0.1 BF Theory: Classical Field equations

- gauge group: Lie group G, with Lie algebra $\mathfrak{g}$ equipped with invariant nondegenerate bilinear form $\langle \cdot\,,\,\cdot \rangle_{\mathfrak{g}}$
- spacetime: smooth, oriented manifold M
- choose principal G-bundle $P \xrightarrow{\pi} M$, vector bundle associated to $P \xrightarrow{\pi} M$ via adjoint action of G on $\mathfrak{g}$ is $\mathrm{ad}(P) \xrightarrow{\pi_a} M$ with $\mathrm{ad}(P) = (P \times \mathrm{Ad}(G))/G$ and $\mathrm{Ad}(G)$ the adjoint representation of G
- basic fields of theory:

  A a connection on P

  $F = dA + A \wedge A$ is curvature of A: an $\mathrm{ad}(P)$-valued 2-form on M

  E an $\mathrm{ad}(P)$-valued $(n-2)$-form on M
- pick local trivialization, can think of A, F, E as $\mathfrak{g}$-valued 1,2,(n - 2)-forms on M, with local coordinates $\{x^j\}$ on M and basis $\{e_m\}$ of $\mathfrak{g} \cong T_{\mathbb{1}}G$

$$\begin{aligned} A &= A^m_a \, dx^a \otimes e_m \\ F &= F^l_{b_1,b_2} \, dx^{b_1} \wedge dx^{b_2} \otimes e_l \\ E &= E^k_{j_1,\ldots,j_{n-2}} \, dx^{j_1} \wedge \ldots \wedge dx^{j_{n-2}} \otimes e_k \end{aligned}$$

- Lagrangian of BF theory:

$$\mathcal{L} = \mathrm{Tr}\,(E \wedge F)$$

  $\mathrm{Tr}\,(E \wedge F)$ is n-form constructed by taking wedge product of form parts of E and F and using bilinear form $\langle \cdot\,,\,\cdot \rangle_{\mathfrak{g}}$ to pair $\mathfrak{g}$-valued parts

$$\mathrm{Tr}\,(E \wedge F) = E^k_{j_1,\ldots,j_{n-2}} \, F^l_{b_1,b_2} \, \langle e_k, e_l \rangle_{\mathfrak{g}} \, dx^{j_1} \wedge \ldots \wedge dx^{j_{n-2}} \wedge dx^{b_1} \wedge dx^{b_2}$$

- field equations: $F = 0$ $\quad d_A E = 0$

  with covariant exterior derivative d_A
- $F = 0$, thus connection A flat, all flat connections locally equivalent, up to gauge transformations
- BF-action invariant under E gauge: $E \mapsto E + d_A\eta$ for some $\mathfrak{g}$-valued $(n-3)$-form η
- since A flat, for any E with $d_A E = 0$ there exists an η such that locally $E = d_A\eta$ since locally all closed forms are exact
- thus locally all solutions of BF theory are equal modulo gauge transformations: BF theory is a topological field theory

- 3d-GR is special case of BF theory: $n = 3$ and $G = \mathrm{SO}(2,1)$ and $\langle\,\cdot\,,\,\cdot\,\rangle_{\mathrm{so}(2,1)}$ is minus its Killling form
- if 1-form $E:\ TM \to \mathrm{ad}(P)$ is bijective, then nondegenerate Lorentzian metric defined by $g(x)(v,w) \overset{\text{def.}}{=} \langle E(x)v,\ E(x)w\rangle_{\mathrm{so}(2,1)}$ for any vectors $v, w \in T_xM$
- E-form can be used to pull back connection A to metric connection Γ on TM, $d_AE = 0$ says Γ is torsion-free, thus Γ is unique Levi-Civita connection on M, $F = 0$ thus Γ flat thus metric g flat
- in 3d spacetime: vacuum Einstein equations say metric is flat
- many different A and E fields correspond to same metric, but all are related by gauge transformations
- thus in 3d spacetime, BF theory with $G = \mathrm{SO}(2,1)$ is alternate formulation of Lorentzian GR without matter fields
- with $G = \mathrm{SO}(3)$ we obtain Riemannian GR, which is easier to quantize than Lorentzian GR

0.2 BF Theory: Classical phase space

- assume spacetime M to be product $\mathbb{R}_{\text{time}} \times S$ with S smooth, oriented $(n-1)$-dim manifold representing space (no loss of generality, since any oriented hypersurface in any n-dim. manifold has neighborhood of this form)
- work in **temporal gauge**: time component of connection A vanishes
- momentum canonically conjugate to A is:

$$\frac{\partial \mathcal{L}}{\partial \dot{A}} = E$$

  (in electromagnetism: electric field is canonically conjugate to vector potential, this is why we use notation E, instead of B as originally done in BF theory, so we can easier exploit analogies)
- $P|_S = \{0\} \times S$ is restriction of bundle P to time-zero slice $\{0\} \times S$
- before imposing constraints, *configuration* space of BF theory is infinite-dim. vector space $\mathcal{A}$ of connections on $P|_S$
- **kinematical phase space** is corresponding classical *phase* space, which is cotangent bundle $T^*\mathcal{A}$, a point in this phase space consists of a connection A on $P|_S$ and an $\mathrm{ad}(P|_S)$-valued $(n-2)$-form E on S
- **symplectic structure** of kinematical phase space $T^*\mathcal{A}$ is

$$\omega\big((\delta A, \delta E),\, (\delta A', \delta E')\big) = \int\limits_S \mathrm{Tr}\,(\delta A \wedge \delta E' - \delta A' \wedge \delta E)$$

- field equations put constraints on initial data A and E on time-zero slice S:

$$0 = B = dA + A \wedge A \qquad\qquad d_A E = 0$$

  B is curvature of connection A, analogous to magnetic field in electromagnetism

- to deal with these constraints, apply symplectic reduction to our kinematical phase space $T^*\mathcal{A}$ to obtain physical phase space
- constraint $d_A E = 0$ is called Gauss law, is analogous to equation in vacuum electromagnetism stating that divergence of electric field vanishes, it generates action of gauge transformations on $T^*\mathcal{A}$
- symplectic reduction with respect to Gauss constraint yields **gauge-invariant phase space** $T^*(\mathcal{A}/\mathcal{G})$ with $\mathcal{G}$ being group of gauge transformations of bundle $P|_S$
- no-curvature constraint $B = 0$ is analogous to requiring magnetic field to vanish, no such requirement exists in electromagnetism, this constraint is special to BF theory, it generates transformations of the form

$$E \mapsto E + d_A \eta$$

  thus these transformations are really gauge symmetries as claimed before

- symplectic reduction with respect to $B = 0$-constraint yields **physical phase space** $T^*(\mathcal{A}_0/\mathcal{G})$ with $\mathcal{A}_0$ being space of flat connections on $P|_S$
- points in physical phase space correspond to physical states of *classical* BF theory

0.3 BF Theory: Canonical quantization

- all phase spaces (kinematical, gauge-invariant, physical) of BF theory are cotangent bundles, hope that quantizing each of them yields Hilbert space of square-integrable functionals on corresponding configuration space

$$\begin{tikzcd}
T^*(\mathcal{A}) \arrow[r, "\text{quantize}"] \arrow[d, "\text{constrain}"'] & \mathcal{H}_{\text{kin}} = L^2(\mathcal{A}) \arrow[d, "\text{constrain}"] \\
T^*(\mathcal{A}/\mathcal{G}) \arrow[r, "\text{quantize}"] \arrow[d, "\text{constrain}"'] & \mathcal{H}_{\text{gauge}} = L^2(\mathcal{A}/\mathcal{G}) \arrow[d, "\text{constrain}"] \\
T^*(\mathcal{A}_0/\mathcal{G}) \arrow[r, "\text{quantize}"] & \mathcal{H}_{\text{phys}} = L^2(\mathcal{A}_0/\mathcal{G})
\end{tikzcd}$$

- since kinematical $T^*\mathcal{A}$ and gauge-invariant phase space $T^*(\mathcal{A}/\mathcal{G})$ are inifinite-dim., it is difficult to rigorously define $L^2(\mathcal{A})$ and $L^2(\mathcal{A}/\mathcal{G})$
- LQG breaks away from Fock space formalism, taking holonomies along paths as basic variables to be quantized, thereby achieves rigorous and background-free, hence diffeomorphism-invariant, definition of these Hilbert spaces
- in resulting picture, basic excitations are not pointlike particles but 1-dim. edges of spin networks, which represent $(n-1)$-dim. space, and 1-dim. Feynman diagrams become replaced by 2-dim. spinfoams representing n-dim. spacetime
- assume compact, connected gauge group G and real, analytic manifold S representing space, smooth S and nonconnected G can be handled, but *concompact* G not yet, thus later consider quantization of vacuum Einstein equation for Riemannian metrics but not for Lorentzian ones, $\mathrm{SO}(n)$ compact, $\mathrm{SO}(1,n)$ not
- $\mathrm{Fun}(\mathcal{A})$ is algebra consisting of all functionals on space of connections $\mathcal{A}$ of the form

$$\begin{aligned} \Psi_{f,\Gamma}[A] &= f(\{h_{\gamma_k}[A] \mid \gamma_k \in \Gamma\}) \\ h_{\gamma_k}[A] &\overset{\text{def.}}{=} \mathcal{P}\exp\int_{\gamma_k} A \end{aligned}$$

  for $A \in \mathcal{A}$, γ_k real-analytic paths in space S, Γ a collection of such paths, $h_{\gamma_k}[A]$ the parallel transport of A along this path with $\mathcal{P}$ denoting path ordering, f continuous, complex-valued function of finitely many such holonomies
- define inner product on $\mathrm{Fun}(\mathcal{A})$, complete it, obtain Hilbert space $\mathcal{H}_{\text{kin}} = L^2(\mathcal{A})$

$$\left\langle \Psi_{f_1,\Gamma_1},\ \Psi_{f_2,\Gamma_2} \right\rangle \overset{\text{def.}}{=} \int \left(\prod_r^{e_r \in \Gamma_{12}} dh_{e_r} \right) \overline{f_1(\{h_{\gamma_k}[A] \mid {}^{\gamma_k \,\in\, \Gamma_{12}}_{\gamma_k \subseteq \tilde{\gamma}_k \in \Gamma_1}\})}\; f_2(\{h_{\gamma_k}[A] \mid {}^{\gamma_k \,\in\, \Gamma_{12}}_{\gamma_k \subseteq \tilde{\gamma}_k \in \Gamma_2}\})$$

  with Γ_{12} being "smallest" graph containing both Γ_1 and Γ_2 and dh being the normalized Haar measure of the gauge group G
- finite collection of real-analytic paths $\gamma_j : [0,1] \to S$ forms a **graph in** S if they are embedded in S and intersect only at their endpoints (if at all), call paths **edges** and endpoints **vertices**, edge γ_j is **outgoing** from a vertex v if $v = \gamma_j(0)$ and **incoming** to v if $v = \gamma_j(1)$
- **closed spin network** $N = (\Gamma, \rho, \iota)$ in S with symmetry group G is triple consisting of:

  1. graph Γ in S

2. edge-labeling ρ of each edge e of Γ by an irrep ρ_e of G
3. vertex-labeling ι of each vertex v of Γ by intertwiner ι_v, which maps tensor product of irreps of incoming edges $e_k^{\text{in}}(v)$ of v to tensor product of irreps of outgoing edges $e_k^{\text{out}}(v)$

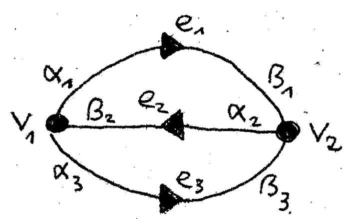
$$\iota_v : \ \left(\rho_{e_1^{\text{in}}(v)} \otimes ... \otimes \rho_{e_{n(v)}^{\text{in}}(v)}\right) \ \rightarrow \ \left(\rho_{e_1^{\text{out}}(v)} \otimes ... \otimes \rho_{e_{o(v)}^{\text{out}}(v)}\right)$$

with $n(v)$ and $o(v)$ = number of incoming/outgoing edges of v.

