Geometric Quantization on curved Spacetime

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Quantum Gravity Seminar

Outline

What is Geometric Quantization good for?

- Better understanding of quantization in general.
- Wider range of applicability.
- Topological invariants of 3-manifolds in applications of geometric quantization to TFTs.

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What is Quantization?

- *"Quantization is putting hats on letters."* $\Rightarrow$???
- "Putting the hat on" means mapping a set of classical observables $\mathcal{O}_1, \ldots, \mathcal{O}_n$ on phase space $\mathcal{M}$ to a set of quantum observables, i.e. hermitian operators $\hat{\mathcal{O}}_1, \ldots, \hat{\mathcal{O}}_n$ acting on a Hilbert space $\mathcal{H}$. $\Rightarrow$ What is the mapping? What is $\mathcal{H}$?
- In Quantum mechanics in the Schrödinger representation we have the Hilbert space $\mathcal{H} = L^2(Q)$. With Q =space-time and the mappings:

$$p_\mu \longrightarrow -i\hbar \frac{\partial}{\partial q^\mu}, \quad q_\mu \longrightarrow q_\mu, \quad [q_\mu, p_\nu] = i\hbar\delta_{\mu\nu}$$

The Quantum from the Classical?

- From a theoretical standpoint, quantization might be viewed as an unnatural procedure since we - as macroscopic beings - are forced to obtain the quantum theory from the classical one.
  However we should view the quantum theory as the truely fundamental theory which is to yield the classical theory through coarse graining.
- There is no reason to believe that any crazy classical Lagrangian should have a corresponding quantum equivalent.

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Requirements for a quantization $\mathcal{Q}$

1. Linearity: $\mathcal{Q}(\lambda f + g) = \lambda \mathcal{Q}(f) + \mathcal{Q}(g)$
2. Unitality: $\mathcal{Q}(1) = \mathbb{1}$
3. Hermiticity: $\mathcal{Q}(f)^* = \mathcal{Q}(f)$
4. Poisson representation: $[\mathcal{Q}(f), \mathcal{Q}(g)] = -i\hbar \mathcal{Q}(\{f, g\})$
5. Irreducibility: $\{f_i\}_{i \in I}$ set of complete observables
   $\Rightarrow \{\mathcal{Q}(f_i)\}_{i \in I}$ set of complete operators.

Product of Observables

Note that we do not demand $\mathcal{P}(f_1 f_2) = \mathcal{P}(f_1)\mathcal{P}(f_2)$.

The crux of quantization

- **Existence**: It is in general not possible to satisfiy both **4** and **5**.
- **Uniqueness**: These axioms do not determine the underlying quantum theory uniquely.

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Symplectic Geometry

- A symplectic manifold M is a manifold with a nondegenerate closed two-form ω called the symplectic form.
- Nondegeneracy $\Leftrightarrow \det(\omega_{\mu\nu}) \neq 0$
  $\Rightarrow \dim(M) = 2n$ (an odd dimensional antisymmetric matirx has zero determinant)
- If ω is exact: $\omega = \mathrm{d}\theta$, then θ is called the symplectic potential.
- To every $f \in C^\infty(M)$ there is an associated Hamiltonian vector field X_f, defined by
  $$\iota_{X_f}\omega = -\mathrm{d}f.$$
- Every Symplectic manifold is Poisson, i.e. $\{f, g\} := \omega(X_f, X_g)$
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Prequantizaton is...

...the construction of a faithful representation of the Poisson algebra of functions by linear operators on a Hilbert space.

- The central observation is that the identity $[X_f, X_g] = X_{\{f,g\}}$ shows that the Hamiltonian vector fields give a representation of the Poisson algebra by first order differential operators on $C^\infty(M)$.
- Indeed, the mapping $f \to -i\hbar X_f$ fulfills the quantization axioms **1,3,4**. However the zero vector field is assigned to any constant function and thus violates **2**: $\mathcal{Q}(1) = \mathbb{1}$.
- The prequantization map, satisfying axioms **1-4** is

$$\tilde{\mathcal{P}} : f \longmapsto \tilde{\mathcal{P}}(f) := -i\hbar X_f - \theta(X_f) + f$$

Note...

...there are no differential operators of order higher then one in geometric quantization.

But...

...what happens if ω is not exact, i.e. globally $\nexists\theta$? Or...
...if we choose a different $\theta' = \theta + \alpha$ with $\mathrm{d}\alpha = 0$?

