_{ab}(t,x), \hat{\pi}^{cd}(t,x')\right] &= \mathrm{i}\hbar\frac{1}{2}(\delta^c_a\ \delta^d_b + \delta^d_a\ \delta^c_b)\ \delta(x-x'), \\
\left[\hat{N}(t,x), \hat{\pi}(t,x')\right] &= \mathrm{i}\hbar\delta(x-x') \\
\left[\hat{N}_a(t,x), \hat{\pi}^b(t,x')\right] &= \mathrm{i}\hbar\delta^b_a\delta(x-x')
\end{aligned}$$

  and all other commutators vanish.

- The operators act on a functional space $\mathcal{F}$ of quantum states. The quantum states are represented in the Schrödinger picture by wave functionals of the 3-metric, $\Psi[q_{ab}]$. The operators are defined by the metric representation

$$\hat{q}_{ab}\Psi[q_{ab}] = q_{ab}\Psi[q_{ab}], \qquad \hat{\pi}^{cd}\Psi[q_{ab}] = \frac{\hbar}{\mathrm{i}}\frac{\delta}{\delta q_{ab}}\Psi[q_{ab}].$$

- The classical constraints become conditions on the quantum states

$$\pi\Psi = 0, \qquad \pi^a\Psi = 0, \qquad \mathcal{H}\Psi = 0, \qquad \mathcal{H}^a\Psi = 0. \tag{1}$$

  The space of solutions of (1), $\mathcal{F}_0$, is a subspace of $\mathcal{F}$. Physical states must be solutions of (1) in order to be invariant under the symmetries of the theory encoded in the constraint operators.

Quantization (II)

- In the metric representation the (quantum) diffeomorphism constraint takes the form

$$\mathcal{H}^a\Psi = -2\nabla_b q_{ac}\frac{\hbar}{\mathrm{i}}\frac{\delta\Psi}{\delta q_{bc}} = 0. \quad (2)$$

  This equation expresses the invariance of Ψ under 3d coordinate transformations: the state vectors are functionals of the geometric properties of the 3-space which are invariant under spatial diffeomorphisms, rather than functionals just of q_{ab}.

- The Hamiltonian constraint results to be

$$\mathcal{H}\Psi = \left(-16\pi G\hbar^2 G_{abcd}\frac{\delta^2}{\delta q_{ab}\delta q_{cd}} - \frac{\sqrt{q}}{16\pi G}R\right)\Psi = 0, \quad (3)$$

  where the inverse of $G_{abcd} = \frac{1}{2\sqrt{q}}(q_{ac}q_{bd} + q_{ad}q_{bc} - q_{ab}q_{cd})$ is the DeWitt metric. (3) is the Wheeler-DeWitt equation. It encodes the full quantum dynamics of gravity.

- Dirac observables must commute with all the first-class constraints generating gauge transformations,

$$[\hat{\mathcal{O}}, \hat{\mathcal{H}}]\Psi = 0,$$

  so the action of an observable on a physical state does not project the state out of the space of physical states $\mathcal{F}_0$.
- An inner product must be defined on $\mathcal{F}_0$ in order to obtain an Hilbert space of physical normalized state vectors.

Problems

- Classical spacetime is the history of 3d geometry evolving deterministically (initial data). The commutation relation between the 3-metric and the conjugate momentum implies uncertainty relation between intrinsic and extrinsic geometry. How to interpret the quantum properties of the spacetime in quantum gravity?
- The constraints must be quantized such that their Poisson relations are consistent with the commutation relations at the quantum level. This process depends on regularizations and factor orderings and does thus not yield a unique result, but gives rise to quantization ambiguities (presence of anomalies).
- Problematic definition of the inner product: the measure in superspace cannot be rigorously defined.
- The Wheeler-DeWitt equation is ill-defined (because of the nonpolynomial dependence on the ADM variables).
- The Wheeler-DeWitt equation has not been solved in general.
- How to interpret the solution to the Wheeler-DeWitt equation? What is $|\Psi[q_{ab}]|^2$?
- No Dirac observable of the system is known.
- How to describe evolution at the quantum level? The coordinate time variable t does not appear in the classical equation nor in the quantum equation $\rightarrow$ problem of time.

The problem of time

The problem originates in the different ways time is treated in the quantum theory and in GR.

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- Time in QM is Newton's absolute time:
  - it is a distinguished classical variable with respect to which the evolution is defined: It has the role to label the evolution of the system;
  - the scalar product on the Hilbert space of physical states, the commutation relations of conjugate variables and operators, the measurement procedures of observables are all taken at fixed times.

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- In GR, time does exist as an objectively measurable independent degree of freedom:
  - the theory is invariant under reparametrizations of the time coordinate;
  - coordinate time is not a physical degree of freedom and evolution in it is not gauge-invariant (time is a pure gauge);
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How do we reconcile this with the manifest change we (seem to) observe?

The problem of time - possible solutions(I)

1) Kuchař:
the relevant observables should not have vanishing Poisson brackets with *all* of the constraints. He agrees with the plan to the level of the diffeomorphism but does not agree that we should apply the same reasoning to the Hamiltonian constraint.