- in case of no outgoing edges for some vertex, we can think of the intertwiner as going from the incoming irreps to the 1-dim. trivial irrep, i.e., the complex numbers
- $\text{Fun}(\mathcal{A}/\mathcal{G})$ is algebra of all gauge-invariant functionals in $\text{Fun}(\mathcal{A})$, complete it in above norm yields gauge-invariant Hilbert space $\mathcal{H}_{\text{gauge}} = L^2(\mathcal{A}/\mathcal{G})$, this space is spanned by **spin network states**:

$$\Psi_N[A] \stackrel{\text{def.}}{=} \prod_r^{e_r \in N} \rho_{e_r}(h_{e_r}[A])^{\alpha_r}_{\beta_r} \prod_j^{v_j \in N} (\iota_{v_j})^{\beta_{j_1}...\beta_{j_{n(j)}}}_{\alpha_{j_1}...\alpha_{j_{o(j)}}}$$

with $n(j)$ the number of incoming edges of the vertex v_j and $o(j)$ the number of its outgoing edges (and sum convention), example:



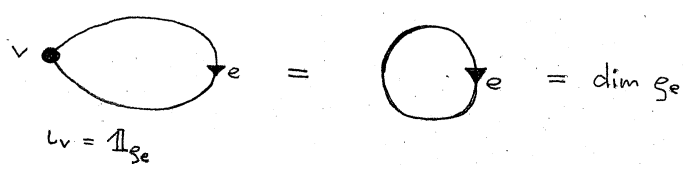
$$\Psi_N[A] = \rho_{e_1}(h_{e_1}[A])^{\alpha_1}_{\beta_1} \ \rho_{e_2}(h_{e_2}[A])^{\alpha_2}_{\beta_2} \ \rho_{e_3}(h_{e_3}[A])^{\alpha_3}_{\beta_3} \ (\iota_{v_1})^{\beta_2}_{\alpha_1\alpha_3} \ (\iota_{v_2})^{\beta_1\beta_3}_{\alpha_2}$$

- arrows indicate orientation of edges, write little alphas at beginning and little betas at ends of edges
- think of holonomy of each edge as a group element, put it into the irrep labeling the edge, pick basis for irrep, think of group element as matrix with alpha of edge as superscript and beta as subscript
- write intertwiners as tensors with alphas of outgoing edges as subscripts and betas of incoming as superscripts
- this recipe ensures each alpha and beta appear exactly once as superscript and once as subscript, take Einstein sum over them, get complex number depending on connection A

- $\Psi_N : \mathcal{A} \to \mathbb{C}$ is continuous functional of connection A, depending on finitely many holonomies, thus $\Psi_N \in \mathrm{Fun}(\mathcal{A})$
- it can be shown that Ψ_N is gauge-invariant, thus $\Psi_N \in \mathrm{Fun}(\mathcal{A}/\mathcal{G})$, moreover it can be proven that the spin network states span $\mathcal{H}_{\text{gauge}} = L^2(\mathcal{A}/\mathcal{G})$
- classically imposing no-curvature constraint yields physical phase space $T^*(\mathcal{A}_0/\mathcal{G})$, however in most cases there is no natural measure on $(\mathcal{A}_0/\mathcal{G})$, so a Hilbert space cannot be defined unambiguously
- have to accept mere vector space instead of Hilbert space, every functional in $\mathrm{Fun}(\mathcal{A}/\mathcal{G})$ restricts to a functional in $\mathrm{Fun}(\mathcal{A}_0/\mathcal{G})$, the space of gauge-invariant functionals on flat connections, call elements of this space **physical states**, even if there is no natural measure on $\mathcal{A}_0/\mathcal{G}$
- space of physical states without inner product is of limited use, proceed with study because its mathematics turn out to be useful for other things
- every spin network N in space S gives a functional in $\mathrm{Fun}(\mathcal{A}_0/\mathcal{G})$ via feeding $\Psi_{N[A]}$ with flat connections only, and with reasonable topology (e.g.: sup norm topology) finite linear combinations of spin network states Ψ_N are dense in $\mathrm{Fun}(\mathcal{A}_0/\mathcal{G})$
- work out rules for spin networks describing same physical state
- spin networks in S related through a homotopy define same physical state, because holonomy of *flat* connection does not change under homotopies
- from representation theory of gauge group G come so called **skein relations**:
- spin network N consisting of contractible loop and one vertex labeled by identity intertwiner (such vertices usually not drawn), corresponding spin network state

  $$\Psi_{N[A]} = \mathrm{Tr}\, \rho_e(h_{e[A]}) \overset{A\,\text{flat}}{=} \dim \rho_e$$

  because holonomy of *flat* connection around contractible loop is identity, thus its trace is dim ρ_e, can replace this part in every spin network containing it and instead multiply remaining spin network state by dim ρ_e and obtaion same physical state, skein relation:

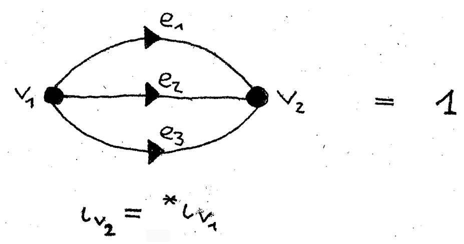
  

- consider trivalent vertices: tensor product of any pair ρ_1, ρ_2 of irreps can be written as direct sum of irreps ρ_k, pick one $\rho_3 \in \{\rho_k\}$, projection from

$\rho_1 \otimes \rho_2$ to ρ_3 is intertwiner which can be used to label trivalent vertices, usually normalized to obtain intertwiner

$$\iota : \ (\rho_1 \otimes \rho_2) \ \rightarrow \ \rho_3 \qquad\qquad \mathrm{Tr}\ \iota\ {}^*\iota \ = \ 1$$

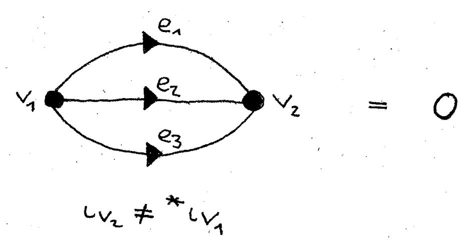
- whenever graph sits in S in contractable way: skein relation



$$\Psi_{N[A]} \overset{A\,\mathrm{flat}}{=} \mathrm{Tr}\ \iota_{v_1}\ {}^*\iota_{v_1} \ = \ 1$$

we can eliminate this part in all spin networks

- if irrep ρ_3 appears more than once in direct sum decomposition $(\rho_1 \otimes \rho_2)$ then there is more than one intertwiner of above form, can always choose basis of such intertwiners obeying $\iota_1^*\iota_2 \ = \ 0$ for any two distinct intertwiners ι_1, ι_2 in basis, skein relation:

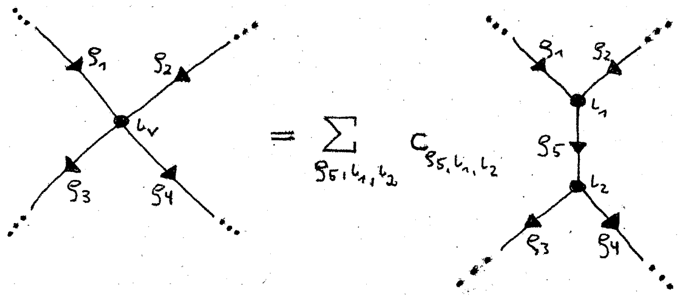


$$\Psi_{N[A]} \overset{A\,\mathrm{flat}}{=} \mathrm{Tr}\ \iota_{v_1}\ {}^*\iota_{v_2} \ = \ 0$$

thus functionals of spin networks containing this part vanish

- choose such a basis of intertwiners for each triple of irreps of gauge group G, sufficient to use these intertwiners (and their duals) to label trivalent vertices in order to get enough states to span $\mathrm{Fun}((\mathcal{A}_0/\mathcal{G}))$

- can break any k-valent vertex into trivalent ones using, e.g.: 4-valent vertex, skein relation:



sum is over irreps ρ_5 and intertwiners ι_1, ι_2 in chosen basis, coefficients c depend on these intertwiners, both sides interpreted as parts of the same larger spin network

- other modifications of spin network in S which do not change physical state:

- can reparametrize any edge of spin network in S by any orientation-preserving diffeomorphism of unit interval

- can reverse orientation of an edge while simultaneously dualizing its irrep and appropriately dualizing intertwiners at its vertices

- can subdivide an edge into two edges labeled by same irrep inserting vertex labeled by identity intertwiner

- can erase edges labeled by trivial irrep

- two spin networks in S define same state in $\mathcal{H}_{\text{gauge}} = L^2(\mathcal{A}/\mathcal{G})$ iff they are related through a sequence of these moves and their inverses

- skein relations can be interpreted both as facts about BF theory and about group representation theory, in first sense edges represent curves embedded in space, in second sense edges are abstract notation for representations of gauge group G, Baez: "fact that both interpretations are possible shows that in some sense BF theory is nothing but a clever way to encode the representation theory of G in a quantum field theory"

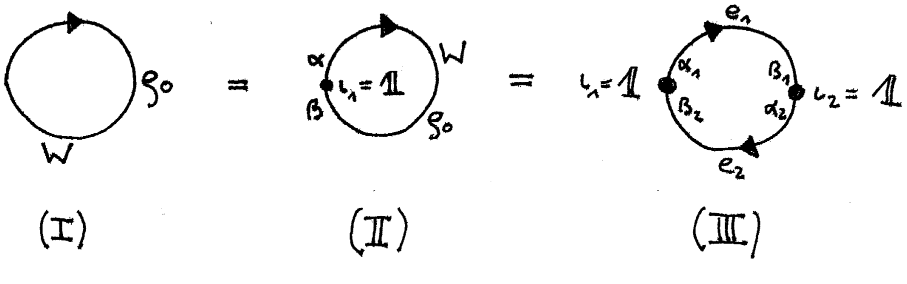
0.4 Observables

- *true* physical observables of BF theory are self-adjoint operators on $\mathcal{H}_{\text{phys}}$ when this is well-defined

- will use term observables for self-adjoint operators on $\mathcal{H}_{\text{gauge}}$ because they are also relevant for other gauge theories

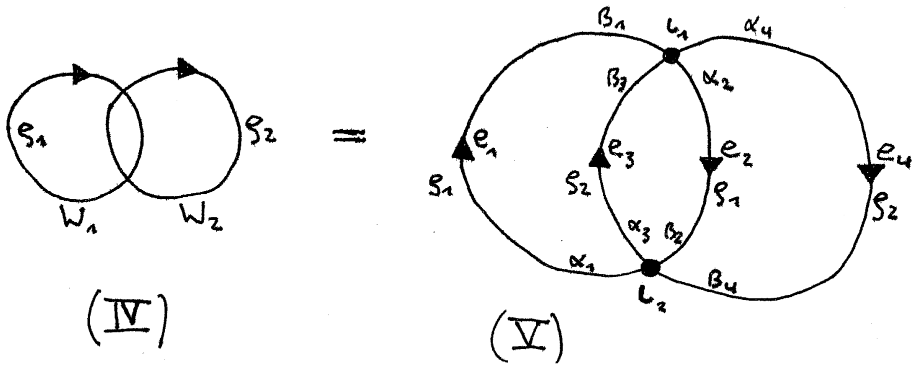
- consider observables of two kinds: gauge-invariant functionals of connection A and of E-field, A analogous to position operator and E to momentum, thus expect A to act as multiplication and E as differentiation

