(2+1)-dimensional (Quantum) Gravity

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June 6th 2008

Quantum Gravity Seminar

Outline

Why do we want to study (2+1)-dim QGr?

- QGr in (3+1) dimensions is hard
- We like playgrounds.
- Playgrounds that hold similar features to the real world (symmetries, black holes and their thermodynamics, ?holography?...).
- Many conceptual problems remain unaltered (problem of time, background independence...)
- Others are solved (nonrenormalizability, implementation of constraints...)
- Can address questions about different approaches to QGr: Do we need topology change? Do we need a TOE? Are there more then one, possibly physically different, but mathematically concise quantum theories of gravity?
- Also mathematically 3-dim gravity and Chern-Simons theory have led to new research fields (TQFT, relation to the Jones polynomial...)

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Flat G-connections

Consider a G-gauge theory on a compact, simply connected n-dim manifold M_g of genus g, whos EL-eqn. allow only flat G-connections A.

- The physical phase space (PPS) of our G-gauge theory under consideration is the so called moduli space

$$\mathrm{PPS} = \mathcal{M} := \{A \in \mathcal{A} | F_A = 0\}/G$$

- We can fully encode the connection in parallel transports and holonomies. But parallel transports of a flat connection is trivial, only on a genus zero M_0 manifold!
- If we have $g > 0$ we get nontrivial holonomies by performing parallel transports around loops that enclose the holes.
- The fundamental group $\pi_1(M_g)$ encodes the curves of distinct homotopy class, and we have the homomorphism

$$H : \pi_1(M_g) = \langle c_1, \ldots, c_{2g} \rangle \to G, \qquad H[\gamma] := \mathrm{P}\exp[\int_\gamma A]$$

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Dimension of the Moduli Space

- This captures almost all DOF of the theory. We are still free to perform a G-conjugation at each base point of the generating loops c_i: $H[c_i] \to g^{-1} H[c_i] g$. Modding out gives

$$\mathcal{M} \simeq \mathrm{Hom}(\pi_1(M_g), G)/G$$

- We have a bound for the dimension of Hom: $\dim(\mathrm{Hom}(\pi_1(M_g), G)) \leq |G|^{2g}$
- The PPS is finite dimensional!
- The PPS depends crucially on the topology of spacetime.
- If $g = 0$, $\pi_1(M_0) = 1$ and we have $\dim(\mathcal{M}) = 0$.

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Why (2+1)-dim gravity is so simple

- In 3dim the Weyl tensor $C_{\alpha\beta\gamma\delta}$ vanishes $\Rightarrow$ The Ricci tensor $R_{\mu\nu}$ determines the Riemann tensor $R_{\alpha\beta\gamma\delta}$:

$$R_{\alpha\beta\gamma\delta} = [\text{Combination of g}'\text{s}]^{\mu\nu}_{\alpha\beta\gamma\delta} R_{\mu\nu}$$

- $\Rightarrow$ solutions to the vacuum Einstein equations are not only Ricci flat but also flat (Riemann flat).
- The PPS=moduli space is finite dimensional!
- Our space is locally Minkowski (dS or AdS in the presents of $\Lambda = \pm|\Lambda|$), and has no local degrees of freedom
- A physicists take: The same follows from a counting argument:
  h_{ab}, P_{ab} have $n(n-1)/2$ DOF each.
  n DOF are eliminated by constraints, n DOF by coordinate choice. $\Rightarrow$

$$n(n-1) - 2n = n(n-3) \quad \text{DOF} \qquad \Rightarrow \ 0 \text{ DOF in n} = 3$$

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The Newtonian Limit in n dimensions

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad \overline{h}_{\mu\nu} := h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h^\sigma_\sigma, \quad h_{\mu\nu} = \overline{h}_{\mu\nu} - \frac{1}{n-2}\eta_{\mu\nu}\overline{h}^\sigma_\sigma$$

- The linearized Einstein equations in the gauge $\partial^\mu \overline{h}_{\mu\nu} = 0$ become

$$-\frac{1}{2}\partial^\sigma\partial_\sigma \overline{h}_{\mu\nu} = 8\pi G T_{\mu\nu} + \mathcal{O}(h^2).$$