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Line Bundles

- Let $\theta' = \theta + \mathrm{d}u$, then $\tilde{\mathcal{P}}'(f)e^{iu/\hbar}\psi = e^{iu/\hbar}\tilde{\mathcal{P}}(f)\psi$. Thus also changing $\psi' := e^{iu/\hbar}\psi$ gives unitarily equivalent operators.
- Since θ and θ' determine u only up to a constant, the global phase remains undefined.
- Is this reminiscent of a gauge theory, or what? But what is the bundle which the wave-functions are sections of?

Definition 1: A complex line bundle $\pi : L \rightarrow M$ is a complex vector bundle with one dimensional fiber.

Definition 2: A Prequantization of a symplectic manifold (M, ω) is a pair (L, ∇) where L is a complex Hermitian line bundle over M and ∇ a compatible connection with curvature $\hbar^{-1}\omega$.

- In a particular trivialization the connection reads: $\nabla = \mathrm{d} - i\hbar^{-1}\theta$.
- Which indeed lets the curvature $\Omega(X,Y)s = i[\nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]}]s$ be $\Omega = \hbar^{-1}\mathrm{d}\theta = \hbar^{-1}\omega$

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Definition 1: A complex line bundle $\pi : L \to M$ is a complex vector bundle with one dimensional fiber.

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- In a particular trivialization the connection reads: $\nabla = \mathrm{d} - i\hbar^{-1}\theta$.
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...is defined to be:

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which of course is well defined, even for non-trivial line bundles, i.e. ω not globally exact and satisfies

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The prequantum Hilbert space is the space of L^2-sections $s : M \to L$.

A crucial question in quantization is:

When is a symplectic manifold (phase space of a classical theory) quantizable? That is: When can we construct a complex line bundle over it which has curvatutre $\hbar^{-1}\omega$?

Short answer:

If $\hbar^{-1}\omega$ is to be the curvature of a connection, then a neccessary and (when M simply connected also sufficient) condition is $(2\pi)^{-1}\int_\Sigma \hbar^{-1}\omega \in \mathbb{Z}$ for any closed oriented two-surface Σ.

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Precise answer:

A symplectic manifold (M, ω) is called quantizable iff the class of $(2\pi\hbar)^{-1}\omega$ in $H^2(M, \mathbb{R})$ lies in the image of $H^2(M, \mathbb{Z})$.

Counter Example

$M = S_r^2 \times S_s^2$ for $r \neq qs$ $q \in \mathbb{Q}$ is not quantizable.

2nd Question:

When are two prequantizations to be viewed as equivalent?

Answer

- Given a prequantization (L, ∇) and a flat line bundle (L_0, ∇_0) the product $(L \otimes L_0, \nabla \otimes \nabla_0)$ is again a prequantization.
- Conversly, given two prequantizations (L_1, ∇_1) and (L_2, ∇_2), they differ by a flat line bundle (L_0, ∇_0).

Isomorphism classes of flat line bundles -and thus prequantizations- are in one-to-one correspondence with the elements of $H^1(M, U(1))$.

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Mechanics on $\mathbb{R}^n$

- Let the space-time be $Q = \mathbb{R}^n$, the phase space is $M = T^*Q$.
  Since $\mathbb{R}^n$ is contractible, all fiber bundles over $\mathbb{R}^n$ are trivial. Thus $L = T^*Q \times \mathbb{C}$.
- The canonical symplectic 2-form is globally exact $\omega = \mathrm{d}\theta = \mathrm{d}(p_\nu \mathrm{d}q^\nu) = \mathrm{d}p_\nu \wedge \mathrm{d}q^\nu$.
  And the connection on L is globally given by $\nabla = \mathrm{d} - i\hbar^{-1}\theta$.

The hamiltonian vector field X_{q^μ} associated to q^μ is determined by

$$\omega(X_{q^\mu}, \cdot) \overset{!}{=} -\mathrm{d}q^\mu \quad \Rightarrow \quad \mathrm{d}p_\nu(X_{q^\mu})\mathrm{d}q^\nu = -\mathrm{d}q^\mu \quad \Rightarrow \quad X_{q^\mu} = -\tfrac{\partial}{\partial p_\mu}.$$

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- This obviously does not reproduce the Schrödinger representation of position and momentum operators of quantum mechanics.
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Cauchy Correspondence for Klein-Gordon Theory

Klein-Gordon fields are sections of the bundle $E = Q \times \mathbb{R} \to Q$ and the theory is given by

$$L_{KG} := \frac{1}{2}(\nabla_a \phi \nabla_b \phi - m^2 \phi^2), \qquad \Rightarrow \qquad [\nabla_a \nabla^a - m^2]\phi = 0.$$

- In arbitrary curved spacetimes Q, the classical existance and uniqueness properties of solutions to the KG-eqn. can be very different from that of Minkowski spacetime.
- However, there is a simple condition on (Q, g) which guarantees that the KG-eqn. has a well posed initial value formulation: It has to be globally hyperbolic (i.e. admit a Cauchy surface).