- in order to obtain operators instead of operator-valued distributions we need to smear these fields, i.e., integrate them over some region of space, A is 1-form, thus integrate over path, E is $(n-2)$-form, thus integrate over $(n-2)$-submanifold
- simplest gauge-invariant functional of connection A is Wilson loop which is functional of form $\mathrm{Tr}\ \rho(h_{\gamma}[A])$ for some loop γ in space S and some irrep ρ of gauge group G, Wilson loop contains gauge-invariant information about holonomy of connection A around this loop
- spin networks are generalization of Wilson loops: any spin network N defines operator $\widehat{\Psi}_N[A]$ acting simply as multiplication by functional $\Psi_N[A]$ on elements of $\mathrm{Fun}(\mathcal{A}/\mathcal{G})$
- $\Psi_N[A]$ is bounded functional on $\mathcal{A}/\mathcal{G}$, thus $\widehat{\Psi}_N[A]$ extends to bounded operator on $\mathcal{H}_{\text{gauge}} = L^2(\mathcal{A}/\mathcal{G})$ called **spin network observable**,
- $\Psi_N[A]$ is also bounded functional on $\mathcal{A}_0/\mathcal{G}$, thus $\widehat{\Psi}_N[A]$ extends to bounded operator on $\mathcal{H}_{\text{phys}} = L^2(\mathcal{A}_0/\mathcal{G})$
- spin network observables on $\mathcal{H}_{\text{gauge}}$ not invariant under homotopies of underlying graph Γ of N, but spin network operators on $\mathcal{H}_{\text{phys}}$ are invariant and satisfy skein relation
- any product of Wilson loop observables can be written as finite linear combination of spin network observables, thus spin network observables measure correlations among holonomies of connection A around collection of loops
- example: one Wilson loop as spin network observable



$$
\begin{aligned}
\text{(I)} &\stackrel{\text{def.}}{=} \mathrm{Tr}\ \rho_0(h_W[A]) \\
= \text{(II)} &= \rho_0(h_W[A])^{\alpha}_{\beta}\ \overbrace{\delta_{\alpha\beta}}^{\iota_1} \\
= \text{(III)} &= \rho_0(h_{e_1}[A])^{\alpha_1}_{\beta_1}\ \rho_0(h_{e_2}[A])^{\alpha_2}_{\beta_2}\ \overbrace{\delta_{\alpha_1\beta_2}}^{\iota_1}\ \overbrace{\delta_{\alpha_2\beta_1}}^{\iota_2} \\
&= \rho_0(h_{e_1}[A])^{\alpha_1}_{\alpha_2}\ \rho_0(h_{e_2}[A])^{\alpha_2}_{\alpha_1} \\
&= \Big(\rho_0(h_{e_1}[A])\,\rho_0(h_{e_2}[A])\Big)^{\alpha_1}_{\alpha_1} \\
&= \Big(\rho_0(h_W[A])\Big)^{\alpha_1}_{\alpha_1}
\end{aligned}
$$

- example: two Wilson loops as spin network observable



Wilson loops are independent of each other, intersect in two points, but do not interact/couple, this is respected by ι_1 linking only e_1 with e_2 and e_3 with e_4 but not e_1 with e_4 or e_3 with e_2, analogue ι_2

$$
\begin{aligned}
&(\mathrm{IV}) = \operatorname{Tr} \rho_1(h_{W_1}[A]) \cdot \operatorname{Tr} \rho_2(h_{W_2}[A]) \\
= \; &(\mathrm{V}) = \rho_1(h_{e_1}[A])^{\alpha_1}_{\beta_1} \, \rho_1(h_{e_2}[A])^{\alpha_2}_{\beta_2} \, \rho_2(h_{e_3}[A])^{\alpha_3}_{\beta_3} \, \rho_2(h_{e_4}[A])^{\alpha_4}_{\beta_4} \; \overbrace{\delta_{\alpha_2\beta_1}\,\delta_{\alpha_4\beta_3}}^{\iota_1} \; \overbrace{\delta_{\alpha_1\beta_2}\,\delta_{\alpha_3\beta_4}}^{\iota_2} \\
&\phantom{(\mathrm{V})} = \Big(\rho_1(h_{e_1}[A])\,\rho_1(h_{e_2}[A])\Big)^{\alpha_1}_{\alpha_1} \cdot \Big(\rho_2(h_{e_3}[A])\,\rho_2(h_{e_4}[A])\Big)^{\alpha_3}_{\alpha_3} \\
&\phantom{(\mathrm{V})} = \Big(\rho_1(h_{W_1}[A]))\Big)^{\alpha_1}_{\alpha_1} \cdot \Big(\rho_2(h_{W_2}[A]))\Big)^{\alpha_3}_{\alpha_3}
\end{aligned}
$$

- for $G = \mathrm{U}(1)$ gauge-invariant functional of E-field given by integral over $(n-2)$-dim. submanifold Σ of S

$$\int_\Sigma E$$

which measures flux of electric field through Σ, but integral is not gauge-invariant in nonabelian case

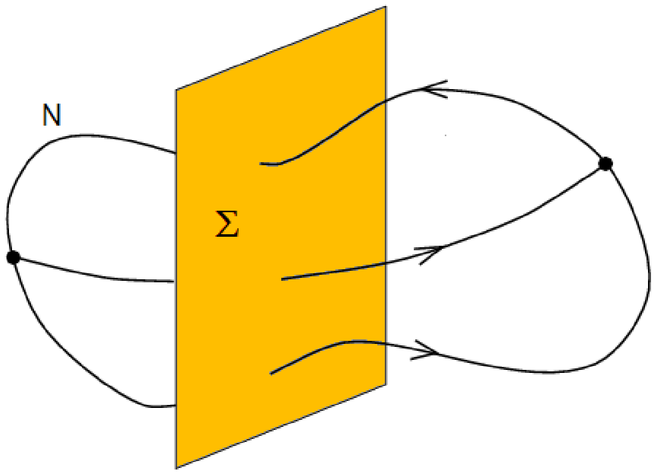
- using some $\mathfrak{g}$-valued function e on Σ and some $(n-2)$-form $d^{\overset{n-2}{}}x$ on Σ compatible with its orientation we can write

$$E|_\Sigma = e\, d^{\overset{n-2}{}}x$$

and a gauge-invariant functional of E independent of the way we write $E|_\Sigma = e\, d^{\overset{n-2}{}}x$ is provided by

$$E(\Sigma) \overset{\text{def.}}{=} \int_\Sigma \underbrace{d^{\overset{n-2}{}}x \, \sqrt{\langle e, e\rangle_{\mathfrak{g}}}}_{|E|}$$

- quantizing $E(\Sigma)$ gives self-adjoint operator $\widehat{E}(\Sigma)$ on $\mathcal{H}_{\text{gauge}} = L^2(\mathcal{A}/\mathcal{G})$ if Σ is real-analytically embedded in space S, give only final results of quantization procedure
- spin network N in space S generically intersects $(n-2)$-submanifold Σ transversely in finitely many points which are not vertices



- then $\widehat{E}(\Sigma)$ acts on states $\Psi_N \in \mathcal{H}_{\text{gauge}} = L^2(\mathcal{A}/\mathcal{G})$ by

$$\widehat{E}(\Sigma)\,\Psi_N = \sum_k \sqrt{C(\rho_k)}\;\Psi_N$$

  with sum over all points p_k of intersection of N and Σ and $C(\rho_k)$ being the Casimir of the irrep ρ_k labeling the intersecting edge (if an edge intersects Σ several times, each point of intersection is counted seperately)
- physical significance of spin network edges is that they represent quantized flux lines of E-field
- 3-dim. BF theory with $G = \mathrm{SU}(2), \mathrm{SO}(3)$ is formulation of 3-dim. Riemannian GR, Σ is curve and $\widehat{E}(\Sigma)$ can be interpreted as length of curve Σ, in 4-dim. BF theory with same gauge groups $\widehat{E}(\Sigma)$ can be interpreted as area of surface Σ
- irreps of SU(2) correspond to spins $j = 0, \frac{1}{2}, 1, \ldots$ and the Casimir of the spin-j irrep is $j(j+1)\mathbb{1}$, thus spin network edge with spin j contributes length/area $\sqrt{j(j+1)}$ to any curve/surface it crosses transversely
- consequence: length/area in 3-dim. Riemannian QG have discrete spectra, since irreps of SO(3) have only integer spins, spectrum is sparser than for SU(2)
- application: blackhole entropy: associate degrees of freedom of event horizon to points of intersection with spin networks, then derive Bekenstein-Hawking entropy proportional to area of event horizon

- if graph Γ of spin network N intersects Σ nongenerically, by subdividing its edges where necessary we can assure:
  - ⊕ if an edge of Γ contains a segment lying in Σ, then the whole edge lies in Σ
  - ⊕ each edge of Γ intersects Σ at most once
  - ⊕ all isolated intersection points are vertices
    (isolated intersection points sit on edges not lying wholely in Σ,
    this is also possible for $\dim \Sigma > 2$)
- divide edges of vertices v lying in Σ in three classes:
  - ⊕ **horizontal edges** lie in Σ, we can achieve that all horizontal edges are outgoing and all nonhorizontal edges are incoming (by orientation reversing where necessary)
  - ⊕ e is called **upward (downward) edge**, if $\{b_1, ..., b_{n-2}, e\}$ is (not) righthanded base of space S, with $\{b_1, ..., b_{n-2}\}$ a righthanded base of Σ (thus for $\dim \Sigma = 2$ upward edges are on the right hand side when looking in direction of Σ)
- can write any intertwiner labeling v as linear combination of intertwiners of special form

$$\iota_v : \ (\rho_v^{\mathrm{up}} \otimes \rho_v^{\mathrm{down}}) \ \rightarrow \ \rho_v^{\mathrm{hor}}$$

  wherein each ρ is is an irreducible summand of the tensor product of all representations labeling upwards/downwards/horizontal edges

- thus can write any spin network state Ψ_N with graph Γ as linear combination of spin network states $\Psi_{N,k}^{\mathrm{spec}}$ with same graph and intertwiners in special form:

$$\Psi_N = \sum_s \Psi_{N,s}^{\mathrm{spec}}$$

- then the general action of $\widehat{E}(\Sigma)$ on states $\Psi_{N,s}^{\mathrm{spec}} \in \mathcal{H}_{\mathrm{gauge}} = L^2(\mathcal{A}/\mathcal{G})$ with intertwiners in special form is given by

$$\widehat{E}(\Sigma)\, \Psi_{N,s}^{\mathrm{spec}} = \sum_{v \,\in\, \Sigma} \frac{1}{2} \sqrt{2C(\rho_v^{\mathrm{up}}) + 2C(\rho_v^{\mathrm{down}}) - C(\rho_v^{\mathrm{hor}})}\; \Psi_{N,s}^{\mathrm{spec}}$$

  with sum over all vertices v in which N intersects Σ (both vertices of horizontal edges sit on Σ and count separately in this sum)

