Algebraic Quantum Field Theory

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Outline

1. Fock quantization in a curved spacetime
2. Algebraic approach to quantization

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1. Fock quantization in a curved spacetime

2. Algebraic approach to quantization

- (M,g) = globally hyperbolic spacetime, $E \to M$ = vector bundle over M, section $\phi \in C^\infty_{\mathrm{comp}}(M,E)$ represents field configuration on spacetime, e.g.: real Klein-Gordon field
- $C^\infty(M,E)$ smooth sections, $C^\infty_{\mathrm{comp}}(M,E)$ smooth sections with compact support = test sections
- equation of motion: wave equation $\widehat{P}\phi = \alpha$
  $\widehat{P}: \ C^\infty(M,E) \to C^\infty(M,E)$ = normally hyperbolic operator
  $\widehat{P}$ and α given, ϕ to be found
- second order linear differential operator $\widehat{P}: \ C^\infty(M,E) \to C^\infty(M,E)$ **normally hyperbolic** if for local coordinates $(x^1,\dots,x^n)$ on M and local trivialization of E it writes

$$\widehat{P} = -\sum_{i,j=1}^{n} g^{ij}(x)\,\frac{\partial^2}{\partial x^i\,\partial x^j} + \sum_{j=1}^{n} A_j(x)\,\frac{\partial}{\partial x^j} + B(x)$$

examples: d'Alembert operator $\square$, square of Dirac operator D^2, $\delta d + d\delta$ with exterior derivative d and codifferential δ

- Theorem (in Bär, Ginoux, Pfäffle):
  (M, g) globally hyperbolic Lorentzian manifold, metric connection ∇,
  $\Sigma \subset M$ spacelike Cauchy surface, n future directed timelike unit normal field along Σ,
  E vector bundle over M, $\widehat{P}$ normally hyperbolic operator acting on sections in E,

  then for each $\phi_0, \chi_0 \in C^\infty_{\mathrm{comp}}(\Sigma, E)$ and each $\alpha \in C^\infty_{\mathrm{comp}}(M, E)$
  exists unique solution $\phi \in C^\infty(M, E)$ of **Cauchy problem**: ϕ satisfies

$$\begin{aligned} \widehat{P}\,\phi &= \alpha \\ \phi|_\Sigma &= \phi_0 \\ (\nabla_n \phi)|_\Sigma &= \chi_0 \end{aligned}$$

  and $\operatorname{supp} \phi \subseteq J^M(K)$ with $K = \operatorname{supp} \phi_0 \cup \operatorname{supp} \chi_0 \cup \operatorname{supp} \alpha$ and map sending (α, ϕ_0, χ_0)

  to unique solution ϕ of Cauchy problem is continuous and linear

- point in classical **phase space** $\mathcal{P}$
  $=$ test field configuration ϕ_0 $+$ test momentum π_0 on Cauchy surface Σ,
  assume π depends on ϕ only via $(\nabla_n \phi)$
- consider only systems with linear equations of motion
  and **configuration space** $\mathcal{C}$ being vector space, thus $\mathcal{P} = T^*\mathcal{C}$ vector space
  therefore at $x \in \mathcal{P}$ can identify $T_x\mathcal{P}$ with $\mathcal{P}$,
  thus symplectic form $\Omega_{\mathcal{P}}$ on $\mathcal{P}$ under this identification bilinear map $\Omega_{\mathcal{P}} : \; \mathcal{P} \times \mathcal{P} \to \mathbb{R}$
- $\mathcal{S}$ **space of solutions** of wave equation arising from initial data in $\mathcal{P}$,
  $=$ vector space because of linear equations of motion
- by theorem above each point $(\phi_0, \pi_0) \in \mathcal{P}$ in phase space
  gives rise to unique element in space $\mathcal{S}$ of solutions,
  thus can identify $\mathcal{P}$ and $\mathcal{S}$,
  moreover $\mathcal{S}$ independent of choice of Cauchy surface Σ
- write $\phi|_{\Sigma} =: \phi^{\Sigma}$

- let phase space $\mathcal{P}$ be equipped with symplectic structure $\Omega_{\mathcal{P}}: \ \mathcal{P} \times \mathcal{P} \ \rightarrow \ \mathbb{R}$, for Klein-Gordon field given by

$$\Omega_{\mathcal{P}}((\overset{\Sigma}{\phi_1}, \overset{\Sigma}{\pi_1}), (\overset{\Sigma}{\phi_2}, \overset{\Sigma}{\pi_2})) \ = \ \int_{\Sigma} dV \ (\pi_1 \phi_2 - \pi_2 \phi_1)$$