Theorem: Let (Q, g) be a globally hyperbolic spacetime and let Σ be a smooth Cauchy surface. Then the KG-eqn. has a well posed initial value formulation:
Given any pair of C^∞-functions $(\phi_0, \dot{\phi}_0)$ on Σ, there exists a unique global solution ϕ to the KG-eqn., such that $\phi|_\Sigma =: \phi_0$ and $[n^a \nabla_a \phi]_\Sigma =: \dot{\phi}_0$.

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Classical Fields

Let M be the phase space of a classical field theory with action

$$S := \int_M vol\ L(\phi^\alpha, \phi^\beta_a, x),$$

where L is a function on the first jet bundle $J^1(E)$ and ϕ is a section of a vector bundle $E \longrightarrow Q$ fullfilling the Euler-Lagrange equations:

$$\frac{\partial L}{\partial \phi^\alpha} = \frac{\partial}{\partial x^a} \frac{\partial L}{\partial \phi^\alpha_a},$$

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- Define a one form θ on M by

$$\iota_X \theta := \int_\Sigma \mathrm{d}\sigma X^\alpha n_a \frac{\partial L}{\partial \phi^\alpha_a}$$

where is $\Sigma \subset Q$ a Cauchy surface with volume form $\mathrm{d}\sigma$ and future orientated normal vector n^a.

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- θ defines a closed 2-form $\omega := \mathrm{d}\theta$ which is independent of the choice of Σ, since for a field X which falls off sufficiently fast at infinity:

$$\iota_X(\theta' - \theta) = \iota_X \mathrm{d}\left(\int_D vol\ L\right) \qquad \Rightarrow \quad \mathrm{d}(\theta' - \theta) = 0,$$

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$$\begin{aligned} \omega(X,Y) &= \mathrm{d}\theta(X,Y) = X(\theta(Y)) - Y(\theta(X)) - \theta([X,Y]) = \int_\Sigma \tilde{\omega}^a n_a \mathrm{d}\sigma \\ \tilde{\omega}^a &= \frac{\partial^2 L}{\partial \phi^\beta \partial \phi^\alpha_a}(X^\beta Y^\alpha - Y^\beta X^\alpha) + \frac{\partial^2 L}{\partial \phi^\beta_b \partial \phi^\alpha_a}(Y^\alpha \nabla_b X^\beta - X^\alpha \nabla_b Y^\beta) \end{aligned}$$

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- If ω is degenerate, we must reduce M by factoring out (TM/K) the characteristic distribution $K := \{X \in \tau_M | \omega(X, \cdot) = 0\}$, which generally involves the removal of gauge freedom.
- Note however that TM/K is not necessarily integrable, i.e. is not necessarily the tangent bundle to a reduced phace space M/G. According to the Frobenius theorem this is only the case if TM/K is involutive.

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Classical Fields 2

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- ω is nondegenerate and thus a symplectic form, given by

$$\omega(\phi, \phi') = \int_\Sigma n^a[\phi'\nabla_a\phi - \phi\nabla_a\phi']\mathrm{d}\sigma,$$

  where we have identified points in the space of solutions (phase space) M with vectors on M. This is possible since M is a vector space.

- Using the Cauchy correspondence $\phi = (\phi_0, \dot{\phi}_0)$, we can rewrite the symplectic form:

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- KG-theory is linear, and thus has a linear space of solutions M (a vector space). Every vector space is contractible and thus all fiber bundles over it are trivial.
- The complex line bundle of prequantization is $L = M \times \mathbb{C}$ and the connection on L is globally $\nabla = \mathrm{d} - i\hbar^{-1}\theta$.

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Klein-Gordon Theory 2

Even in classical field theory in general one cannot define a $L^2(Q, E)$ field at any given point $x \in Q$ since the field may be infinite on a null set. One thus makes use of an $f \in L^2(Q, \mathbb{R})$ with compact support and resorts to smeared field configurations

$$A_f(\phi) := \int_\Sigma \mathrm{d}\sigma \ f\phi = \int_\Sigma \mathrm{d}\sigma \ f\phi_0.$$

The linear functional $A_f(\cdot)$ on M is the object one wants to quantize.