- in generic case considered before: $C(\rho_v^{\mathrm{up}}) = C(\rho_v^{\mathrm{down}})$ and $C(\rho_v^{\mathrm{hor}}) = 0$, thus general action of $\widehat{E}(\Sigma)$ reduces to simpler generic formula given before

0.5 Path integral quantization

- quantization of BF theory via path integral: partition function

$$\mathcal{Z} = \int \mathcal{D}A \int \mathcal{D}E \; \exp i \int_M \mathrm{Tr}\,(E \wedge F)$$

  with $\mathcal{D}A$, $\mathcal{D}E$ denoting some measure over space of connections and E-fields

- in order to give sense to this formal expression, introduce discretization and hope that path integral becomes well defined

- instead of whole spacetime manifold M, define theory only on discrete, finite set of points in spacetime, which we call **vertices**, choose vertices dense enough such we can assume all fields to be varying slowly between vertices

- basic variable of theory is connection A, can be implemented in discrete version by considering parallel transports from one vertex to another one, therefore we need lines connecting the vertices along which we consider the parallel transport to take place, call these lines **edges**

- in order to implement curvature, consider holonomies along closed loops formed by edges, can specify one of these loops by introducing surface bounded by the edges forming the loop, call these surfaces **faces**

- vertices, edges and faces together form structure we call a **lattice** L on spacetime

- equip lattice L with orientation of edges and faces, define discretized connection A on L by assigning group element g_e to each edge e representing parallel transport along edge in direction of its orientation

$$g_e = \mathcal{P} \exp - \int_e A$$

- space of discretized connections is space of all such assignments, thus for lattice with N edges this space is just G^N, measure on space of connections under this discretization becomes product over Haar measure of all edges:

$$\mathcal{D}A \;\; \rightsquigarrow \;\; \prod_e dg_e$$

- define discretized gauge transformation h by assigning element h_v of gauge group G to each vertex v, changes discretized connection:

$$h: \; g_e \;\mapsto\; g'_e \;=\; h^{-1}_{v_i(e)}\, g_e h_{v_f(e)}$$

with $v_i(e)$ initial and $v_f(e)$ final vertex of edge e induced by edge's orientation

- orientation of face f induces cyclical order on its k vertices and edges: $..., v_1, e_1, v_2, e_2, v_3, ..., v_k, e_k, v_1, ...$
- discretized curvature represented by holonomy along edges around face f, assign to f group element g_f, which is product of group elements g_e of edges $(e_1, ..., e_k)$ of f in cyclical order induced by orientation of f, use inverse group element, if orientation of edge is negative (disagrees with orientation of f), write orientation of edge e_j as $o_j \;=\; \pm 1$ then

$$g_{f,1} \;\stackrel{\text{def.}}{=}\; g_{e_1}^{o_1} \cdots g_{e_k}^{o_k}$$
$$g_{f,a} \;\stackrel{\text{def.}}{=}\; g_{e_a}^{o_a} \cdots g_{e_{a+k}}^{o_{a+k}}$$

from which we see that g_f depends on vertex v_a from which we choose to travel around f in direction of its orientation, the different g_f are related by conjugacy:

$$g_{f,b} \;=\; \underbrace{g_{e_b}^{o_b} \cdots g_{e_{a-1}}^{o_{a-1}}}_{g}\; g_{f,a}\; \underbrace{g_{e_{a+k}}^{-o_{a+k}} \cdots g_{e_{b+k+1}}^{-o_{b+k+1}}}_{g^{-1}}$$

- under discretized gauge transformations g_f also transforms by conjugation:

$$h: \; g_{f,a} \;\mapsto\; g'_{f,a} \;=\; h^{-1}_{v_a}\, g_{f,a}\, h_{v_a}$$

- thus gauge invariant information about curvature contained in conjugacy class of g_f, in order to obtain gauge-invariant Lagrangian, apply class function σ to g_f, which means we consider a Wilson loop around face f, which replaces curvature F, but is not considered an observable
- **class functions** σ of group elements g are invariant under conjugation: $\sigma(g) \;=\; \sigma(h^{-1}gh)$, any class function is a linear combination of characters of irreps of G
- the **character** $\chi_\rho(g)$ of a group element g for a group representation ρ acting on a vector space V is the trace of the matrix $t^\rho(g)$ derived from the representation matrix $\rho(g)$ via

$$\chi_\rho(g) \;=\; \mathrm{Tr}\; t^\rho(g)$$
$$t^\rho_{ab}(g) \langle \phi_a | \rho(g) | v_b \rangle$$

with $\{v_a\}$ a basis of representation space V, $\{\phi_a\}$ a dual basis and $\rho(g)$ the representation matrix

- thus oriented loop γ with an irrep ρ_γ living on it gives rise to observable $\chi_{\rho_\gamma}(g_\gamma)$ depending on connection A, which is called Wilson loop
- assume we have metric g on M, not appearing in Lagrangian, but will be needed for Hodge $\star$-operator
- choose hypercubic lattice with slowly varying lattice constant a in order to allow nonflat metric
- holonomy g_f around face f expressed by approximately constant curvature 2-form F on f by

$$g_f \approx \exp(-a_f^2 F_{\mu\nu} + \mathcal{O}(a_f^4)) \tag{0.1}$$

  with μ, ν unit directions spanning f from chosen basepoint with a_C local value of lattice parameter
- A-field is 1-form, associated to edges, F is 2-form, associated to faces, E-field is $(n{-}2)$-form, should be associated to something $(n{-}2)$-dimensional, associate it to dual faces of lattice, then $E \wedge F$ lives on full n-dim. cell of lattice, yielding a volume form which can be integrated
- lattice face and its dual face in one-to-one correspondence, thus can associate discretized E-field directly to faces f, in continuum this corresponds to using Hodge $\star$ operator of our auxiliary metric to transform the $(n{-}2)$-form E into a 2-form $\widetilde{E} = \star E$
- for action on M with slowly varying curvature we obtain

$$\begin{aligned} S &= \int\limits_M \mathrm{Tr}\,(E \wedge F) = \int\limits_M \mathrm{Tr}\,(\star\widetilde{E} \wedge F) \\ &\approx \sum_C a_C^n \sum_{(\mu,\nu)}^{C} \mathrm{Tr}\,(\widetilde{E}_{\mu\nu} F_{\mu\nu}) \end{aligned}$$

  with sum over all cells C of lattice and unordered pairs (μ, ν) of distinct unit directions from a basepoint of C, with a_C local value of lattice parameter
- since after discretization $\widetilde{E}$ and F live on faces, switch to summation over all faces, which corresponds to summation over all cells and unit directions (however fields are still continuous at this stage)

$$\begin{aligned} S &\approx \sum_f \mathrm{Tr}\,(\widetilde{E}^f_{\mu\nu} F^f_{\mu\nu})\, a_f^{n-2} a_f^2 \\ &= \sum_f \mathrm{Tr}\,(\widehat{E}^f_{\mu\nu}\, \underbrace{a_f^2 F^f_{\mu\nu}}_{\widehat{F}^f_{\mu\nu}}) \end{aligned} \tag{0.2}$$

with $F^f_{\mu\nu}$ the component in unit directions μ, ν spanning face f, $\widehat{E}$ a local rescaling of $\widetilde{E}$ absorbing weight and area quotient, a_f^2-factor remains in order to match (0.1),

- in discretization, curvature $a_f^2 F^f_{\mu\nu} = \widehat{F}^f_{\mu\nu}$ on face f becomes group element g_f according to (0.1), thus linear function $\mathrm{Tr}\,(\widehat{E}^f_{\mu\nu}\,\cdot)$ on Lie algebra $\mathfrak{g}$ must become function on group G, make substitution

$$\exp i\mathrm{Tr}\,(\widehat{E}^f_{\mu\nu}\,\cdot) \;\rightsquigarrow\; \chi_{\rho_f}(\cdot)$$

characters of irreps ρ of G appear naturally, because we want a gauge-invariant Lagrangian and they form a basis of gauge-invariant class functions

- thus d.o.f. of E-field appear in discretized BF theory as representation valued d.o.f. attached to faces of lattice
- discretized version of exponentiated action:

$$\exp iS \;\rightsquigarrow\; \prod_f \chi_{\rho_f}(g_f)$$

- dynamical variables are now discrete connection in form of group elements g_e on edges e and discrete E-field in form of irreps ρ_f on faces f
- auxiliary nature of metric reflected by disappearance lattice parameter a in final expression, they are absorbed during local rescaling of E-field
- back to calculation of partition function

$$\mathcal{Z} = \int \mathcal{D}A \int \mathcal{D}E\, \exp\, i \int_M \mathrm{Tr}\,(E \wedge F)$$

first perform integration over E field, which upon discretization becomes valued in irreps of G living on faces f of lattice, thus integration over E should turn into sum over irreps, but weight for each irrep still unknown, to find it, formerly perform integral over E-field in form (0.2) for one face f, here E still continuous, thus integral makes sense

$$\int d\widehat{E}^f_{\mu\nu}\, \exp\, i\mathrm{Tr}\,(\widehat{E}^f_{\mu\nu}\widehat{F}^f_{\mu\nu}) = \delta(\widehat{F}^f_{\mu\nu})$$

wherein a normalization factor is left out and the l.h.s. is the Fourier representation of the delta function

- under discretization delta function on Lie algebra becomes delta function on group

$$\delta(g_f) = \sum_{\rho_f} \chi_{\rho_f}(g_f) \ \dim \rho_f$$

with the sum over all irreps of gauge group G, we see searched weight per irrep is its dimension, thus schematically replace integral over E-field by

$$\int \mathcal{D}E \ ... \ \rightsquigarrow \ \sum_{\{\rho_f\}} \left(\prod_f \dim \rho_f \right) ...$$

with sum over set of all possible combinations of irreps labeling faces

- thus inserting all discrete analogues we have constructed a discrete version of the partition function:

$$\begin{aligned}\mathcal{Z} &= \int \prod_e dg_e \sum_{\{\rho_f\}} \left(\prod_f \dim \rho_f \right) \prod_f \chi_{\rho_f}(g_f) \\ &= \int \prod_e dg_e \sum_{\{\rho_f\}} \prod_f \chi_{\rho_f}(g_f) \ \dim \rho_f\end{aligned}$$

- since

$$\begin{aligned}\int \mathcal{D}E \ \exp i \int_M \mathrm{Tr}\,(E \wedge F) &= \int \left(\prod_f d\widehat{E}^f_{\mu\nu} \right) \ \exp i \sum_f \mathrm{Tr}\,(\widehat{E}^f_{\mu\nu} \widehat{F}^f_{\mu\nu}) \\ &= \prod_f \delta(\widehat{F}^f_{\mu\nu}) \\ &\rightsquigarrow \prod_f \delta(g_f)\end{aligned}$$

there is another way of arriving at the partition function:

$$\begin{aligned}\mathcal{Z} &= \int \prod_e dg_e \ \prod_f \delta(g_f) \\ &= \int \prod_e dg_e \prod_f \sum_{\rho_f} \chi_{\rho_f}(g_f) \ \dim \rho_f\end{aligned}$$

with the sum running over possible irreps ρ_f on the face f

- both versions of partition function likely to diverge, first because of sum taken over infinitely many irreps, second because of product of delta functions, but since only observable quantities need be finite in physics and the partition function is not observable, this is not necessarily a physical problem

- since BF theory does not need a metric for its definition, partition function should depend only on topology of spacetime manifold M

- in order to have a good quantization we require independence of *observables* from discretization