- if ϕ_1 and ϕ_2 are solutions of the equation of motion, then $\Omega_{\mathcal{P}}((\overset{\Sigma}{\phi_1}, \overset{\Sigma}{\pi_1}), (\overset{\Sigma}{\phi_2}, \overset{\Sigma}{\pi_2}))$ is conserved in time: yields same value for all Cauchy surfaces Σ_t of chosen foliation of spacetime M,
- therefore $\Omega_{\mathcal{P}}$ induces symplectic mapping $\Omega_{\mathcal{S}}: \ \mathcal{S} \times \mathcal{S} \ \rightarrow \ \mathbb{R}$ via

$$\Omega_{\mathcal{S}}(\phi_1, \phi_2) \ \coloneqq \ \Omega_{\mathcal{P}}((\overset{\Sigma}{\phi_1}, \overset{\Sigma}{\pi_1}), (\overset{\Sigma}{\phi_2}, \overset{\Sigma}{\pi_2}))$$

- fixing $(\phi^{\Sigma}, \pi^{\Sigma}) \in \mathcal{P}$ can view $\Omega_{\mathcal{P}}((\phi^{\Sigma}, \pi^{\Sigma}), \bullet)$ as linear function on $\mathcal{P}$
  and fundamental Poisson brackets on $\mathcal{P}$ then can be expressed as

$$\left\{\Omega_{\mathcal{P}}((\phi_1^{\Sigma}, \pi_1^{\Sigma}), \bullet),\ \Omega_{\mathcal{P}}((\phi_2^{\Sigma}, \pi_2^{\Sigma}), \bullet)\right\} = -\Omega_{\mathcal{P}}((\phi_1^{\Sigma}, \pi_1^{\Sigma}), (\phi_2^{\Sigma}, \pi_2^{\Sigma}))$$

- want construct bosonic QFT in which functions $\Omega_{\mathcal{P}}((\phi^{\Sigma}, \pi^{\Sigma}), \bullet)$ on classical phase space $\mathcal{P}$
  are irreducibly represented by operators $\widehat{\Omega}_{\mathcal{P}}((\phi^{\Sigma}, \pi^{\Sigma}), \bullet)$
  satisfying commutation relations corresponding to fundamental Poisson brackets:

$$[\widehat{\Omega}_{\mathcal{P}}((\phi_1^{\Sigma}, \pi_1^{\Sigma}), \bullet), \widehat{\Omega}_{\mathcal{P}}((\phi_2^{\Sigma}, \pi_2^{\Sigma}), \bullet)] = -i\hbar\, \Omega_{\mathcal{P}}((\phi_1^{\Sigma}, \pi_1^{\Sigma}), (\phi_2^{\Sigma}, \pi_2^{\Sigma}))\ \widehat{\mathbb{1}}$$

- using correspondence between phase space $\mathcal{P}$ and solution space $\mathcal{S}$
  equivalently we can look for operators

$$\left[\widehat{\Omega}_{\mathcal{S}}(\phi_1, \bullet),\ \widehat{\Omega}_{\mathcal{S}}(\phi_2, \bullet)\right] = -i\hbar\, \Omega_{\mathcal{S}}(\phi_1, \phi_2)\ \widehat{\mathbb{1}}$$

- introduce compatible **complex structure** $\widehat{J}$ on symplectic vector space $(\mathcal{S}, \Omega_{\mathcal{S}})$:
  linear operator $\widehat{J}: \ \mathcal{S} \to \mathcal{S}$ fulfilling $\widehat{J}^2 = -\widehat{\mathbb{1}}$,
  and $\Omega_{\mathcal{S}}(\phi_1, \phi_2) = \Omega_{\mathcal{S}}(\widehat{J}\phi_1, \widehat{J}\phi_2)$ for all $\phi_{1,2} \in \mathcal{S}$,
  thus defining a positive definite metric μ on $\mathcal{S}$ via $\mu(\cdot, \cdot) = \Omega_{\mathcal{S}}(\widehat{J}\, \cdot, \cdot)$
- $\widehat{J}$ induces Hermitian complex inner product on solution space $\mathcal{S}$:

$$\langle \cdot \, , \, \cdot \rangle_{\mathcal{S}} = \frac{1}{2\hbar}\mu(\cdot, \cdot) - \frac{i}{2\hbar}\Omega_{\mathcal{S}}(\cdot, \cdot) = \frac{1}{2\hbar}\Omega_{\mathcal{S}}\Big((\hat{J} - i\hat{\mathbb{1}})\cdot, \cdot\Big)$$

- $\widehat{J}$ naturally splits complexification of solution space $\mathcal{S}_{\mathbb{C}} = \mathcal{S} \oplus i\mathcal{S}$
  into positive and negative frequency subspaces:

$$\text{positive frequency vectors: } \phi^+ := \frac{1}{2}(\phi - i\widehat{J}\phi) = \overline{\phi^-}$$

$$\text{negative frequency vectors: } \phi^- := \frac{1}{2}(\phi + i\widehat{J}\phi) = \overline{\phi^+}$$