- The Hamiltonian vector field $X_A = (x_A, y_A)$ of A_f is defined by

$$\begin{aligned} \omega[(x_A, y_A), (\tilde{\phi}_0, \dot{\tilde{\phi}}_0)] &= -\mathrm{d}A_f(\tilde{\phi}_0, \dot{\tilde{\phi}}_0) = -A_f(\tilde{\phi}_0, \dot{\tilde{\phi}}_0) \\ \int_\Sigma \mathrm{d}\sigma[\tilde{\phi}_0 y_A - x_A \dot{\tilde{\phi}}_0] &= -\int_\Sigma \mathrm{d}\sigma \, f\tilde{\phi}_0, \end{aligned}$$

  which gives $X_A = (x_A, y_A) = (0, -f)$.

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$$\mathcal{P}(A_f) = -i\hbar[\mathrm{d} - i\hbar^{-1}\theta]_{(0,-f)} + A_f = i\hbar(0, f) + A_f$$

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Analog to the functional A_f we can define a second functional

$$B_f(\phi) := \int_\Sigma \mathrm{d}\sigma\, f n^a \nabla_a \phi = \int_\Sigma \mathrm{d}\sigma\, f \dot{\phi}_0$$

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We can now calculate the commutator of the two operators A_f and B_f using

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[\mathcal{P}(A_f), \mathcal{P}(B_g)] &= -i\hbar\mathcal{P}(\{A_f, B_g\}) = -i\hbar\mathcal{P}(\omega(X_A, X_B)) \\
&= -i\hbar\mathcal{P}(-\int_\Sigma \mathrm{d}\sigma\, fg) = i\hbar\left[\int_\Sigma \mathrm{d}\sigma\, fg\right]\mathcal{P}(1) \\
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To eastablish contact with the physicists language, set $f(x)$"="$\delta(x - x_1)$, $g(x)$"="$\delta(x - x_2)$ to obtain

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Polarization

- We saw that prequantization is not fully satifactory, as -in the case of mechanics - it fails to reproduce the Schrödinger representation of the position operator.
- This is because the prequantum Hilbert space $\mathcal{H} := L^2(M, L)$ is to large. Wave-functions ψ depend on position and momentum. Whereas in the Schrödinger representation they are to depend only on position.
- Solution: Demand that $\psi \in \mathcal{H}$ is constant (i.e. parallel) along n linearly independent vector fields in M:

  $$\nabla_X \psi = 0 \quad \forall X \in P.$$

  where $P \subset TM$ is an n-dimensional distribution.

$$\Rightarrow \forall X, Y \in P, \quad \Omega(X, Y) = -i\nabla_{[X,Y]} = \hbar^{-1}\omega(X, Y)$$

which is automatically satisfied if

1. P is integrable (i.e. a foliation) and thus involutive: $\forall X, Y \in P : \quad [X, Y] \in P$
2. The integral manifolds are isotropic (i.e. Lagrangian, since $\dim(P) = n$): $\forall X, Y \in P : \quad \omega(X, Y) = 0$

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Definition: Polarization

A real Polarization of a symplectic manifold (M, ω) is a foliation of M by Lagrangian submanifolds.

Polarization of Mechanics on $\mathbb{R}^n$

Prequantization left us with

$$\mathcal{P}(q^\mu) = i\hbar \tfrac{\partial}{\partial p_\mu} + q^\mu, \qquad \mathcal{P}(p^\mu) = -i\hbar \tfrac{\partial}{\partial q_\mu}.$$

The idea is now to reduce $\mathcal{H} = L^2(M, L)$ to $L^2(Q, Q \times \mathbb{C})$ via polarization, so that the sections of $L \to M \simeq Q \times T^*_m Q$ are restricted to the ones that depend only on position.

- Choose $P = T_p(T^*_m Q) \simeq T^*_m Q$, then $\forall X, Y \in T^*_m Q \quad \Rightarrow \quad X, Y = X(p), Y(p) \quad \Rightarrow$

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Momentum Representation

The position (or Schrödinger) representation amounts to a vertical polarization ($P \simeq T^*_m Q$). Since the prequantum operators are asymmetic, the momentum representation is not just a horizontal ($P \simeq Q$) polarization with the prequantum operators previously deduced, but one adaitionally has to alter the choice of the connection on the line bundle.

