Relativistic Mechanics: spacetime as the arena for dynamics

References: 1 Quantum Gravity, by C. Rovelli (Ch. 3)

2 Mathematical Methods of Clas. Mech. by V.I. Arnold (Ch 9. secs. 45, 46)

Basic concepts:

C — (relativistic) configuration space

[In the case of an unconstrained particle C is spacetime]

Motions of particles are unparametrized curves, $\gamma \subset C$.

[world lines]

Γ_{phys} — space of physical states.

A physical state is a motion which follows the dynamics of the system, $\gamma_{phys} \in \Gamma_{phys}$
(a point in)

Hamiltonian Formalism

(Applicable in optics ?, non-rel. mechanics and many other areas)

Variational principle:

A curve $\gamma \subset C$ connecting the events $q_1, q_2 \in C$ is a physical motion ($\gamma \in \Gamma_{phys}$) iff it is the projection of a curve $\tilde{\gamma}$ on T^*C ($\boldsymbol{\gamma} = \pi(\tilde{\gamma})$) which extremizes the action

$$S_\Sigma[\tilde{\gamma}] = \int_{\tilde{\gamma}} p\, dq$$

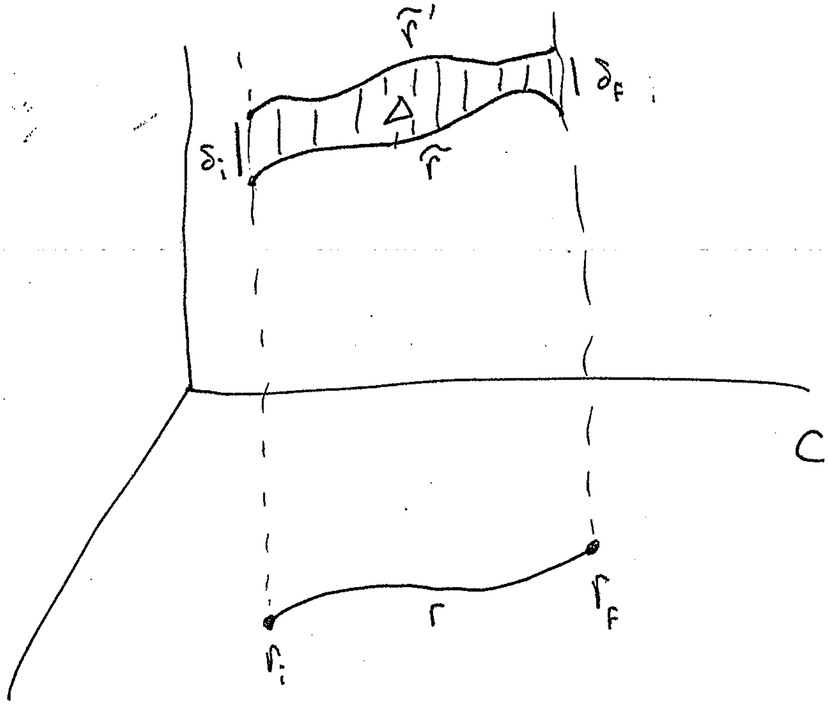
among the curves satisfying $\tilde{\gamma} \subset \Sigma \subset T^*C$, $\pi(\partial\tilde{\gamma}) = q_1 \cup \{q_2\}$.

[Σ is usually locally specified by a Hamiltonian constraint, $H \simeq 0$.
$\Sigma = H^{-1}(0)$ for $H : T^*C \to \mathbb{R}$.]

Symplectic formalism $\left(T^*C \,,\ \omega = d(p\,dq) \right)$

Theorem: The physical motions defined by the action principle can also be characterized as the integral curves of the ~~null~~ degenerate directions of $\omega\big|_\Sigma$.

Sketch of proof.



$$\int_\Delta \omega = \int_{\tilde\Gamma} p\,dq - \int_{\tilde\Gamma'} p\,dq + \underbrace{\left(\int_{\delta_f} p\,dq - \int_{\delta_i} p\,dq \right)}_{=0}$$

$= 0$ since $\pi(\partial\tilde\Gamma) = \pi(\partial\tilde\Gamma') = q_1 \cup (-q_2)$

$\Rightarrow$ ...

$\Leftarrow$ ...

Definition: $\Gamma_{phys} = \Sigma / \sim_{\ker \omega|_\Sigma}$

It is a manifold for some $\Sigma \subset T^*C$.

Must make sense at least locally in the region of interest (see local measurements).

Theorem: The 2-form $\omega_{phys} = \sigma^* \omega|_\Sigma$

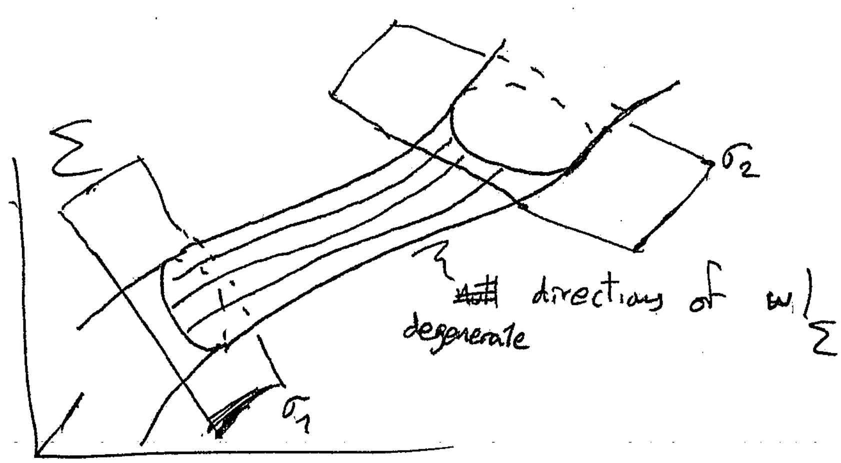
$$\Gamma_{phys} \underset{\sigma}{\overset{[\;]_{\ker}}{\rightleftarrows}} \Sigma \qquad , \qquad [\;]_{\ker} \circ \sigma = id_{\Gamma_{phys}}$$

is independent of the choice of representative/"gauge choice"

and it is non degenerate, making the space of physical states (motions)

$$\left(\Gamma_{phys}, \ \omega_{phys} = \sigma^* \omega|_\Sigma \right) \qquad \text{a symplectic space.}$$

