Connection Variables in General Relativity

Mauricio Bustamante Londoño

Instituto de Matemáticas
UNAM Morelia

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- We can think the connection as an object which give us information about curvature (by mean of the parallel transport), it suggests to look for a formulation of general relativity, not in metric variables, but in connection variables.
- Connection variables are fundamentals variables in the gauge theories. Maybe we could take some tools of gauge theories in understanding general relativity. As we will see, it will happen.

The frame fields

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On the tangent space in a point x of the spacetime $\mathcal{M}$ we can choose a usual basis of partial derivatives $\{(\partial_\mu)_x\}$ and an orthonormal basis of *tetrads* $\{e_I(x)\}$ such that $e_I(x) = e_I^\mu(x)\partial_\mu$ and $e_I^\mu e_J^\nu g_{\mu\nu} = \eta_{IJ}$.

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It is easy to check that the frame fields (tetrads) transforms, locally, under the Lorentz group, that is to say, the symmetry of such vectors is $SO(4)$ (Euclidean case).

Note that more technically we are dealing with the frame bundle on $\mathcal{M}$ with structure group $SO(4)$ and then we can define an one-form conncetion ω as:

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A volume element can be written as $\epsilon_{IJKL} e^I \wedge e^J \wedge e^K \wedge e^L$ where ϵ_{IJKL} is a totally antisymmetric symbol with $\epsilon_{0123} = 1$.

Palatini's action

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Now we have all the elements needed to write the Hilbert-Einstein action in terms of frame fields. A simple calculation gives us

$$S[g] = \int d^4x \sqrt{g} R \longrightarrow S[e, \omega] = \frac{1}{2} \int \epsilon_{IJKL} e^I \wedge e^J \wedge R[\omega]^{KL}$$

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By taking variations of the action we obtain the equations of motion:

$$\epsilon_{IJKL}e^J \wedge e^K \wedge R^{KL} = 0 \longrightarrow \text{ Eisntein's equations}$$
$$D(e^I \wedge e^J) = 0 \longrightarrow \text{ Free torsion}$$

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- The Palatini's formalism is a *first order* formalism.
- We can recover the usual Einstein's equation.
- As the torsion is defined as $T^I = De^I$ is easy to check that the second equation is equivalent to the free torsion condition.

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If we note that $\star^2 = 1$ then the eigenvectors of such operator are:

$$\underbrace{T^{+}_{\mu\nu} = \frac{1}{2}\left(T_{\mu\nu} + \star T_{\mu\nu}\right)}_{self-dual} \quad \text{and} \quad \underbrace{T^{-}_{\mu\nu} = \frac{1}{2}\left(T_{\mu\nu} - \star T_{\mu\nu}\right)}_{antiself-dual}$$

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$$\left[T_i^+, T_j^+\right] = \sum_{i=1}^{3} \epsilon_{ijk} T_k^+$$

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But the generators of the Lie algebra of $SO(4)$ are precisely antisymmetric matrices, then all that means, in this context, that

$$so(4) \cong so(3) \oplus so(3)$$

The last fact let us to write the connection as

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And then one prove that

$$R^{IJ}[\omega] = R^{IJ}[\omega^{+} + \omega^{-}] = R^{IJ}[\omega^{+}] + R^{IJ}[\omega^{-}],$$

hence,

$$S[e,\omega] = \underbrace{S[e,\omega^{+}]}_{S^{+}} + \underbrace{S[e,\omega^{-}]}_{S^{-}}.$$

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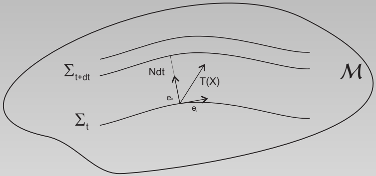
That means that the equations of motion obtained with S are the same that the obtained with S^+, in other words, we just need the information contained in the half of the Palatini's action to describe the dynamics of the gravitational field!.

The self-dual action

As in ADM formalism, consider a foliation of spacetime $\mathcal{M}$ given by the diffeomorphism $\mathbb{R} \times \Sigma$, where Σ is a compact, orientable 3-manifold.



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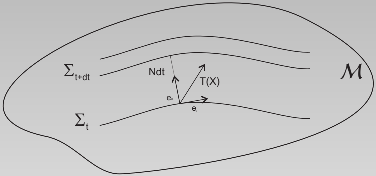

The foliation vector $T(X)$ can be written in terms of the shift vector and lapse function as $Ne_0 + N^i e_i$.
Before to write the action in terms of N and N^i it is better for our proposes to take into account the following relations:

- If $q_{ab} = \delta_{ij} e^i_a e^j_b$ is the induced metric on Σ then is easy to see that $\sqrt{-g} = N\sqrt{q}$.

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or

$$S = \int dx^0 \int d^3x \left(\tilde{N} R^{ij}{}_{ab} E^a_i E^b_j - 2N^a R^{0i}{}_{ab} E^b_i + 2R^{0i}{}_{0a} E^a_i \right)$$

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and let's "make-up" her a little:
Define a $su(2)$- valued one form connection as $A^i_a = 2\omega^i_a$ and note that the two-form curvature associated with this connection is

$$F^i_{ab} = \left(dA^i\right)_{ab} + [A, A]^i_{ab} = 2R^{0i}{}_{ab}$$

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Also $2R^{0i}{}_{0a} = \partial_0 A_a^i + D_a \lambda^i$, where $\lambda^i = -2\omega_0^i$.

Finally, the action for general general relativity in terms of connection variables or Ashtekar variables is

$$S = \int dt \int d^3x \left[\left(\partial_0 A^i_a\right) E^a_i - \left(N^a F^k{}_{ab} E^b_k + \lambda^i D_a E^a_i - \frac{1}{2} N F^{ij}{}_{ab} E^a_i E^b_j \right) \right].$$

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and the Hamiltonian of the theory is

$$H = \int_\Sigma d^3x \left(\underbrace{N^a F^i{}_{ab} E_i^b}_{\mathcal{D}[N^a]} + \underbrace{\lambda^i D_a E_i^a}_{\mathcal{G}[\lambda^i]} - \frac{1}{2} \underbrace{N F^{ij}{}_{ab} E_i^a E_j^b}_{\mathcal{S}[N]} \right)$$

Gauss constraint

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Gauss constraint generate $SU(2)$ gauge transformations. General relativity looks like a Yang-Mills theory!.

Diffeomorphisms constraint

Variations on N^a produce the *diffeomorphisms constraint*

$$F^i{}_{ab} E^b_i = 0.$$

Defining $\tilde{\mathcal{D}}[N^b] = \mathcal{D}[N^b] - \mathcal{G}[N^b A^j_b]$, we find

$$\left\{ A^i_a, \tilde{\mathcal{D}}[N^b] \right\} = \mathfrak{L}_{N^b} A^i_a$$
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This constraint generate the diffeomorphisms on the 3-manifold Σ.

Scalar constraint

Variations on N produce the *Scalar constraint*

$$NF^{ij}{}_{ab}E_i^a E_j^b = 0.$$

Its Poisson brackets with the phase space variables is:

$$\begin{aligned}
\{A_a^i(x), \mathcal{S}[N]\} &= N\epsilon^{ij}{}_k F_{ab}^k(x) E_j^b(x) \\
\{E_i^a(x), \mathcal{S}[N]\} &= D_b\left(N\epsilon^{jk}{}_i E_j^b E_k^a\right).
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Although is not evident, this constraint generate transverse moves of Σ and together with the last constraint we obtain the diffeomorphisms on $\mathcal{M}$.

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Now let's go to the quantization...

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