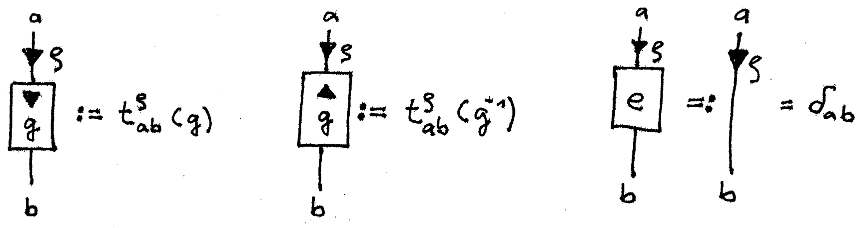
- remember character $\chi_\rho(g)$ of group element g for group representation ρ acting on vector space V is trace of the matrix $t^\rho(g)$ derived from representation matrix $\rho(g)$ via

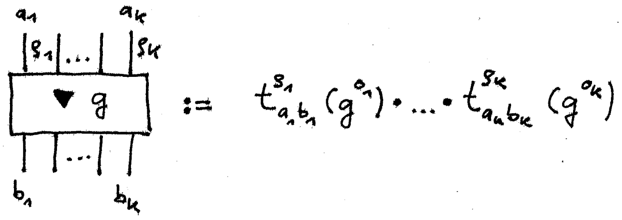
$$\begin{aligned} \chi_\rho(g) &= \mathrm{Tr}\, t^\rho(g) \\ t^\rho_{ab}(g) &= \langle \phi_a | \rho(g) | v_b \rangle \end{aligned}$$

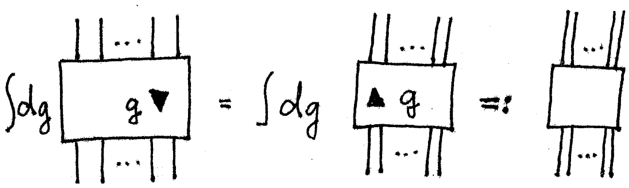
  with here $\{v_a\}$ an ONB of representation space V, $\{\phi_a\}$ a dual ONB and $\rho(g)$ the representation matrix, inserting $\mathbb{1} = \sum_k |v_k\rangle\langle\phi_k|$ in definition of matrix element we get with $o_e = \pm 1$ again denoting the relative orientation of edge e to the orentation of the face f:

$$\begin{aligned} t^\rho(g_1 g_2) &= t^\rho(g_1)\, t^\rho(g_2) \\ \chi_{\rho_f}(g_f) &= \mathrm{Tr}\, t^{\rho_f}(g_{e_1}^{o_1} \ldots g_{e_k}^{o_k}) = \mathrm{Tr}\left(t^{\rho_f}(g_{e_1}^{o_1}) \ldots t^{\rho_f}(g_{e_k}^{o_k}) \right) \end{aligned}$$

- now introduce **circuit diagrams**: consist of oriented **wires** (lines) labeled by irreps of group G and carrying one index at each end which run through oriented **cables** (boxes) labeled by group elements, white cables indicate the group integral over the element living on the cable

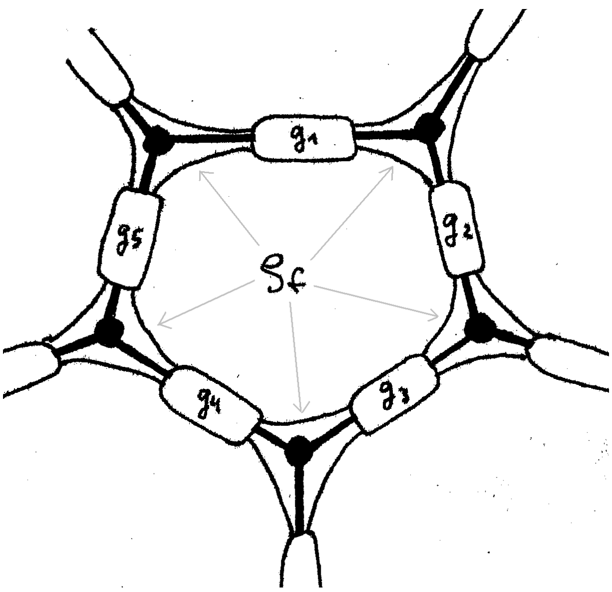






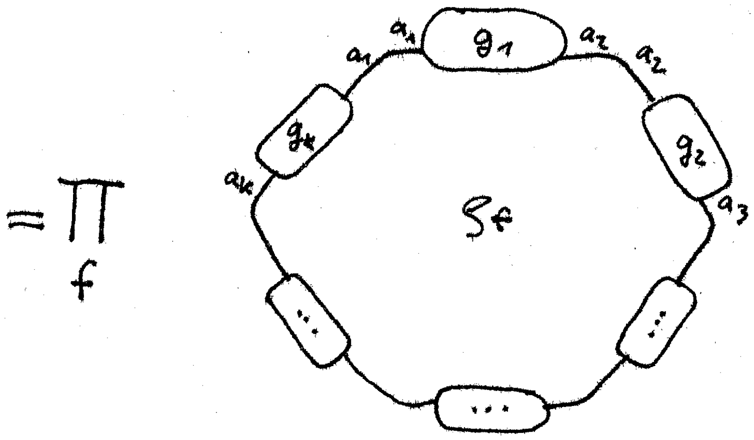
- complex circuit diagrams can be composed of simpler ones by connecting ends of wires and contracting their indices

- can now draw our lattice as big circuit diagram: a wire runs around face f, labeled by the irrep ρ_f and inherits orientation of f while a cable sits on edge e, labeled by group element g_e and inherits orientation of e

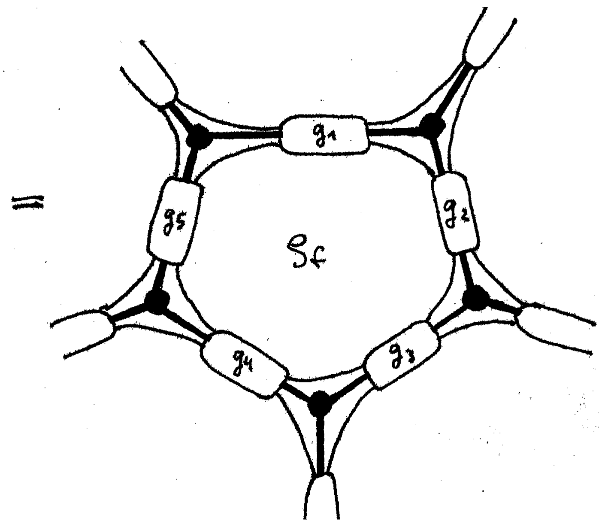


- since

$$\prod_f \chi_{\rho_f}(g_f) = \prod_f \mathrm{Tr}\left(t^{\rho_f}(g_{e_1}^{o_1})\ldots t^{\rho_f}(g_{e_k}^{o_k})\right)$$
$$= \prod_f \sum_{a_1,\ldots,a_k} t^{\rho_f}_{a_1 a_2}(g_{e_1}^{o_1})\, t^{\rho_f}_{a_2 a_3}(g_{e_2}^{o_2}) \ldots t^{\rho_f}_{a_k a_1}(g_{e_k}^{o_k})$$

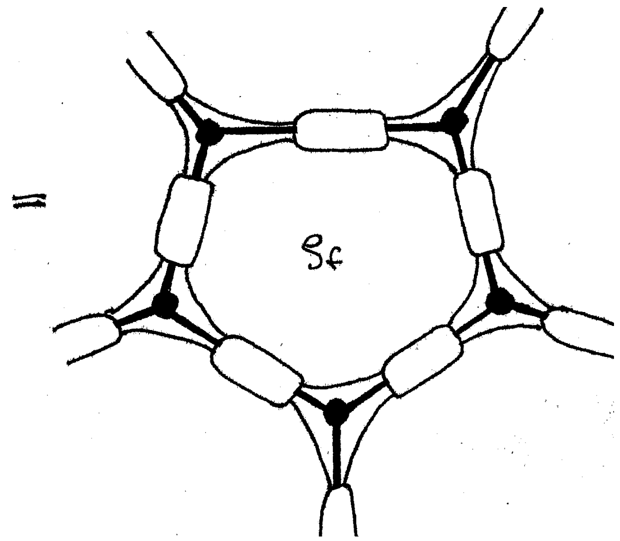


we find that the product of characters over all faces is the value of the circuit diagram of the lattice with cables labeled by the group elements g_e of the lattice's edges e

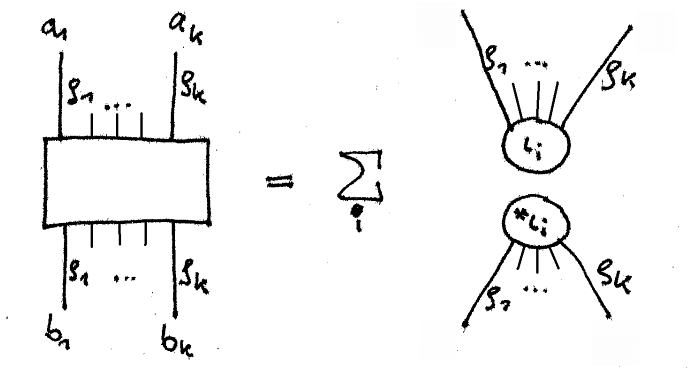


thus the group integrals over all edges of the product of characters is the circuit diagram of the lattice labeled by white cables

$$\int \prod_e dg_e \prod_f \chi_{\rho_f}(g_f)$$

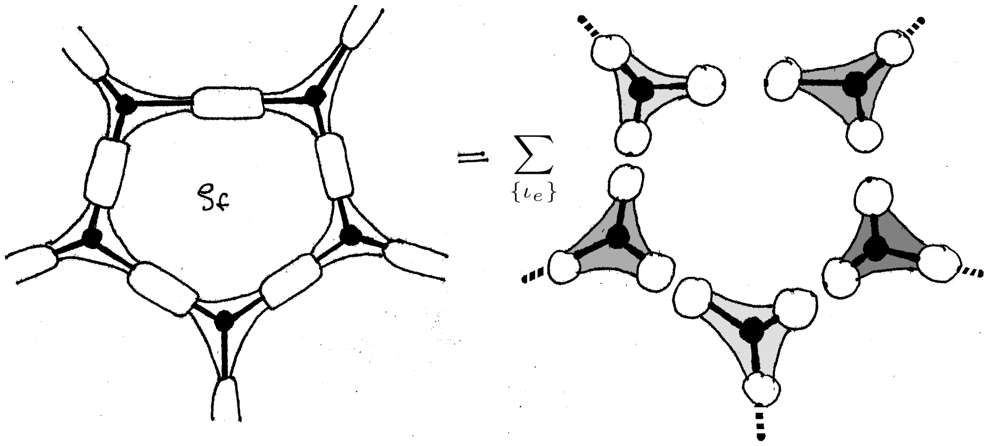


- there is a skein relation (with the sum over a basis of intertwiners ι, with ι mapping the tensor product of irreps of all incident edges to the trivial irrep, the $o_x = \pm 1$ again denoting orientations, and the convention ${}^{+1}\rho = \rho$ and ${}^{-1}\rho = {}^{*}\rho$, i.e., disagreeing orientation of edge and face means taking the dual of the face's irrep)