Sketch of proof.



$$\int_{tube \cap \sigma_1} \omega|_\Sigma = \int_{tube \cap \sigma_2} \omega|_\Sigma \qquad \forall \ \sigma_1, \sigma_2 \ \text{gauge choices.}$$

Remark (Muller and Corichi): ω_{phys} can also be defined as the only 2-form on Γ_{phys} such that $[\,\mathcal{I}^*_{Ker}(\omega_{phys}) = \omega|_\Sigma$.

Newtonian systems:

$$C = C_0 \times \mathbb{R} \qquad , \qquad T^*C = T^*C_0 \times T^*\mathbb{R}$$
$$\ni (p^i, q^i ; p_t, t)$$

$\Sigma = H^{-1}(0)$ with the Hamiltonian constr. $H = p_t + \underset{\text{non-rel. Hamilt./energy f}^n}{H_0}$

$\Rightarrow \Sigma = T^*C_0 \times \mathbb{R}_t$ (extended phase space of Arnold Ch. 9)

$$\omega|_\Sigma = d\,(p^i dq^i + p_t\, dt)\big|_\Sigma = d\,(p^i dq^i - H_0\, dt)\big|_\Sigma .$$

variational principle using $S(\tilde{\gamma}) = \int_{\tilde{\gamma}} p^i dq^i - H_0\, dt$

$\Leftrightarrow$ Integral curves of deg. dir. of $d(p^i dq^i - H_0 dt)$

Theorem: The int. curves of the deg. dir. of $d(p^i dq^i - H_0 dt)$ on $T^*C_0 \times \mathbb{R}_t$ have a 1-1 projection onto the t-axis, $p^i = p^i(t)$, $q^i = q^i(t)$. These f^{ns} satisfy the canonical eqs.

$$\frac{dp^i}{dt} = -\frac{\partial H}{\partial q^i} \quad , \quad \frac{dq^i}{dt} = \frac{\partial H}{\partial p^i}$$

Proof left to the students. (Solution in Arnold p. 236)

Γ_{phys} = space of orbits in $T^*C_0 \times \mathbb{R}_t$

Time gauge fixing σ_{t_i}

Γ_{t_i} = slice of Σ of equal time $= t_i$

$$\omega|_\Sigma \Big|_{\Gamma_{t_i}} = d(p^i dq^i - H_0 dt)\Big|_{\Gamma_{t_i}} = dp^i \wedge dq^i = \omega_{t_i}$$

$$\Gamma_{phys} \xrightarrow{\sigma_{t_i}} \Gamma_{t_i} \subset \Sigma$$

$$(p^i(t), q^i(t)) \longmapsto (p^i(t_i), q^i(t_i); t_i)$$

σ_{t_i} is a good gauge fixing and it is a symplectomorphysm

$$\Gamma_{t_i} \xrightarrow{\sigma_{t_f} \circ \sigma_{t_i}^{-1}} \Gamma_{t_f} \qquad \text{"time evolution"}$$

Free relativistic particle of mass $= m$

$C = \mathbb{R}^4$ Minkowski space

$\Sigma_m = H^{-1}(0)$ and $p_t > 0$ with $H = \eta^{\mu\nu} p_\mu p_\nu + m^2$

$= \mathbb{R}^4 \times K_m^+$ future branch of the mass$=m$ hyperboloid in momentum space.

Variational princ. using $S(\tilde{\gamma}) = \int_{\tilde{\gamma} \subset \mathbb{R}^4 \times K_m^+} p\, dq$

extrema coincide with int. curves of deg. dir. of $dp \wedge dq \big)_{\mathbb{R}^4 \times K_m^+}$.

Consider $\begin{cases} (p,q) \in \Sigma_m = \mathbb{R}^4 \times K_m^+ \\ X_{p,q} = \eta^{\mu\nu} p_\mu \dfrac{\partial}{\partial q^\nu} \in (T\Sigma_m)_{p,q} \end{cases}$

"the tangent vectors of rel. free particles that we know!"

$$\omega\big|_\Sigma (X, -)_{p,q} = dp_\mu \wedge dq^\mu \left(\eta^{\rho\sigma} p_\rho \frac{\partial}{\partial q^\sigma}, - \right)_{p,q} = -\eta^{\rho\sigma} p_\rho\, dp_\sigma (-)_{p,q} = dH_{p,q} = 0$$

$\therefore$ this formalism agrees with what we know from SR,
AND it is Lorentz invariant.

"Time gauge fixings" are possible but break Lorentz inv.

$$\Gamma_{phys} = \mathbb{R}^4 \times K_m^+ \Big/ \text{int curves of } X$$

Today's plan:

- A very simple example ($\ddot{x} = -x$)

  Objectives: i) review the formalism

  ii) visualization aid for the discussion of observables

- Observables: { - Dirac obs.
  - partial obs.
  - ~~local obs.~~ What do we measure?

- Remark about classical vs quantum and the quantization problem

- A generalization to ~~the~~ (relativistic) classical field theory

  * Reference

    Momentum maps and class. rel. fields ...

    by Gotay, Isenberg, Marsden and Montgomery

    physics/980~~1~~1019