- positive and negative frequency subspaces are orthogonal $\langle \phi_1^+, \phi_2^- \rangle_{\mathcal{S}} = 0$ for all $\phi_{1,2} \in \mathcal{S}$,
  and complementary $\phi = \phi^+ + \phi^-$

- **one-particle Hilbert space** $\mathcal{H}_0$ of theory is positive frequency subspace of $\mathcal{S}_{\mathbb{C}}$ with inner product induced by the one of $\mathcal{S}$:

$$\langle \phi_1, \phi_2 \rangle_{\mathcal{H}_0} = \langle \phi_1, \phi_2 \rangle_{\mathcal{S}} = -\frac{i}{\hbar}\, \Omega_{\mathcal{S}}(\phi_1^-, \phi_2^+) = -\frac{i}{\hbar}\, \Omega_{\mathcal{S}}(\overline{\phi_1^+}, \phi_2^+)$$

  (respectively its Cauchy-completion with respect to norm induced by inner product)
- map $K: \ \mathcal{S} \to \mathcal{H}_0$ defined by $K\phi \coloneqq \phi^+$ is linear bijection, image = dense subspace of $\mathcal{H}_0$
- Hilbert space $\mathcal{H}$ of QFT is **symmetric Fock space**:

$$\mathcal{F}_{\mathsf{sym}}(\mathcal{H}_0) \coloneqq \bigoplus_{n=0}^{\infty} \left(\bigotimes_{\mathsf{sym}}^{n} \mathcal{H}_0 \right)$$

- symmetrized tensor product $\otimes_{\mathsf{sym}}^n \mathcal{H}_0$ of $\mathcal{H}_0$ is subspace of n-fold tensor product consisting of totally symmetric maps

$$\alpha : \ \bigoplus_{1}^{n} \overline{\mathcal{H}_0} \to \mathbb{C}$$

$$\sum_{j_1, \ldots, j_n = 1}^{n} \left| \alpha(\bar{e}_{j_1}, \ldots, \bar{e}_{j_n}) \right|^2 < \infty$$

$$\bigotimes^{0} \mathcal{H}_0 \coloneqq \mathbb{C}$$

- abstract index notation for Hilbert spaces: given some Hilbert space $\mathcal{H}$ can construct complex conjugate $\overline{\mathcal{H}}$, dual space $\mathcal{H}^*$ and complex conjugate dual $\overline{\mathcal{H}^*}$
- denote elements of Hilbert spaces by

$$\begin{array}{cccc} \mathcal{H} & \overline{\mathcal{H}} & \mathcal{H}^* & \overline{\mathcal{H}^*} \\ \phi^A & \phi^{A'} = \overline{\phi}^A & \phi_A & \phi_{A'} = \overline{\phi}_A \end{array}$$

- **Riesz' Lemma**: for every element ϕ_A of $\mathcal{H}^*$ exists unique element $\phi^A \in \mathcal{H}$ with

$$\phi_A(\bullet) = \langle \phi^A, \bullet \rangle_{\mathcal{H}}$$

  provides antilinear bijection between Hilbert space and its dual
- by Riesz lemma can identify $\overline{\mathcal{H}}$ with $\mathcal{H}^*$, i.e. $\phi^{A'} = \overline{\phi}^A$ with ϕ_A, and $\overline{\mathcal{H}^*}$ with $\mathcal{H}$, i.e. ϕ^A with $\phi_{A'} = \overline{\phi}_A$, therefore do not need primed indices, can write inner product

$$\langle \psi^A, \phi^A \rangle_{\mathcal{H}} = \overline{\psi}_A \phi^A$$

- denote elements of $\otimes^n \mathcal{H}$ as $\phi^{A_1,\ldots,A_n}$ and elements of $\otimes^n \mathcal{H}^*$ as $\psi_{A_1,\ldots,A_n}$
- elements of $\otimes^n_{\mathsf{sym}} \mathcal{H}$ satisfy $\phi^{A_1,\ldots,A_n} = \phi^{(A_1,\ldots,A_n)}$
- vector in symmetric Fock $\mathcal{F}_{\mathsf{sym}}(\mathcal{H}_0)$ space in abstract index notation written as

$$\Psi = (\psi, \psi^{A_1}, \psi^{A_1 A_2}, \ldots, \psi^{A_1 \cdots A_n}, \ldots)$$

norm given by

$$\| \Psi \|^2 = \langle \Psi, \Psi \rangle = \overline{\psi}\psi + \overline{\psi}_{A_1}\psi^{A_1} + \overline{\psi}_{A_1 A_2}\psi^{A_1 A_2} + \ldots < \infty$$

