The program of Loop Quantum Gravity

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Contents

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But, how can we integrate over a space of connections? It is mandatory to endow this space with a measure.

Making a measure

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The key idea is to assign a group element $g_e \in G$ to each edge e of a graph embedded into the spatial manifold Σ.
If we achieve to identify $\mathcal{A}$ with something like G^∞ then we can take the fact that the infinite product of *probability* measure spaces is well defined and because G is a compact group we could obtain $\mathcal{DA}$ through the products of the Haar measure on G, in other words:

$$\mathcal{D}\mathcal{A} = \prod_e dg_e$$

To make it, define the space $\mathcal{A}_e$ by

$$\mathcal{A}_e = \left\{ F : \mathcal{P}_{e(0)} \to \mathcal{P}_{e(1)} | F(xg) = F(x)g \right\} .$$

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Then, fixing a trivialization of the bundle in the endpoints of the curve e we may identify $\mathcal{A}_e$ with a copy of the group G and hence,

$$\mathcal{A} = \mathcal{A}_{e_1} \times \cdots \times \mathcal{A}_{e_n} \cong G^n.$$

With all that we define a space $Fun_0(\mathcal{A})$ as the space generated by functionals $\psi(A)$ of the form

$$\psi(A) = f\left(\mathrm{P}\exp\int_{e_1} A, \ldots, \mathrm{P}\exp\int_{e_n} A\right).$$

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And $Fun(\mathcal{A})$ as the completion of $Fun_0(\mathcal{A})$ in the sup norm

$$\|\psi\|_\infty = \sup_{A\in\mathcal{A}}|\psi(A)|.$$

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The invariant states under gauge transformations are wave functions $\Psi[A]$ on the quotient of $\mathcal{A}$ by the group $\mathcal{G}$ of gauge transformations, namely, $L^2(\mathcal{A}/\mathcal{G})$.

Spin network states

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Consider an graph Γ embedded in the manifold Σ. Choose an orientation for each link e in the graph and assign it an irreducible representation ρ_e of G. Then, $H_e[A] = \rho_e\left(\mathrm{P}\exp\int_e A\right)$ can be seen as a matrix with components $H_e[A]^i_j$. Now, take the tensor product of all these matrices and obtain a "big tensor" $H(A)$.

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To each vertex assign a tensor in the tensor product of all representations incoming and outcoming (an *intertwining* operator) of the vertex. Another "big tensor" I is obtained with the tensor product of all the intertwining operators.
We get a *spin network state* by the contraction of all the indices of $H(A) \otimes I$.

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- By the Peter-Weyl theorem $(L^2(G) \cong \oplus\rho \otimes \rho^*)$, these states span the Hilbert space of gauge invariant states in $\mathcal{H}(kin)$, namely, they span $L^2(\mathcal{A}/\mathcal{G})$
- By the later, *spin network states* are solutions to Gauß constraint!.

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But we have experience with this kind of situations. We seek states in the space of distributions $\mathcal{H}^*_{kin}$ of the kindematical Hilbert space $\mathcal{H}_{kin}$ which are diffeomorphism invariant.
What do we mean when we talk about diffeomorphism invariant states?

Representation of $Diff(\Sigma)$ on $\mathcal{H}^*_{kin}$

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Consider an operator U_ϕ, $\phi \in Diff(\Sigma)$, associated with the diffeomorphism transformations and such that:

$$(\mathcal{U}_\phi \Phi)\, \Psi_{\mathsf{S}} = \Phi\, (\mathcal{U}_\phi \Psi_{\mathsf{S}}) = \Phi\left(\Psi_{\phi(\mathsf{S})}\right),$$

where $\Phi \in \mathcal{H}^*_{kin}$ and $\Psi_{\phi(\mathsf{S})}$ is the spin network state which results by applying ϕ to each one of his labels $S = (\Gamma, j_e, i_n)$ in a specific manner.

The diffeomorphism invariant states are those which satisfies

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And the solutions of this equation are given by states of the form

$$(\Psi_{\mathsf{S}}| = \sum_{\Psi_{\mathsf{S}'} \in \mathcal{O}(\Psi_{\mathsf{S}})} \langle \Psi_{\mathsf{S}'}|$$

where $\mathcal{O}(\Psi_{\mathsf{S}'})$ denotes the orbit of $\Psi_{\mathsf{S}'}$ under the $\mathit{Diff}(\Sigma)$-action.
These states don't change when $\mathcal{U}_\phi$ acts on them.

Now, to conclude the quatization of diffeomorphism constraint, we define the scalar product between two of such states as

$$(\Psi_S|\Psi_{S'}) = \sum_{\Psi_{S''} \in \mathcal{O}(\Psi_S)} \langle \Psi_{S''}|\Psi_{S'}\rangle \,.$$

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Hence, the states in $\mathcal{H}^*_{kin}$ which don't change with the action of $\mathcal{U}_\phi$ are the states which solve the diffeomorphism constraint!.

Quantization of scalar constraint

The geneal idea is to put the constraint in the form

$$\mathcal{S}[N] = \int_\Sigma d^3x \; N\epsilon^{abc}\delta_{ij}F^i_{ab}\left\{A^j_c, \mathbf{V}\right\}$$

where $\mathbf{V}$ is the volume of Σ.

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where $\mathbf{V}$ is the volume of Σ.
Taking into account that for an infinitesimal loop α on a plane, the curvature and the Poisson bracket above are regulatizated as follow

$$H_\alpha[A] - H^{-1}_\alpha[A] = \epsilon^2 F^i_{ab}\tau_i + O(\epsilon^4)$$
$$H^{-1}_{e_a}\left\{H_{e_a}[A], V\right\} = \epsilon\left\{A^i_a, V\right\} + O(\epsilon^2),$$

the operator becomes:

$$\hat{S}[N] = \lim_{\epsilon\to 0}\sum_I N_I\epsilon^{abc}\mathrm{tr}\left[\left(\hat{H}_\alpha[A] - \hat{H}^{-1}_\alpha[A]\right)\hat{H}^{-1}_{e_a}\left\{\hat{H}_{e_a}[A], \hat{V}\right\}\right]$$

This limit doesn't exists in general, but does in the Hilbert space of diffeomorphism invariant states because there, the dependence of ϵ becomes trivial. The analysis of scalar constraint involves a lot of subtleties and technical procedures that I am not going to show here (see the references).

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References

- J.Baez, *Spin networks in gauge theory*, Advances in Mathematics 117, 253-272, 1996.
- J.Baez, *Spin networks in non perturbative quantum gravity*, arXiv:gr-qc/9504036.
- A.Perez, *Introduction to loop quantum gravity and spin foams*, arXiv:gr-qc/0409061.
- C.Rovelli, *Quantum Gravity*, Cambridge University Press, 2004.
- Jose Antonio Zapata's and Robert Oeckl's comments.