- The Newtonian Limit is obtained by setting $T_{00} \approx \rho$, all other $T_{\mu\nu} \approx 0$ and $\partial/\partial t = 0$

$$-\frac{1}{4}\nabla^2 \overline{h}_{00} = \nabla^2\Phi = 4\pi G\rho$$

- The geodesic equation reduces to

$$\frac{\mathrm{d}^2 x^i}{\mathrm{d}t^2} - \frac{1}{2}\partial_i h_{00} = 0, \quad \Leftrightarrow \quad \frac{\mathrm{d}^2 x^i}{\mathrm{d}t^2} + \frac{2(n-3)}{(n-2)}\partial_i \Phi = 0$$

In $n = 3$ test particles experience no Newtonian Force.

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First Order Formalism

- Independent variables are the Dreibein e^a_μ and the spin connection ω^{ab}_μ.

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Which introduces an additional $SO(2,1)$-invariance under $e^a_\mu \to O^a_b e^b_\mu$.

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$$S_{\mathrm{EH}} = 2\int_M \left[e^a \wedge \mathrm{d}\omega_a + \frac{1}{2}\epsilon_{abc} e^a \wedge \omega^b \wedge \omega^c + \frac{\Lambda}{6}\epsilon_{abc} e^a \wedge e^b \wedge e^c \right],$$

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$$0 = \mathrm{d}e_a + \epsilon_{abc}\omega^b \wedge e^c \quad (1)$$

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What are the Diffeomorphisms?

- Let $\phi \in \mathrm{Diff}(M)$ and furthermore locally $\phi = F_t^X$ flow of a vector field X.

$$\begin{aligned} \delta e^a &= L_X e^a = \mathrm{d}\rho^a + \epsilon_{abc}\rho^c\omega^b - \epsilon_{abc}\tau^b e^c \\ \delta\omega^a &= L_X \omega^a = \mathrm{d}\tau^a + \epsilon_{abc}\tau^c\omega^b - \Lambda\epsilon_{abc}\rho^c e^b, \end{aligned}$$

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$$S_{\mathrm{CS}}[A] := \int_M \mathrm{tr}\left[A \wedge \mathrm{d}A + \frac{2}{3} A \wedge A \wedge A\right]$$

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$$S_{\mathrm{CS}} = S_{\mathrm{EH}}|_{\Lambda=0} .$$

- Where an $ISO(2,1)$ gauge transformation by an infinitesimal parameter $u = \rho^a P_a + \tau^a J_a$ is determined by

$$\begin{aligned} \delta A_\mu &= \delta e^a_\mu P_a + \delta \omega^a_\mu J_a \\ \delta A_\mu &= D_\mu u = \partial_\mu u + [A_\mu, u] \\ &= [\partial_\mu \rho_a - \epsilon_{abc} \tau^c e^b_\mu + \epsilon_{abc} \rho^c \omega^b_\mu] P^a + [\partial_\mu \tau_a + \epsilon_{abc} \tau^c \omega^b_\mu] J^a \end{aligned}$$

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Under the gauge tranformation

$$A^g := g^{-1}\mathrm{d}g + g^{-1}Ag$$

the Chern-Simons action transformes as (hint: use $\mathrm{d}(g^{-1}g) = 0$):

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For closed M...

- ...the boundary term vanishes.
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- ...the boundary term does not vanish and the Chern-Simons action is not gauge invariant!
- ... even greater problem arises from the variational principle:

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- The surface term does not vanish for either Dirichlet or Neumann conditions.

Making the variational principle work I

- CS-theory alone is not a well defined physical theory on closed manifolds.
- Cure: We choose a complex structure on ∂M, fix an apropriate mixed boundary condition A_z or $A_{\bar{z}}$, and add a suitable boundary term $S_{\partial M}[A_z, A_{\bar{z}}]$ that compensates the boundary term above.

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Gauge transformation of the modified CS-action

$$\tilde{S}_{\mathrm{CS}}[A^g] = \tilde{S}_{\mathrm{CS}} + S^{+}_{\mathrm{WZW}}[g, A_z]$$

where $S^{+}_{\mathrm{WZW}}[g, A_z]$ is a chiral Wess-Zumino-(Novikov)-Witten action on ∂M:

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- The number of physical degrees of freedom of a CS-theory depends strongly on whether spacetime has a boundary.
- If M has a boundary, gauge invariance is broken at ∂M and the "would-be pure gauge degrees" g become dynamical on the boundary.
- This adds an infinite-dimensional space of inequivalent solutions.