> *[$\mathcal{H}$] generates the dynamical change of data from one hypersurface to another. The hypersurface itself is not directly observable, just as the points $x \in \Sigma$ are not directly observable. However, the collection of the canonical data $(q_{ab}(1), \pi^{ab}(1))$ on the first hypersurface is clearly distinguishable from the collection $(q_{ab}(2), \pi^{ab}(2))$ of the evolved data on the second hypersurface. If we could not distinguish between those two sets of data, we would never be able to observe dynamical evolution.*

Kuchař concludes that "if we could observe only constants of motion, we could never observe any change". On this basis he distinguishes between two types of variable: *observables* and *perennials*. The former class are dynamical variables that remain invariant under spatial diffeomorphisms but *do not* commute with the Hamiltonian constraint; while the latter are observables that *do* commute with the Hamiltonian constraint. Kuchařs key claim is that one can observe dynamical variables that are not perennials.
The key point is that the Hamiltonian constraint should not be seen as a generator of gauge transformations. Consequently the observables do not act on the space of solutions $\mathcal{F}_0$.

The problem of time - possible solutions (II)

Kuchařs method involves:

- finding four (scalar) fields $X^A(x)(A = 0, 1, 2, 3)$ representing a spacelike embedding $X^A : \Sigma \to \mathcal{M}$ of a hypersurface Σ in the spacetime manifold $\mathcal{M}$. These kinematical variables are to be understood as positions in the manifold, and the dynamical variables (separated out from the former variables within the phase space and representing the true degrees of freedom of gravity) are observables evolving along the manifold;
- interpreting the constraints as conditions that identify the momenta P_A conjugate to X^A with the energy-momenta of the remaining degrees of freedom: they thus determine the evolution of the true gravitational degrees of freedom between hypersurfaces. (solving the constraints on the classical level, internal time)

The quantization leads to a Tomonaga-Schwinger equation

$$\mathrm{i}\hbar \frac{\delta \Psi[\phi^r(x)]}{\delta X^A(x)} = h_A\left(x; X^B, \phi^r, p^s\right] \Psi[\phi^r(x)], \qquad (r, s = 1, 2) \tag{4}$$

in which the variables X^A stay classical (as t in the Schrödinger equation).

There are many problems: multiple-choice, no global time, problem in defining the Hamiltonian h_A,...

The problem of time - possible solutions (III)

2) Matter as an internal time (Brown and Kuchař):
Introduction of matter variables coupled to spacetime geometry instead of (functionals) of the gravitational variables. One consider a dust field filling all space. The dust play the role of time. These variables are used to label spacetime points. This includes an internal time variable against which systems can evolve, and which can function as the fixed background for the construction of the quantum theory. A Schrödinger equation can be written and the Hamiltonian appearing there does not depend on the dust variables.

3)Unimodular gravity (Unruh):
It is a modification of general relativity, according to which the cosmological constant is taken to be a dynamical variable for which the conjugate is taken to be cosmological time.4° The result is that the Hamiltonian constraint is augmented by a cosmological constant term $\lambda + \sqrt{q(x)}$ giving the Hamiltonian constraint $\lambda + \sqrt{q(x)}\mathcal{H} = 0$ The presence of this extra term (and its conjugate τ) unfreezes the dynamics, thus allowing for a time-dependent Schrödinger equation describing evolution with respect to τ.

4) Definition of time after quantization:
It is the problem of extracting a notion of time from timeless dynamics described by the Wheeler-DeWitt equation. A consequence of the timeless nature of this equation is the problematic implementation of an inner product for state vectors. DeWitt (inspired by the Klein-Gordon inner product) proposed the following definition for the inner product of two solutions of the Wheeler-DeWitt equation

$$(\Psi_1, \Psi_2) = \mathrm{i} \int \prod_x \mathrm{d}\Sigma^{ab}(x) \Psi_1^*[q_{ab}] \left(G_{abcd} \frac{\overrightarrow{\delta}}{\delta q_{cd}} - \frac{\overleftarrow{\delta}}{\delta q_{cd}} G_{abcd} \right) \Psi_2[q_{ab}]$$

The product is taken over all the points of 3d-space, the integration is over a $5 \times \infty^3$-dimensional surface in the space of metric q_{ab}, $\mathrm{d}\Sigma^{ab}$ is the surface element.
This inner product is invariant under the deformation of the $5 \times \infty^3$ surface. But it is not positive definite, it vanishes for real solutions of the Wheeler-DeWitt equation. Moreover no separation into positive and negative frequencies, as for the Klein-gordon equation, is available in general for the Wheeler-DeWitt equation, so the problem of "negative probability" can not be avoided.

WKB approximation

Starting point is the ansatz

$$\Psi[q_{ab}] = C[q_{ab}] \exp\left(\frac{\mathrm{i}}{\hbar} S[q_{ab}]\right) , \tag{5}$$

where C is slowly varying with respect to S. Then one seeks a solution of the Wheeler-DeWitt equation as a power-series in $S[q_{ab}]$.
One obtains for the leading-order:

1. $S[q_{ab}]$ obeys the Hamilton-Jacobi equation for the gravitational field (the same equation for the classical action). This happens also in standard quantum theories.
2. If one finds a state depending on a parameter playing the role of internal time, then $\Psi[q_{ab}]$ obeys a Schrödinger that uses a time-derivative with respect to this internal time (cosmological models).

There are problems:

- It is unclear how the fact that S obeys the Hamilton-Jacobi equation of GR is meant to yield classical spacetimes (decoherence not available).
- The next term in the WKB expansion do not obey a Schrödinger equation.
- What is the meaning of superpositions of WKB states?

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