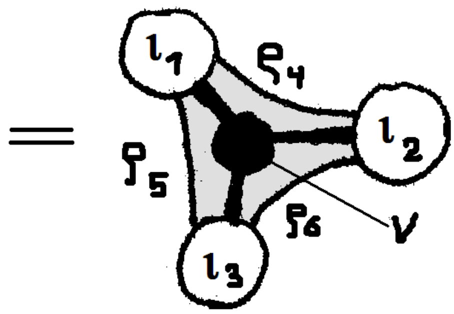
$$\left(\bigotimes_{x=1}^{k} {}^{o_x}\rho_x \right) \underset{{}^{*}\iota_i}{\overset{\iota_i}{\rightleftarrows}} \mathbb{C}$$

which turns the whole circuit diagram of our lattice into a disconnected set of small circuit diagrams, called here vertex diagrams, with one vertex diagram sitting on each vertex, now with sums of intertwiners labeling the edges, value of big circuit diagram labeled by group elements on edges and irreps on faces now becomes sum (over bases of intertwiners for each edge) of products of values of vertex diagrams:



- value of vertex diagram at vertex v depends on irreps labeling incident faces and their orientations, and all intertwiners labeling incident edges, denote value of vertex diagram by

$$A_v(\{\rho_{f_v}\},\{o_{f_v}\},\{\iota_{e_v}\})$$



- thus obtain for partition function

$$\begin{aligned}\mathcal{Z} &= \sum_{\{\rho_f\}}\Big(\prod_f \dim\rho_f\Big)\int\Big(\prod_e dg_e\Big)\Big(\prod_f \chi_{\rho_f}(g_f)\Big)\\ &= \sum_{\{\rho_f\}}\sum_{\{\iota_e\}}\Big(\prod_f \dim\rho_f\Big)\Big(\prod_v A_v(\{\rho_{f_v}\},\{o_{f_v}\},\{\iota_{e_v}\})\Big)\end{aligned}$$

with first sum over set of all possible combinations of irreps labeling faces and second sum over bases of intertwiners labeling each edge

- now discretizing BF theory we naturally have arrived at a lattice consisting of vertices, edges and faces, discretized gauge transformation sitting on vertices, discretized connection and intertwiners living on edges, E-field and discretized curvature in form of irreps live on faces, this is precisely a spinfoam defined in next section
- also partition function is that of spinfoam model, can interpret the dimension of an irrep as a face amplitude, value of vertex diagram as vertex amplitude, then product over faces and vertices of these ingredients is amplitude of a spinfoam, and sum over all labelings is sum over all spinfoams living on the lattice, in addition one can further introduce edge amplitudes

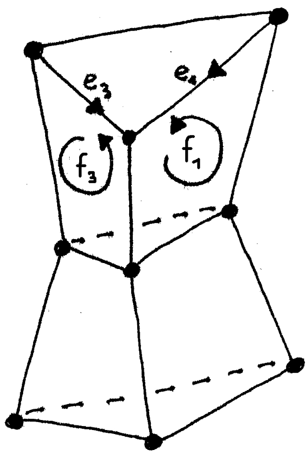
0.6 Spin foams

- **closed spinfoam** $F = (\kappa, \rho, \iota)$ with symmetry group G is triple consisting of:
  1. oriented 2-complex κ,
  2. face-labeling ρ of each face f of κ by an irrep ρ_f of G and
  3. edge-labeling ι of each edge e of κ by an intertwiner ι_e, which maps the tensor product of the irreps of the incoming faces $f_q^{\text{in}}(e)$ of e to the tensor product of the irreps of the outgoing faces $f_q^{\text{out}}(e)$

$$\iota_e : \left(\rho_{f_1^{\text{in}}(e)} \otimes ... \otimes \rho_{f_{n(e)}^{\text{in}}(e)}\right) \to \left(\rho_{f_1^{\text{out}}(e)} \otimes ... \otimes \rho_{f_{o(e)}^{\text{out}}(e)}\right)$$

  with $n(e)$ being the number of incoming and $o(e)$ of outgoing faces of e.

- face **incoming** to edge if orientation of edge agrees with orientation induced by face

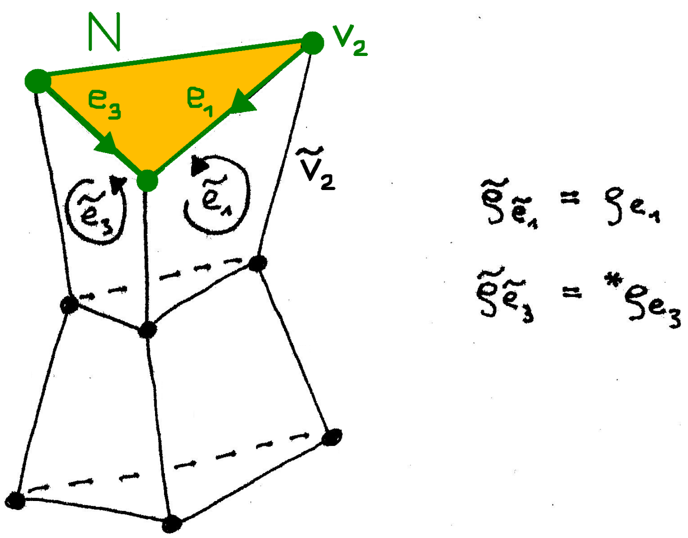


- same definition as for spin network, but everything one dimension higher, spin networks are slices of spinfoams

- in closed spinfoams *all* faces and edges are labeled by irreps and intertwiners, however we want to use spinfoams in order to connect spin networks
- therefor we need spinfoams, in which the edges lying in the spin networks to be connected are not labeled by intertwiners, but inherit the irrep of the face they are bordering, corresponding to the irrep labeling the edge within the spin network, while the spinfoam edges ending at a spin network vertex must correspond to the intertwiner as this vertex
- if a spin network N with graph Γ borders a spinfoam F with 2-complex κ, then each vertex v of N is the end of a unique edge $\widetilde{v}$ of F and each edge e of N is bordering a unique face $\widetilde{e}$ of F
- spinfoam $\big(F = (\kappa, \widetilde{\rho}, \widetilde{\iota})\big) : \ \emptyset \ \rightarrow \ \big(N = (\Gamma, \rho, \iota)\big)$ with symmetry group G connecting the empty spin network $\emptyset$ with N is triple consisting of:
    1. oriented 2-complex κ bordered by Γ,
    2. face-labeling $\widetilde{\rho}$ of each face f of κ by irrep $\widetilde{\rho}_f$ of G and
    3. edge-labeling $\widetilde{\iota}$ of each edge e of κ not contained in Γ by an intertwiner $\widetilde{\iota}_e$, which maps the tensor product of the irreps of the incoming faces $f_q^{\text{in}}(e)$ of e to the tensor product of the irreps of the outgoing faces $f_q^{\text{out}}(e)$

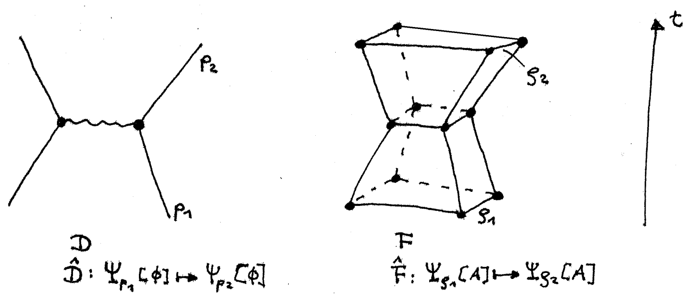
       $$\widetilde{\iota}_e : \ \big(\rho_{f_1^{\text{in}}(e)} \otimes ... \otimes \rho_{f_{n(e)}^{\text{in}}(e)}\big) \ \rightarrow \ \big(\rho_{f_1^{\text{out}}(e)} \otimes ... \otimes \rho_{f_{o(e)}^{\text{out}}(e)}\big)$$

       with $n(e)$ being the number of incoming and $o(e)$ of outgoing faces of e such that
    4. for any edge e of Γ, if $\widetilde{e}$ is incoming to e then we have $\widetilde{\rho}_{\widetilde{e}} = \rho_e$, but if $\widetilde{e}$ is outgoing from e then we have $\widetilde{\rho}_{\widetilde{e}} = {}^*\rho_e$ and
    5. for any vertex v of Γ after appropriate dualizations we have $\widetilde{\iota}_{\widetilde{v}} = \iota_v$.



- **dual spin network** $^*N$ of N consists of the same oriented 1-complex Γ, but with each edge e labeled by the dual irrep $^*\rho_e$ and with each vertex v labeled by the appropriately dualized version of the intertwiner ι_v.
- **tensor product** $N^1 \otimes N^2$ of two spin networks $N^1 = (\Gamma^1, \rho^1, \iota^1)$ and $N^2 = (\Gamma^2, \rho^2, \iota^2)$ is defined to be the disjoint union of N^1 and N^2 i.e., the tensor product consists of the 1-complex $\Gamma^1 \dot{\cup} \Gamma^2$, the parts of Γ^1 labeled using ρ^1 and ι^1 and those of Γ^2 by ρ^2 and ι^2
- **open spin foam** is a spinfoam $F: \ N^1 \rightarrow N^2$ connecting two nonempty spin networks N^1 and N^2 defined to be $F: \ \emptyset \rightarrow \left({}^*N^1 \times N^2\right)$.

0.6.1 Analogy spinfoams - Feynman diagrams



- standard QFT: transition amplitudes using Feynman diagrams = graphs with edges labeled by unitary irreps of symmetry group, (product of Poincaré group and internal symmetry group). Thus edges are labeled by momenta, spins and the internal quantum numbers.
- a Feynman diagram determines an operator on Fock space mapping an initial state to a final one, however it does not only contain the information of what will be the final state, but also in which way the transition happens, i.e., information about intermediate states, thus its edges and vertices tell the quantum history of the transition process, in which various virtual particles are created and annihilated
- one Feynman diagram gives an amplitude for a transition to happen in a certain way, total transition amplitude is sum over all amplitudes of graphs labeled by representations connecting initial with final set of points (states), i.e., the sum over all possible histories
- contribution of any graph to total amplitude given by product of subamplitudes associated to its edges and vertices, each edge amplitude (**propagator**) depends only on label of edge itself, each vertex amplitude depends on various irreps labeling all incoming edges of that vertex.
- propagators computed using free theory, while interactions represented merely by vertices
- Spinfoams can be defined in analogy to Feynman diagrams for arbitrary symmetry groups. Spinfoams replace Feynman diagrams, spin networks replace ends of ingoing and outcoming legs

- total transition amplitude between two spin networks: sum over contributions of spinfoams connecting first spin network with second. Each spinfoam in sum contributes to total amplitude amount given by product of subamplitudes associated to its faces, edges and vertices

- By analogy with Feynman diagrams, face and edge amplitudes can be thought of as some kind of propagators, vertices represent interactions, with their amplitudes characterizing nontrivial dynamics of theory

- two spin networks can be connected in many ways by different spinfoams, corresponding to different Feynman diagrams connecting same in/out particles

- Spinfoams and networks can be studied either abstractly or embedded in manifolds, edges and faces can be labeled by irreps of either groups or quantum groups

- spin networks merge concepts of **quantum state** and **geometry of space**, while spinfoams merge concepts of **quantum history** and **geometry of spacetime**

- a spin network state $\Psi_{N[A]}$ gives complex amplitudes of connections A on space, a spinfoam describes the evolution of the spin network state

- in analogy to Feynman diagrams, spinfoams define operators mapping initial spin network states to final ones, but also contain the information of how the transition happens, the quantum history of spacetime