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- The number of physical degrees of freedom of a CS-theory depends strongly on whether spacetime has a boundary.
- If M has a boundary, gauge invariance is broken at ∂M and the "would-be pure gauge degrees" g become dynamical on the boundary.
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Gauge transformation of the modified CS-action

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Einstein Hilbert and Moduli Spaces

$$\begin{aligned} S_{\mathrm{EH}} &= \int_M \epsilon^{\rho\mu\nu} e_{\rho a}[\partial_\mu \omega_\nu^a - \partial_\nu \omega_\mu^a + [\omega_\mu, \omega_\nu]^a] \\ 0 &= \partial_\mu \omega_\nu^a - \partial_\nu \omega_\mu^a + \epsilon^{abc}\omega_{\mu b}\omega_{\nu c} \\ 0 &= \partial_\mu e_\nu^a - \partial_\nu e_\mu^a + \epsilon^{abc}(\omega_{\mu b} e_{\nu c} - \omega_{\nu b} e_{\mu c}) \end{aligned}$$

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Naive Quantization

$$Z(M) = \int \mathcal{D}[e,\omega] \exp[iS_{\text{EH}}]$$

Using $\int \mathrm{d}x e^{ixy} = \delta[y]$

$$Z(M) = \int \mathcal{D}[\omega] \prod_{\mu,\nu,a} \delta[F^a_{\mu\nu}]$$

We use the splitting

$$\omega = \overline{\omega} + \Omega, \qquad e = \overline{e} + E$$

where $\overline{\omega}$ and $\overline{e}$ are flat, and applying

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Still: Naive Quantization

we find

$$Z(M) = \frac{1}{|\det(\overline{D}_\mu \Omega_\nu - \overline{D}_\nu \Omega_\mu)|}$$

with the covariant exterior derivative

$$\overline{D}\beta^a := \mathrm{d}\beta^a + \epsilon^{abc}\overline{\omega}_b \wedge \beta_c$$

The above operator has ininitly many zero modes (for which $\det^{-1}$ diverges), of the form $\Omega_\mu = \overline{D}_\mu \epsilon$!

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Quantization, a little less naive

Solution: Add Gauge fixing term! Does this not break Diffeo? We shall see...

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\begin{aligned}
\mathcal{L}_{\text{fix}} &:= -v_a \wedge \star D \star \Omega^a - u_a \wedge \star D \star E^a \\
Z_{\text{tot}}[\mathcal{M}] &:= Z_{\text{FP}} \int \mathcal{D}[\Omega, u, E, v] \exp\left[i \int_M e \wedge F_\omega + \mathcal{L}_{\text{fix}} \right] \\
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with the pair of three-form Lagrange multipliers u, v.
Integral is linear in v and E, solve these integrations first.
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A toy model

$$S_{\text{toy}} := \int_M \mathrm{d}^3x \alpha \Delta \beta$$

Expanding α, β in the base of orthonormal modes of Δ:

$$\Delta\phi_n = \lambda_n \phi_n, \quad \alpha = \sum_m a_m \phi_m, \quad \beta = \sum_n b_n \phi_n,$$

we get a sum $\sum'$ over non-zero modes ($\lambda_n \neq 0$):

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- But: Integration over non-zero modes will give $\delta[\overline{D}\Omega_a + \frac{1}{2}\epsilon_{abc}\Omega^b \wedge \Omega^c + \star D \star u_a]$.
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What have we learned?

- In (2+1)-dim the PPS=moduli space of gravity becomes finite dimensional, we have no local degrees of freedom.
- (2+1)-dim gravity is Chern-Simons theory with the gauge group depending on Λ
- For manifolds with boundary the action principle of CS-theory is not well defined.
- By defining a complex structure on the boundary and adding the right boundary terms and conditions the modified CS-theory can be made well defined.
- The "would-be pure gauge degrees" become dynamical on the boundary of the modified theory. They are described by a chiral WZW term which adds an infinite-dim space of solutions.
- First order (2+1)-dim gravity was quantized using path integrals by Witten in '89. Despite gauge fixing, the partition function was found to be a topological invariant, thus preserving diffeo.

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