- for element $\xi^A \in \mathcal{H}_0$ and corresponding $\overline{\xi}_A \in \overline{\mathcal{H}_0}$
define **creation** $\hat{a}^\dagger$ and **annihilation** $\hat{a}$ **operators**: $\mathcal{F}_{\mathsf{sym}}(\mathcal{H}_0) \rightarrow \mathcal{F}_{\mathsf{sym}}(\mathcal{H}_0)$

$$\hat{a}^\dagger(\xi)\,\Psi \coloneqq (0,\, \psi\xi^{A_1},\, \sqrt{2}\,\xi^{(A_1}\psi^{A_2)},\, \sqrt{3}\,\xi^{(A_1}\psi^{A_2 A_3)},\, \ldots)$$
$$\hat{a}(\overline{\xi})\,\Psi \coloneqq (\overline{\xi}_{A_1}\psi^{A_1},\, \sqrt{2}\,\overline{\xi}_{A_1}\psi^{A_1 A_2},\, \sqrt{3}\,\overline{\xi}_{A_1}\psi^{A_1 A_2 A_3},\, \ldots)$$

satisfying commutation relations $[\hat{a}(\overline{\xi}), \hat{a}^\dagger(\eta)] = \overline{\xi}_A \eta^A\, \hat{\mathbb{1}}$

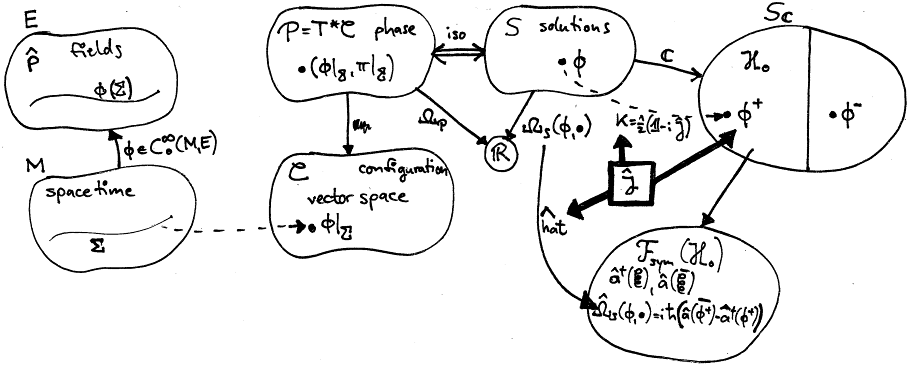
- represent linear classical observables $\Omega_{\mathcal{S}}(\phi, \bullet)$ by operator

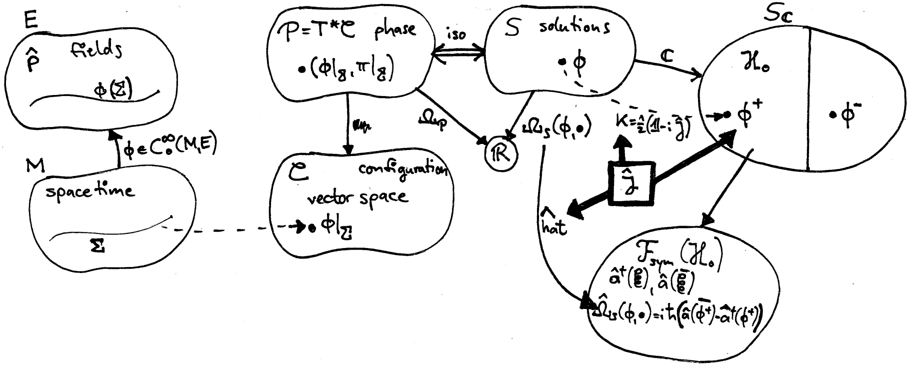
$$\widehat{\Omega}_{\mathcal{S}}(\phi, \bullet) \coloneqq i\hbar\Big(\widehat{a}(\overline{K\phi}) - \widehat{a}^{\dagger}(K\phi)\Big) = i\hbar\Big(\widehat{a}(\overline{\phi^{+}}) - \widehat{a}^{\dagger}(\phi^{+})\Big)$$

  then operators are self-adjoint and satisfy desired commutation relations:

$$[\widehat{\Omega}_{\mathcal{S}}(\eta, \bullet), \widehat{\Omega}_{\mathcal{S}}(\xi, \bullet)] = -i\hbar\Omega_{\mathcal{S}}(\eta, \xi)\,\widehat{\mathbb{1}}$$

- calculating commutation relations only need general properties of inner product, which is induced by complex structure $\widehat{J}$,
  thus one gets representation of the CCR for any choice of complex structure,
  freedom to choose $\widehat{J}$ is freedom to chose quantum representation of the CCR