Hilbert Space

- Define: $P_L := \{\psi : M \to L | \nabla_X \psi = 0, \forall X \in P\}$
- Ideally one would like to define the Hilbert space of the quantum theory to be

$$\mathcal{H}_P := L^2(M, L) \cap P_L.$$

However, in general this simple definition faces severe problems:

- For the vertical polarization of mechanics one gets $L^2(M, L) \cap P_L = \emptyset$, since the Schrödinger wave functions depend only on coordinates and thus the momentum integrals diverge.
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Complex polarization

Turn TM into a Kähler manifold, by defining a complex structure J (linear map $J : TM \to TM$ with $J^2 = -1$) on TM which is compatible with the symplectic structure ω (i.e. $\omega(Jv, Jw) = \omega(v, w)$).

Kähler polarization

J can be diagonalized in the complexification $TM^{\mathbb{C}} := \{X + iY | X, Y \in TM\}$ of TM.

- The $+i, -i$ eigenspaces of J are denoted $T^{(1,0)}M = \{X - iJX | X \in TM\}$ and $T^{(0,1)}M = \{X + iJX | X \in TM\}$ which are Lagrangian subspaces since J is compatible with ω.
- The polarization $P = T^{(0,1)}M$ is called Kähler, since
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- For Kähler polarizations the straightforward construction $\mathcal{H}_P := L^2(M, L) \cap P_L$ works, as it can be shown that $P(L) \subset L^2(M, L)$ closed and thus a Hilbert space of its own right.
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Turn TM into a Kähler manifold, by defining a complex structure J (linear map $J : TM \to TM$ with $J^2 = -1$) on TM which is compatible with the symplectic structure ω (i.e. $\omega(Jv, Jw) = \omega(v, w)$).

Kähler polarization

J can be diagonalized in the complexification $TM^{\mathbb{C}} := \{X + iY | X, Y \in TM\}$ of TM.

- The $+i, -i$ eigenspaces of J are denoted $T^{(1,0)}M = \{X - iJX | X \in TM\}$ and $T^{(0,1)}M = \{X + iJX | X \in TM\}$ which are Lagrangian subspaces since J is compatible with ω.
- The polarization $P = T^{(0,1)}M$ is called Kähler, since

$$\overline{P} \cap P = T^{(1,0)}M \cap T^{(0,1)}M = \{0\}$$

- For Kähler polarizations the straightforward construction $\mathcal{H}_P := L^2(M, L) \cap P_L$ works, as it can be shown that $P(L) \subset L^2(M, L)$ closed and thus a Hilbert space of its own right.
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Polarization

Let (Q, g) be a globally hyperbolic spacetime, which is stationary (it admits a one-parameter group of isometries whose orbits are timelike). Let ξ be the Killing vector field which generates these isometries.

- Complexifying M gives $M^{\mathbb{C}}$, on which we define the inner product

$$\begin{aligned} \langle \phi_1, \phi_2 \rangle &:= \int_\Sigma T_{ab}(\phi_1, \phi_2) \xi^a n^b \mathrm{d}\sigma \\ T_{ab}(\phi_1, \phi_2) &= \nabla_a \overline{\phi}_1 \nabla_b \phi_2 + \nabla_b \overline{\phi}_1 \nabla_a \phi_2 - \frac{1}{2} g_{ab} [\nabla^c \overline{\phi}_1 \nabla_c \phi_2 + m^2 \overline{\phi}_1 \phi_2] \end{aligned}$$

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Polarization

On M we now have the antisymmetric 2-form $\omega(\cdot,\cdot)$ and the symmetric 2-form $\langle\cdot,\cdot\rangle$. We thus get the relation

$$(\star) \qquad \omega(X,\cdot) = C\,\langle X,\cdot\rangle\,, \quad \text{for} \quad C\in\mathbb{R}\setminus\{0\}$$

- This generates two disjunct subspaces $V^{\pm} := \{X\in TM^{\mathbb{C}} | C \gtrless 0 \quad \text{in} \quad (\star)\}$.
- $V^{\pm}$ are Lagrangian since $\forall X,Y\in V^{+}$ and $C^{+}\in(0,+\infty)$

$$-\omega(Y,X) = \omega(X,Y) = C^{+}\,\langle X,Y\rangle = C^{+}\,\langle Y,X\rangle\,,$$

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Thanx to my "coach" Olaf and all the listeners...

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