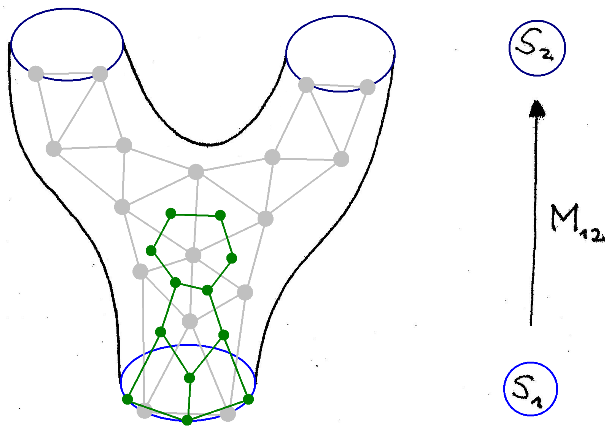
- in 4-dim. BF theory with gauge group SU(2):

  - ⊕ spin network edges give area to surfaces they cross, since they are slices of spinfoam faces, these give area to surfaces they intersect
  - ⊕ spin network vertices give 3-volume to regions of space they lie in, since they are slices of spinfoam edges, these give volume to 3-surfaces they cross
  - ⊕ spinfoam vertices expected to give 4-volume to regions of spacetime they lie in, but computations not finished

0.6.2 Transition amplitudes

- region of n-dim. spacetime given by compact oriented cobordism $M_{12} : S_1 \rightarrow S_2$ with S_1, S_2 being $(n-1)$-dim. compact oriented manifolds representing space

- choose n-dim. triangulation Δ of spacetime M,
  induces $(n-1)$-dim. triangulations $\partial\Delta_1$ and $\partial\Delta_2$ on S_1, S_2 with dual 1-skeletons Γ_1, Γ_2



- construction of gauge-invariant Hilbert spaces for S_1 and S_2:
- given graph Γ, define connection on Γ by assigning element g of gauge group G to each edge, space of such connections is $\mathcal{A}_\Gamma$, assignment can be thought of as representing parallel transport along edge γ, if graph were embedded in space with connection $\mathcal{A}$,

$$g_{\gamma[A]} = h_{\gamma[A]} \stackrel{\text{def.}}{=} \mathcal{P}\exp\int_\gamma A$$

- define gauge transformation on Γ by assigning element of G to each vertex, group of these gauge transformations is $\mathcal{G}_\Gamma$, acts in natural way on $\mathcal{A}_\Gamma$
- can use normalized Haar measure of G to define Hilbert spaces $\mathcal{H}_{\text{kin}} = L^2(\mathcal{A}_\Gamma)$ and $\mathcal{H}_{\text{gauge}} = L^2(\mathcal{A}_\Gamma/\mathcal{G}_\Gamma)$, latter one spanned by spin network states
- ONB of $\mathcal{H}_{\text{gauge}} = L^2(\mathcal{A}_\Gamma/\mathcal{G}_\Gamma)$ formed by spin network states Ψ_N with $N = (\Gamma, \rho, \iota)$ generated by ρ ranging over all possible labelings of the edges of Γ by unitary irreps of G, and the intertwiners of each vertex ranging over an ONB of the space of intertwiners connecting incoming with outgoing edges of the vertex
- relation between this combinatorial construction and spin networks embedded in space: if graph Γ is embedded in space S, then trivializing the principal G-bundle at the vertices gives a map from $\mathcal{A}$ to $\mathcal{A}_\Gamma$ and a homomorphism from $\mathcal{G}$ to $\mathcal{G}_\Gamma$, thus there are inclusions

$$\begin{aligned} L^2(\mathcal{A}_\Gamma) &\hookrightarrow L^2(\mathcal{A}) \\ L^2(\mathcal{A}_\Gamma/\mathcal{G}_\Gamma) &\hookrightarrow L^2(\mathcal{A}/\mathcal{G}) \end{aligned}$$

- now use as gauge-invariant Hilbert spaces for S_1 and S_2:

$$\begin{aligned}\mathcal{H}^1_{\text{gauge}} &= L^2(\mathcal{A}_{\Gamma_1}/\mathcal{G}_{\Gamma_1})\\ \mathcal{H}^2_{\text{gauge}} &= L^2(\mathcal{A}_{\Gamma_2}/\mathcal{G}_{\Gamma_2})\end{aligned}$$

  and describe **time evolution** as an operator

$$\widehat{Z}(M): \;\; L^2(\mathcal{A}_{\Gamma_1}/\mathcal{G}_{\Gamma_1}) \;\to\; L^2(\mathcal{A}_{\Gamma_2}/\mathcal{G}_{\Gamma_2})$$

- since spin network states Ψ_N form a basis of $\mathcal{H}_{\text{gauge}} = L^2(\mathcal{A}_\Gamma/\mathcal{G}_\Gamma)$ it is sufficient for specifying this operator to know the transition amplitudes

$$\langle \Psi_{N_2}, \widehat{Z}(M)\, \Psi_{N_1}\rangle_2$$

  (the subscript 2 indicates the scalar product of spin network states on S_2) from spin network states Ψ_{N_1} on S_1 to Ψ_{N_2} on S_2 for all spin networks N^1 having Γ_1 as underlying graph and all N^2 with Γ_2, because then we can write with an ONB $\{\Psi_{N_2}\}$

$$\widehat{Z}(M)\, \Psi_{N_1} = \sum_{N_2} \Psi_{N_2}\, \langle \Psi_{N_2}, \widehat{Z}(M)\, \Psi_{N_1}\rangle_2 \tag{0.3}$$

- write these transition amplitudes as sum over amplitudes $Z(F)$ of spinfoams F going from N_1 to N_2

$$\langle \Psi_{N_2}, \widehat{Z}(M)\, \Psi_{N_1}\rangle_2 \overset{\text{def.}}{=} \sum_{F}^{N_1 \to N_2} Z(F) \tag{0.4}$$

  where the sum here is restricted to spinfoams whose underlying 2-complex is the dual 2-skeleton of the fixed triangulation of spacetime M

- amplitude for closed spinfoam $F = (\kappa, \rho, \iota)$ is product of amplitudes of its faces f , edges e and vertices v

$$Z(F) = N(F) \prod_{f\in\kappa} Z_f(\rho_f) \prod_{e\in\kappa} Z_e(\rho_{f(e)}) \prod_{v\in\kappa} Z_v(\rho_{f(v)})$$

  with N being a normalization factor, the face amplitude being the dimension of the irrep labeling the face, the edge amplitude can be normalized to 1 and the vertex amplitude depending on the irreps labeling all incident faces of the vertex

- amplitude formula for open spinfoams differs from the one for closed spinfoams only in three points:
  - $\oplus$ edges and vertices lying in the spin networks to be connected are excluded from the product of amplitudes
  - $\oplus$ for spinfoam edges $\widetilde{v}$ ending in vertices v of the spin networks N_1 or N_2 we use the square root of the usual edge amplitude
  - $\oplus$ for spinfoam faces $\widetilde{e}$ ending in edges e of the spin networks N_1 or N_2 we use the square root of the usual face amplitude
- reason for these modifications is achieving **product rule for spinfoam amplitudes**

$$Z(F_{13}) \;=\; Z(F_{23})\, Z(F_{12})$$

for all spinfoams $F_{13}: \;\; N_1 \;\to\; N_3$ obtained by gluing together $F_{12}: N_1 \;\to\; N_2$ and $F_{23}: \;\; N_2 \;\to\; N_3$ along their common border N_2 whereby edges and vertices lying in N_2 become erased, which then assures (provided the sum over spinfoam amplitudes converges) the **composition property (gluing rule)** for composable cobordisms $M_{12}: \;\; S_1 \;\to\; S_2$ and $M_{23}: \;\; S_2 \;\to\; S_3$:

$$\widehat{Z}(M_{23}M_{12}) \;=\; \widehat{Z}(M_{23})\, \widehat{Z}(M_{12})$$

$$\begin{aligned} \Big\langle \Psi_{N_3},\; \widehat{Z}(M_{13})\, \Psi_{N_1} \Big\rangle_3 &\stackrel{\text{def.}}{=} \sum_{F_{13}}^{N_1\to N_3} Z(F_{13}) \\ \stackrel{!}{=} \Big\langle \Psi_{N_3},\; \widehat{Z}(M_{23})\widehat{Z}(M_{12})\, \Psi_{N_1} \Big\rangle_3 &= \sum_{\xi_2}^{\text{ONB}} \Big\langle \Psi_{N_3},\; \widehat{Z}(M_{23})\, \xi_2 \Big\rangle_3 \; \Big\langle \xi_2,\; \widehat{Z}(M_{12})\, \Psi_{N_1} \Big\rangle_2 \\ &\stackrel{\text{def.}}{=} \sum_{\xi_2}^{\text{ONB}} \left(\sum_{F_{23}}^{\xi_2\to N_3} Z(F_{23}) \right) \left(\sum_{F_{12}}^{N_1\to \xi_2} Z(F_{12}) \right) \\ &= \sum_{F_{23}F_{12}}^{N_1\to N_3} Z(F_{23})\, Z(F_{12}) \end{aligned}$$

- for spinfoam $F_{12}: \;\; N_1 \;\to\; N_2$ define a spinfoam operator $\widehat{F}_{12}: L^2(\mathcal{A}_{\Gamma_1}/\mathcal{G}_{\Gamma_1}) \;\to\; L^2(\mathcal{A}_{\Gamma_2}/\mathcal{G}_{\Gamma_2})$ acting on arbitrary spin network states $\Psi_1 \in L^2(\mathcal{A}_{\Gamma_1}/\mathcal{G}_{\Gamma_1})$ as

$$\begin{aligned} \widehat{F}_{12}\, \Psi_1 &\stackrel{\text{def.}}{=} \Psi_{N_2}\, \langle \Psi_{N_1},\, \Psi_1 \rangle_1 \\ \Rightarrow \quad \widehat{F}_{12}\, \Psi_{N_1} &= \Psi_{N_2}\, \| \Psi_{N_1} \|^2 \end{aligned}$$

such that for arbitrary spin network states $\Psi_1 \in L^2(\mathcal{A}_{\Gamma_1}/\mathcal{G}_{\Gamma_1})$ and $\Psi_2 \in L^2(\mathcal{A}_{\Gamma_2}/\mathcal{G}_{\Gamma_2})$

$$\langle \Psi_2, \widehat{F}_{12}\, \Psi_1 \rangle_2 \;=\; \langle \Psi_2, \Psi_{N_2} \rangle_2 \, \langle \Psi_{N_1}, \Psi_1 \rangle_1$$

so that consistently with (0.3) and (0.4) we can write the time evolution operator as

$$\begin{aligned}
\widehat{Z}(M_{12})\Psi_{N_1} &= \sum_{N_2}^{\Gamma_2} \sum_{F_{12}}^{N_1 \to N_2} Z(F_{12})\, \widehat{F}_{12}\, \Psi_{N_1} \,/\, \| \, \Psi_{N_1} \|^2 \\
&= \sum_{N_2}^{\Gamma_2} \Psi_{N_2} \sum_{F_{12}}^{N_1 \to N_2} Z(F_{12}) \\
&= \sum_{N_2}^{\Gamma_2} \Psi_{N_2}\, \langle \Psi_{N_2}, \widehat{Z}(M_{12})\, \Psi_{N_1} \rangle_2
\end{aligned}$$