E
$\hat{P}$ fields
$\phi(\Sigma)$
$\phi \in C_0^\infty(M,E)$
M
space time
Σ
$P = T^*C$ phase
$\bullet(\phi|_\Sigma, \pi|_\Sigma)$
ω_P
C configuration
vector space
$\bullet \phi|_\Sigma$
iso
S solutions
$\bullet \phi$
Ω_P
$\mathbb{R}$
$\Omega_S(\phi, \bullet)$
$\mathbb{C}$
$S_\mathbb{C}$
$\mathcal{H}_0$
$\bullet \phi^+$
$\bullet \phi^-$
$K = \frac{1}{2}(\mathbb{1} - i\hat{J})$
$\hat{J}$
hat
$\mathcal{F}_{sym}(\mathcal{H}_0)$
$\hat{a}^\dagger(\xi), \hat{a}(\bar{\xi})$
$\hat{\Omega}_S(\phi, \bullet) = i\hbar(\hat{a}(\bar{\phi^+}) - \hat{a}^\dagger(\phi^+))$



- completely free to choose complex structure $\widehat{J}$, no naturally preferred $\widehat{J}$,
  complex structure exactly determines Fock space construction,
  which defines notion of particles,
  thus no natural notion of particles in general, curved spacetime

- QFT consisting of Hilbert space $\mathcal{H}$ with operators $\widehat{\mathcal{O}}_\alpha : \mathcal{H} \to \mathcal{H}$
unitary equivalent to quantum theory $(\mathcal{H}', \widehat{\mathcal{O}}'_\alpha)$
if exists unitary map $U : \mathcal{H} \to \mathcal{H}'$ such that $\widehat{\mathcal{O}}_\alpha = U^{-1} \widehat{\mathcal{O}}'_\alpha U$
- unitary equivalent QFT are physically equivalent: state $\Psi \in \mathcal{H}$ in quantum theory $(\mathcal{H}, \widehat{\mathcal{O}}_\alpha)$ has same physical properties as state $U\Psi \in \mathcal{H}'$ in QFT $(\mathcal{H}', \widehat{\mathcal{O}}'_\alpha)$
- working with relation $\left[\widehat{\Omega}_{\mathcal{S}}(\phi_1, \bullet),\ \widehat{\Omega}_{\mathcal{S}}(\phi_2, \bullet)\right] = -i\hbar\,\Omega_{\mathcal{S}}(\phi_1, \phi_2)\ \widehat{\mathbb{1}}$
difficulties arise: operators can be unbounded and thus not everywhere defined, therefore their composition and commutators need not be well defined
- more convenient: exponentiated version, write $W(\phi) = \exp i\Omega_{\mathcal{S}}(\phi, \bullet)$ and look for map turning $W(\phi)$ into operator $\widehat{W(\phi)}$ such that it is unitary, varies continuously with ϕ and as equivalent of commutation relations satisfies **Weyl relations**:

$$
\begin{aligned}
\widehat{W(0)} &= \mathbb{1} \\
\widehat{W(-\phi)} &= \widehat{W(\phi)}^\dagger \\
\widehat{W(\phi)}\,\widehat{W(\psi)} &= e^{i\omega(\phi,\psi)/2}\,\widehat{W(\varphi+\psi)}
\end{aligned}
$$

- **Stone-von Neumann Theorem:**
  If $(\mathcal{S}, \Omega)$ is finite-dimensional symplectic vector space and $(\mathcal{H}, \widehat{W(\phi)})$ and $(\mathcal{H}', \widehat{W'(\phi)})$ are strongly continuous, irreducible, unitary representations of the Weyl relations, then they are unitarily equivalent.
- if Stone-von Neumann **were** valid also for QFT of infinite-dimensional phase space $\mathcal{P}$, then choice of complex structure $\widehat{J}$ would lead to unitarily equivalent theories
- however if $\mathcal{P}$ is of infinite dimension, then different choices of $\widehat{J}$ can yield unitarily inequivalent theories: Stone-von Neumann does not hold here
- therefore in order to uniquely define **the** QFT for a general, curved spacetime essential to find preferred unique unitary equivalence class of complex structures $\widehat{J}$
- for general curved spacetimes: no known criterion to find unique preferred equivalence class of complex structures
- problem is circumvented by algebraic approach to QFT which does not require specification of preferred unitary equivalence class of $\widehat{J}$'s

Outline

- usual approach: first states, then observables acting on these states
- algebraic approach reverses roles of states and observables:
  first observables constructed as elements of abstract algebra
  then states defined as objects acting on observables by assigning real numbers to them
- advantage: allows treatment all states, also in unitarily inequivalent QFTs, on equal footing, thereby possible to define theory without selecting preferred construction
- key observation justifying algebraic approach:
  even if $(\mathcal{F}_{\mathsf{sym}}(\mathcal{H}_0^1), \{\widehat{\Omega}_{\mathcal{S}}^1(\phi, \bullet)\})$ and $(\mathcal{F}_{\mathsf{sym}}(\mathcal{H}_0^2), \{\widehat{\Omega}_{\mathcal{S}}^2(\phi, \bullet)\})$ unitarily inequivalent, algebraic relations satisfied by observables $\{\widehat{\Omega}_{\mathcal{S}}^1(\phi, \bullet)\}$ are same as of $\{\widehat{\Omega}_{\mathcal{S}}^2(\phi, \bullet)\}$

- $\mathbb{C}$-algebra $\mathcal{A}$ = vector space over $\mathbb{C}$
  with bilinear, associative vector multiplication: $\mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}, \qquad (a, b) \mapsto ab$

- C^*-**algebra** is $\mathbb{C}$-algebra with complete norm $\| \ \| : \ \mathcal{A} \to \mathbb{R}_0^+$ and antilinear star map $* : \ \mathcal{A} \to \mathcal{A}, \quad a \mapsto a^*$ fulfilling for all $a, b \in \mathcal{A}$:

$$a^{**} = a \qquad \text{* is involution} \tag{1}$$

$$(ab)^* = b^* a^* \tag{2}$$

$$\| ab \| \leqslant \| a \| \| b \| \qquad \text{norm is submultiplicative} \tag{3}$$

$$\| a^* \| = \| a \| \qquad \text{* is isometry} \tag{4}$$

$$\| a^* a \| = \| a \|^2 \qquad C^*\text{-property} \tag{5}$$

- C^*-subalgebra is subset of C^*-algebra closed under all its operations
- C^*-algebra has at most one unit $\mathbb{1}$, i.e. $a\mathbb{1} = \mathbb{1}a = a$ for all $a \in \mathcal{A}$ it fulfills $\mathbb{1}^* = \mathbb{1}$ and $\| \mathbb{1} \| = 1$
- in C^*-algebras all operations continuous on domains of definition
- BLop$(\mathcal{H}) = C^*$-algebra of **bounded linear operators** on Hilbert space $\mathcal{H}$, star operation here is taking adjoint of operator

- **Weyl system** $(\mathcal{A}, W)$ of symplectic vector space (V, ω) consists of C^*-algebra $\mathcal{A}$ with unit $\mathbb{1}$ and Weyl map $W: \;\; V \;\rightarrow\; \mathcal{A}$ such that for all $\varphi, \psi \in V$ Weyl relations are fulfilled:

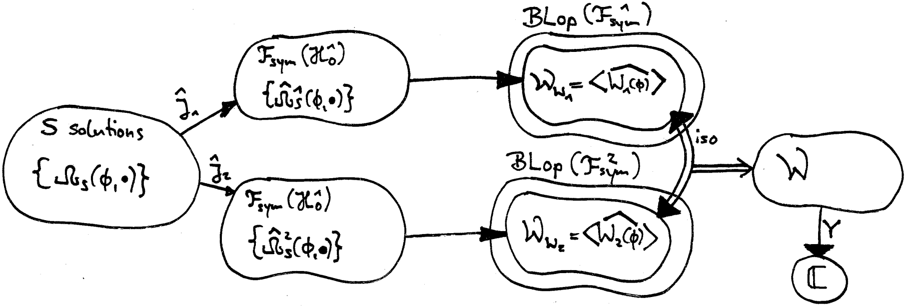
$$W(0) \;=\; \mathbb{1} \tag{6}$$

$$W(-\varphi) \;=\; W(\varphi)^* \tag{7}$$

$$W(\varphi)\, W(\psi) \;=\; e^{i\omega(\varphi,\psi)/2}\, W(\varphi + \psi) \tag{8}$$

  $W(\varphi)$ is unitary for all $\varphi \in V$, $\{W(\varphi)\}_{\varphi \,\in\, V}$ are linear independent

- $\langle W(V) \rangle \subset \mathcal{A} =$ complex linear span of all elements $\{W(\varphi\}_{\varphi \,\in\, V}$, is closed under vector multiplication and star/adjoint, completing it in norm of C^*-algebra $\mathcal{A}$ yields C^*-subalgebra: the **Weyl algebra** $\mathcal{W}_W(\mathcal{A})$ of $\mathcal{A}$ with respect to W


S solutions
$\{\omega_S(\phi,\bullet)\}$
$\hat{j}_1$
$\hat{j}_2$
$\mathcal{F}_{sym}(\mathcal{H}_0^1)$
$\{\hat{\omega}_S^1(\phi,\bullet)\}$
$\mathcal{F}_{sym}(\mathcal{H}_0^1)$
$\{\hat{\omega}_S^2(\phi,\bullet)\}$
$BLop(\mathcal{F}_{sym}^1)$
$\mathcal{W}_{\omega_1} = \langle \widehat{W_1(\phi)} \rangle$
$BLop(\mathcal{F}_{sym}^2)$
$\mathcal{W}_{\omega_2} = \langle \widehat{W_2(\phi)} \rangle$
iso
$\mathcal{W}$
Y
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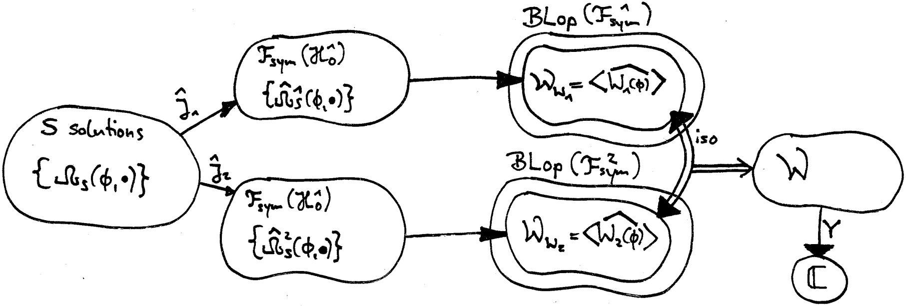