with sum running over all spin networks N_2 based on graph Γ_2

0.7 4-dim. Quantum Gravity

- BF theory has no local degrees of freedom, flat connections, thus no use in 4-dim. GR with local d.o.f. and curvature
- Palatini formulation of 4-dim. GR: spacetime is 4-dim. oriented smooth manifold M
- choose bundle $\mathcal{T}$ over M, is isomorphic to TM but not in canonical way, thus $\dim \mathcal{T} = 8$, bundle resp. fibres called **internal space**, equip $\mathcal{T}$ with orientation and metric η, either Lorentzian or Riemannian
- P is oriented orthonormal frame bundle of M, is a principal G-bundle with G either SO(3,1) or SO(4) correspondsing to metric η
- basic fields in Palatini formulation:
  - $\oplus$ a $\mathcal{T}$-valued 1-form e on M: $\quad e: \; TM \;\to\; \mathcal{T}$
  - $\oplus$ a connection A on P
  - $\oplus$ curvature F of A: an ad(P)-valued 2-form
- adjoint vector bundle ad(P) isomorphic to $\Lambda^2\mathcal{T}$, thus can think of F also as $\Lambda^2\mathcal{T}$-valued 2-forms, also $(e \wedge e)$, with local coordinates $\{x^k\}$ on M and basis $\{b_m\}$ of $T^*\mathcal{T}$:

$$\begin{aligned}
e \wedge e &= e^{l_1}_{k_1} e^{l_2}_{k_2}\; dx^{k_1} \wedge dx^{k_2} \otimes b_{l_1} \wedge b_{l_2} \\
F &= F^{m_1 m_2}_{j_1 j_2}\; dx^{j_1} \wedge dx^{j_2} \otimes b_{m_1} \wedge b_{m_2} \\
e \wedge e \wedge F &= e^{l_1}_{k_1} e^{l_2}_{k_2} F^{m_1 m_2}_{j_1 j_2}\; dx^{k_1} \wedge dx^{k_2} \wedge dx^{j_1} \wedge dx^{j_2} \otimes b_{l_1} \wedge b_{l_2} \wedge b_{m_1} \wedge b_{m_2} \\
&= \underbrace{f(x)\; dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4}_{\mathrm{Tr}\,(e \wedge e \wedge F)} \otimes \underbrace{b_1 \wedge b_2 \wedge b_3 \wedge b_4}_{\mathrm{vol}_{\mathrm{int}}}
\end{aligned}$$

- wedge simultaneously differential form parts and internal parts yields $\Lambda^4\mathcal{T}$-valued 4-forms $(e\wedge e\wedge F)$, metric and orientation of $\mathcal{T}$ give internal volume form $\mathrm{vol}_{\mathrm{int}}$, that is, nonvanishing section of $\Lambda^4\mathcal{T}$, can write $(e\wedge e\wedge F)$ as product of internal volume form and ordinary 4-form, call this 4-form $\mathrm{Tr}\,(e\wedge e\wedge F)$

- Lagrangian is $\mathcal{L} = \mathrm{Tr}\,(e\wedge e\wedge F)$

- field equations: $\quad e\wedge F = 0 \qquad d_A(e\wedge e) = 2\,e\wedge d_A e = 0$

- define spacetime metric on M via e and internal metric η: $g(v,w) \overset{\text{def.}}{=} \eta(ev, ew)$

- if $e: \; TM \to \mathcal{T}$ is bijective, then spacetime metric g is nondegenerate and inherits signature of internal metric η

- can pull back connection A to a metric connection Γ on TM, if $e: \; TM \to \mathcal{T}$ is bijective, then $d_A e = 0$, saying Γ torsion-free, thus Γ is Levi-Civita connection of spacetime metric g

- rewriting $e\wedge F$ in terms of Riemann curvature tensor, one sees it is proportional to Einstein tensor, thus $e\wedge F = 0$ is vacuum Einstein equation

- thus Palatini field equations are extension of vacuum Einstein equation to case of degenerate metrics

- setting $E = e\wedge e$ makes Palatini Lagrangian look like BF Lagrangian, difference: not every $\mathrm{ad}(P)$-valued 2-form E is of form $e\wedge e$, thus allowed variations of E field in Palatini GR are more restricted than those in BF theory, thus Palatini field equations weaker than BF equations:

$$
\begin{aligned}
F &= 0 &\qquad\qquad e\wedge F &= 0 \\
d_A E &= 0 & d_A E = 2e\wedge d_A e &= 0
\end{aligned}
$$

- relation between Palatini GR and BF theory suggests that one could develop a spinfoam model of QG by taking a spinfoam model of BF theory and then impose quantum analogues of constraint that E is of form $e\wedge e$

- since BF theory is well understood only for compact gauge groups, at the moment we are limited to Riemannian QG

- transition amplitudes in BF theory computed for fixed triangulation of spacetime, then independence from triangulation can be shown because flat BF theory has no local degrees of freedom, but no reason why this should hold in GR which is curved and has local d.o.f., unsolved problem, work with spinfoams which are dual 2-skeleton of fixed triangulation

- consider at classical level constraints that must hold in order to have E field of form $e\wedge e$: pick spin structure for spacetime and take $\mathrm{Spin}(4)$ as gauge group, which is double cover of $\mathrm{SO}(4)$

- locally can think of E field as taking values in $\mathfrak{so}(4) \cong \mathfrak{so}(3) \oplus \mathfrak{so}(3)$, thus can write $E = E^+ + E^-$ as sum of left-handed and right-handed parts taking values in $\mathfrak{so}(3)$

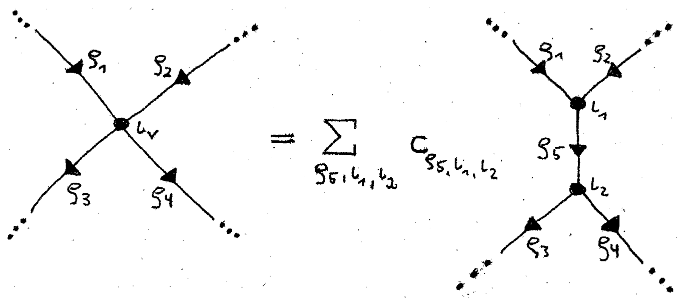
- if $E = e \wedge e$ then constraint $\| E^+(v,w) \| = \| E^-(v,w) \|$ holds for all vector fields v, w on M with the norm induced by the $\mathfrak{so}(3)$ Killing form, constraint guarantees $E = e \wedge e$ up to sign and Hodge star on $\Lambda^2 \mathcal{T}$, ignore this ambiguity, impose constraint in spinfoam model

- recall facts of 4-dim. BF theory with gauge group SU(2): spinfoam is dual 2-skeleton of triangulated 4-manifold, each dual face labeled by spin, each dual edge by intertwiner, corresponds to labeling each triangle by spin and each tetrahedron by 4-valent intertwiner connecting four spins

- consider general trivalent intertwiners: tensor product of any pair ρ_1, ρ_2 of irreps can be written as direct sum of irreps ρ_k, pick one $\rho_3 \in \{\rho_k\}$, projection from $\rho_1 \otimes \rho_2$ to ρ_3 is trivalent intertwiner ι which can be used to label trivalent vertices or edges, usually normalized to obtain intertwiner Tr $\iota\, {}^*\iota = 1$

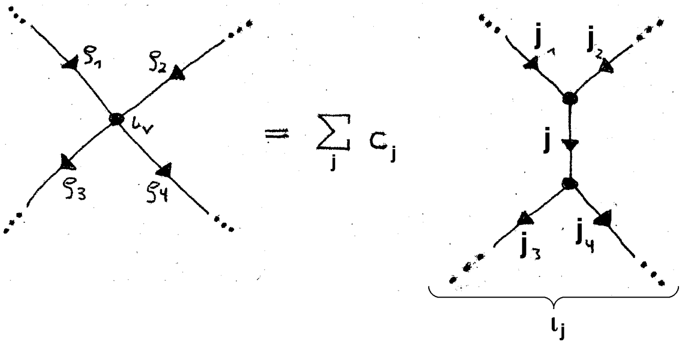
- if irrep ρ_3 appears more than once in direct sum decomposition $(\rho_1 \otimes \rho_2)$ then there is more than one intertwiner of above form, can always choose ONB $\{\iota_k\}$ of such intertwiners: Tr $(\iota_k\, {}^*\iota_l) = \delta_{kl}$

- can break any 4-valent intertwiner into trivalent ones using skein relation:



sum is over irreps ρ_5 and intertwiners ι_1, ι_2 in ONB as above, coefficients c depend on these intertwiners, both sides interpreted as parts of the same larger spin network

- now SU(2) irreps satisfy $j_1 \otimes j_2 \cong |j_1 - j_2| \oplus ... \oplus j_1 + j_2$, thus each basis of intertwiners $\iota : \quad (j_1 \otimes j_2) \rightarrow j_3$ consists of exactly one (normalized) element iff $|j_1 - j_2| \leq j_3 \leq (j_1 + j_2)$ and else it is empty, thus in SU(2) case we do not need to explicitly label trivalent vertices by intertwiners, sufficient to label edges by spins, thus skein relation simplifies to



- in 4-dim. triangulation with each tetrahedron labeled by a 4-valent intertwiner, this skein relation corresponds to chopping each tetrahedron in half by a parallelogram labeled by the sum over spins j on the right-hand side of the skein relation, thus all data encoded in spins labeling surfaces, each spin describing integral of $\| E \|$ over its surface
- describe 4-dim. Riemannian QG as BF theory with gauge group Spin(4) with extra constraint, since $\mathrm{Spin}(4) \cong \mathrm{SU}(2) \times \mathrm{SU}(2)$, irreps are of form $j^+ \otimes j^-$ for arbitrary spins $j^\pm$, thus label triangles and parallelograms by pair (j^+, j^-) of spins, describing integral of $\| E^\pm \|$ over surface
- in order to on quantum level impose constraint $\| E^+(v,w) \| = \| E^-(v,w) \|$, restrict to labeling surfaces with equal spins
- label each triangle a by irrep of form $j_a \otimes j_a$ and each tetrahedron by intertwiner of form $\sum\limits_j c_j \, \iota_j \otimes \iota_j$ with $\iota_j : \; j_{a_1} \otimes j_{a_2} \xrightarrow{j} j_{a_3} \otimes j_{a_4}$ and sum over spins j labeling the parallelogram
- however exist three ways of splitting tetrahedron in half by parallelogram P, want constraint $\int\limits_P \| E^+ \| = \int\limits_P \| E^- \|$ to hold for all three, thus must label tetrahedra by intertwiners $\sum\limits_j c_j \, \iota_j \otimes \iota_j$ which maintain this form during switching to different splitting
- unique solution: $\iota = \sum\limits_j (2j+1) \, \iota_j \otimes \iota_j$
- problem: sums over spinfoams in partition function and transition amplitudes for this spinfoam model diverge (probably), there exists a q-deformed version where sums become finite, but this version is not triangulation-independent

0.8 Conclusions

- state:
  - ⊕ have proposal for spinfoam model of QG with quantized values for area and volume

  - ⊕ quantum state of space is linear combination of spin networks
  - ⊕ transition amplitudes computed as sum over spinfoams connecting spin networks
  - ⊕ in q-deformed version of theory these sums are finite and explicitly computable

- problems:
  - ⊕ only Riemannian QG, no Lorentzian version
  - ⊕ theory depends on fixed triangulation of spacetime
  - ⊕ ability of computations with theory too poor to tell if large-scale limit is classical Riemannian GR

- tasks:
  - ⊕ develop spinfoam models of Lorentzian QG
  - ⊕ determine role which triangulations should play in spinfoam models with local d.o.f.
  - ⊕ develop computational techniques for studying large-scale limit