- Weyl map $W: \ \mathcal{S} \ \rightarrow \ \mathrm{BLop}(\mathcal{F}_{\mathrm{sym}}(\mathcal{H}_0))$ yielding unitary operators given by

$$\widehat{W(\phi)} \ = \ \exp i\widehat{\Omega}_{\mathcal{S}}(\phi, \bullet)$$

- key fact: although symmetric Fock space $\mathcal{F}_{\mathrm{sym}}(\mathcal{H}_0)$ and observables $\widehat{\Omega}_{\mathcal{S}}(\phi, \bullet)$ do depend on choice of complex structure $\widehat{J}$, the Weyl algebra $\mathcal{W}_W\Big(\mathrm{BLop}(\mathcal{F}_{\mathrm{sym}}(\mathcal{H}_0))\Big)$ does not:

  even if complex structures $\widehat{J}_1$ and $\widehat{J}_2$ define unitarily inequivalent QFTs, induced Weyl algebras $\mathcal{W}_1$ and $\mathcal{W}_2$ are isomorphic
- this allows to define fundamental observables for QFT in curved spacetime as elements of the Weyl algebra $\mathcal{W} \ = \ \mathcal{W}_W\Big(\mathrm{BLop}(\mathcal{F}_{\mathrm{sym}}(\mathcal{H}_0))\Big)$ using arbitrary complex structure $\widehat{J}$

- **algebraic state** of quantum field defined as linear map $Y: \ \mathcal{W} \rightarrow \mathbb{C}$ satisfying

$$\text{positivity condition:} \qquad Y(w^* w) \geqslant 0 \qquad \forall w \in \mathcal{W}$$
$$\text{normalization:} \qquad Y(\mathbb{1}) = 1$$

- algebraic state Y called **mixed** if it can be written as sum of states $Y_1 \neq Y_2$

$$Y = c_1 Y_1 + c_2 Y_2 \qquad c_{1,2} > 0$$

  else called **pure**
- $\mathcal{W}$ contains only fundamental (linear) observables,
  there are other physically relevant observables in theory,
  thus we should view $\mathcal{W}$ as minimal set of observables, sufficient to formulate theory
- to get additional observables: enlarge Weyl algebra and/or restrict abstract notion of state

- given any Hilbert space $\mathcal{F}_{\mathsf{sym}}(\mathcal{H}_0)$ carrying representation $R: \ \mathcal{W} \ \rightarrow \ \mathsf{BLop}(\mathcal{F}_{\mathsf{sym}}(\mathcal{H}_0))$ of considered Weyl algebra $\mathcal{W}$, for any state in $\mathcal{F}_{\mathsf{sym}}(\mathcal{H}_0)$ exists unique density matrix $\hat{\rho}: \ \mathcal{F}_{\mathsf{sym}}(\mathcal{H}_0) \ \rightarrow \ \mathcal{F}_{\mathsf{sym}}(\mathcal{H}_0)$
- obtain algebraic state $Y_{\hat{\rho}}: \ \mathcal{W} \ \rightarrow \ \mathbb{C}$ by

$$Y_{\hat{\rho}}(w) \ \coloneqq \ \mathsf{tr}\ \big(\hat{\rho}\, R(w)\big)$$

  thus for each state in each possible Fock construction there is a corresponding algebraic state
- **GNS construction:** (Gelfand-Naimark-Segal) let $\mathcal{W}$ be C^*-algebra with unit and a state $Y: \ \mathcal{W} \ \rightarrow \ \mathbb{C}$, then there exist Hilbert space $\mathcal{H}$, representation $R: \ \mathcal{W} \ \rightarrow \ \mathsf{BLop}(\mathcal{H})$ and vector $|\psi_Y\rangle \in \mathcal{H}$ such that

$$Y(w) \ = \ \langle\psi_Y|R(w)|\psi_Y\rangle \qquad \forall w \in \mathcal{W}$$

  Hilbert space, representation and vector are unique up to unitary equivalence, additional property: $|\psi_Y\rangle$ is **cyclic**: vectors $\{R(w)|\psi_Y\rangle\}_{w \,\in\, \mathcal{W}}$ are dense subspace of $\mathcal{H}$

- sketch of GNS construction: first use state Y to define non-negative, bilinear map

$$\begin{aligned} \langle \cdot\,, \ \cdot \rangle_{\mathcal{W}} : \ \mathcal{W} \times \mathcal{W} &\to \mathbb{C} \\ \langle v, \, w \rangle_{\mathcal{W}} &\coloneqq Y(v^* w) \end{aligned}$$

  (after factoring out kernel of Y) this defines positive definite inner product on (quotient space of) Weyl algebra $\mathcal{W}$

- complete (quotient space of) $\mathcal{W}$ in norm induced by inner product, thereby get GNS Hilbert space $\mathcal{H} = \overline{\mathcal{W}}$
- letting (quotient space of) $\mathcal{W}$ act upon itself by vector multiplication and extending this action continuously to $\mathcal{H}$ get representation $R : \ \mathcal{W} \to \mathsf{BLop}(\mathcal{H})$ by $R(w) = w$ for all $w \in \overline{\mathcal{W}}$
- cyclic vector $|\psi_Y\rangle \in \mathcal{H}$ is unit $\mathbb{1}$ of $\mathcal{W}$
- GNS construction expresses pure and mixed algebraic states as pure states in GNS Hilbert space, however GNS representation of $\mathcal{W}$ irreducible iff algebraic state is pure

- usual Hilbert space approach: observable represented by self-adjoint operator $\widehat{A}: \mathcal{H} \to \mathcal{H}$, with real eigenvalues α_k and eigenvectors $|a_k\rangle$,
  by spectral theorem it has associated family of projection operators $\widehat{P}_k: \mathcal{H} \to V_k$ projecting onto the eigenspace $V_k \subset \mathcal{H}$ spanned by eigenvectors of eigenvalue α_k
- if system in normalized state $|\psi\rangle$, then probability that measurement of $\widehat{A}$ yields value in interval $I \subset \mathbb{R}$ is given by $\| \widehat{P}_I |\psi\rangle \|^2$
  where $\widehat{P}_I$ is projection operator of $\widehat{A}$ for interval I:

$$\widehat{P}_I = \sum_k^{\alpha_k \in I} \widehat{P}_k$$

- more general: in Heisenberg representation: state represented by density matrix $\hat{\rho}$, normalized: tr $\hat{\rho} = 1$, then probability that measurement of self-adjoint observables $\widehat{A}_1, ..., \widehat{A}_n$ made at times $t_1 < ... < t_n$ will yield results lying in intervals $I_1, ..., I_n$ given by

$$\text{tr}\,(\widehat{P}_n ... \widehat{P}_1 \; \hat{\rho} \; \widehat{P}_1 ... \widehat{P}_n)$$

  $\widehat{P}_k$ = projection operator of $\widehat{A}_k(t_k)$ on interval I_k,
  equation contains all available information
  in standard quantum mechanical measurement theory

- in algebraic approach: for arbitrary normalized state Y probability that measurement of self-adjoint observables $\widehat{A}_1, ..., \widehat{A}_n$ made at times $t_1 < ... < t_n$ will yield results lying in intervals $I_1, ..., I_n$ can be defined by

$$\lim_{j_1,...,j_n \to \infty} Y\Big((Q_1)_{j_1}(\hat{A}_1)...(Q_n)_{j_n}(\hat{A}_n)(Q_n)_{j_n}(\hat{A}_n)...(Q_1)_{j_1}(\hat{A}_1)\Big)$$

$\{(Q_k)_{j_k}(\hat{A}_k)\}_{j_k \in \mathbb{N}}$ is any sequence of polynomials in $\widehat{A}_k$
such that polynomials $\{(Q_k)_{j_k}(x)\}$ are uniformly bounded on spectrum of $\widehat{A}_k$
and converge on spectrum of $\widehat{A}_k$ to characteristic function 1_{I_k} of interval I_k

$$1_{I}(x) = \begin{cases} 1: & x \in I \\ 0: & x \notin I \end{cases}$$

- evaluating this definition of probability in GNS representation shows that limit exists and equals what would be obtained from usual QM formula in GNS representation, or any other representation of $\mathcal{W}$ in which Y realized as density matrix
- thus algebraic definition of probability equivalent
  to putting observables into any representation and use standard Hilbert space rule
- however algebraic definition of measurement probability ensures independence of representation

THANKS FOR YOUR ATTENTION!

- R. Wald: QFT in curved spacetime and black hole thermodynamics, university of Chicago Press, 1994
- A. Corichi, J. Cortez, H. Quevedo: Schrödinger and Fock representation for a field theory on curved spacetime, Annals of Physics 313 (2004) 446-478, [arXiv:hep-th/0202070]
- C. Bär, N. Ginoux, F. Pfäffle: Wave equations on Lorentzian manifolds and Quantization, EMS Publishing House, 2007
  downloadable online [http://users.math.uni-potsdam.de/ baer/homepage.html]