1 Fock quantization in a curved spacetime

- let (M, g) be globally hyperbolic spacetime, $E \to M$ be vector bundle over M such that section $\phi \in C^\infty_{\text{comp}}(M, E)$ represents a field configuration on spacetime, e.g., real Klein-Gordon field

- $C^\infty(M, E)$ smooth sections, i.e., maps $s: M \to E$ such that $\pi \circ s = \text{Id}_M$, $C^\infty_{\text{comp}}(M, E)$ smooth sections of E with compact support = **test sections** on E

- consider as equation of motion **wave equation** $\hat{P}\phi = \alpha$ with normally hyperbolic operator $\hat{P}: C^\infty(M, E) \to C^\infty(M, E)$ acting on sections ϕ in fiber bundle $M \to E$, $\hat{P}$ and α given, ϕ to be found

- linear differential operator of second order $\hat{P}: C^\infty(M, E) \to C^\infty(M, E)$ is called **normally hyperbolic** if for local coordinates $(x^1, ..., x^n)$ on M and a local trivialization of E it writes as

$$\hat{P} = -\sum_{i,j=1}^{n} g^{ij}(x) \frac{\partial^2}{\partial x^i \partial x^j} + \sum_{j=1}^{n} A_j(x) \frac{\partial}{\partial x^j} + B(x)$$

  with A_j and B matrix-valued coefficients depending smoothly on x and $(g^{ij}$ the inverse matrix of $(g_{ij} = \langle \partial_{x^i}, \partial_{x^j} \rangle_{TM}$

- examples of normally hyperbolic operators: d'Alembert operator $\Box$, square of Dirac operator D^2, $\delta d + d\delta$ with exterior derivative d and codifferential δ

- **Theorem (in Bär, Ginoux, Pfäffle):** let (M, g) globally hyperbolic Lorentzian manifold with metric connection ∇, $\Sigma \subset M$ spacelike Cauchy surface, n future directed timelike unit normal field along Σ, E vector bundle over M and $\hat{P}$ normally hyperbolic operator acting on sections in E,

  then for each $\phi_0, \chi_0 \in C^\infty_{\text{comp}}(\Sigma, E)$ and each $\alpha \in C^\infty_{\text{comp}}(M, E)$ there exists unique solution $\phi \in C^\infty(M, E)$ of the **Cauchy problem**, i.e., ϕ satisfies

$$\begin{aligned} \hat{P}\phi &= \alpha \\ \phi|_\Sigma &= \phi_0 \\ (\nabla_n \phi)|_\Sigma &= \chi_0 \end{aligned}$$

  and $\operatorname{supp}\phi \subseteq J^M(K)$ with $K = \operatorname{supp}\phi_0 \cup \operatorname{supp}\chi_0 \cup \operatorname{supp}\alpha$

  and the map sending (α, ϕ_0, χ_0) to unique solution ϕ of Cauchy problem is continuous and linear

- point in classical phase space $\mathcal{P}$ consists of test field configuration ϕ_0 and test momentum π_0 on Cauchy surface Σ, we assume that π depends on ϕ only via $(\nabla_n \phi)$

- moreover consider only systems with linear equations of motion and configuration space $\mathcal{C}$ being vector space, $\mathcal{P} = T^*\mathcal{C}$ thus vector space, too, therefore at any point $x \in \mathcal{P}$ we can identify $T_x\mathcal{P}$ with $\mathcal{P}$ itself, thus symplectic form $\Omega_\mathcal{P}$ on $\mathcal{P}$ under this identification can be seen as bilinear map $\Omega_\mathcal{P} : \mathcal{P} \times \mathcal{P} \to \mathbb{R}$
- $\mathcal{S}$ is space of solutions of wave equation which arise from initial data in $\mathcal{P}$, also vector space because equations of motion linear
- by theorem above each point $(\phi_0, \pi_0) \in \mathcal{P}$ in phase space gives rise to unique element in space $\mathcal{S}$ of solutions, thus we can identify $\mathcal{P}$ and $\mathcal{S}$, moreover $\mathcal{S}$ is independent of choice of Cauchy surface Σ
- write $\phi|_\Sigma =: \overset{\Sigma}{\phi}$
- let phase space $\mathcal{P}$ be equipped with symplectic structure $\Omega_\mathcal{P} : \mathcal{P} \times \mathcal{P} \to \mathbb{R}$, for Klein-Gordon field given by

$$\Omega_\mathcal{P}((\overset{\Sigma}{\phi}_1, \overset{\Sigma}{\pi}_1), (\overset{\Sigma}{\phi}_2, \overset{\Sigma}{\pi}_2)) = \int_\Sigma dV \, (\pi_1\phi_2 - \pi_2\phi_1)$$

- if ϕ_1 and ϕ_2 are solutions of the equation of motion, then $\Omega_\mathcal{P}((\overset{\Sigma}{\phi}_1, \overset{\Sigma}{\pi}_1), (\overset{\Sigma}{\phi}_2, \overset{\Sigma}{\pi}_2))$ is conserved in time, i.e., yields same value for all Cauchy surfaces Σ_t of chosen foliation of spacetime M,
- therefore $\Omega_\mathcal{P}$ induces symplectic mapping $\Omega_\mathcal{S} : \mathcal{S} \times \mathcal{S} \to \mathbb{R}$ via

$$\Omega_\mathcal{S}(\phi_1, \phi_2) := \Omega_\mathcal{P}((\overset{\Sigma}{\phi}_1, \overset{\Sigma}{\pi}_1), (\overset{\Sigma}{\phi}_2, \overset{\Sigma}{\pi}_2))$$

- fixing $(\overset{\Sigma}{\phi}, \overset{\Sigma}{\pi}) \in \mathcal{P}$ we can view $\Omega_\mathcal{P}((\overset{\Sigma}{\phi}, \overset{\Sigma}{\pi}), \bullet)$ as linear function on $\mathcal{P}$ (and thus as linear observable) and fundamental Poisson brackets on $\mathcal{P}$ then can be expressed as

$$\left\{\Omega_\mathcal{P}((\overset{\Sigma}{\phi}_1, \overset{\Sigma}{\pi}_1), \bullet), \ \Omega_\mathcal{P}((\overset{\Sigma}{\phi}_2, \overset{\Sigma}{\pi}_2), \bullet)\right\} = -\Omega_\mathcal{P}((\overset{\Sigma}{\phi}_1, \overset{\Sigma}{\pi}_1), (\overset{\Sigma}{\phi}_2, \overset{\Sigma}{\pi}_2))$$

- want to construct bosonic QFT in which functions $\Omega_\mathcal{P}((\overset{\Sigma}{\phi}, \overset{\Sigma}{\pi}), \bullet)$ on classical phase space $\mathcal{P}$ are irreducibly represented by operators $\widehat{\Omega}_\mathcal{P}((\overset{\Sigma}{\phi}, \overset{\Sigma}{\pi}), \bullet)$ (whole object is operator even though we write the hat only above Ω) satisfying commutation relations corresponding to fundamental Poisson brackets:

$$[\widehat{\Omega}_\mathcal{P}((\overset{\Sigma}{\phi}_1, \overset{\Sigma}{\pi}_1), \bullet), \widehat{\Omega}_\mathcal{P}((\overset{\Sigma}{\phi}_2, \overset{\Sigma}{\pi}_2), \bullet)] = -i\hbar\, \Omega_\mathcal{P}((\overset{\Sigma}{\phi}_1, \overset{\Sigma}{\pi}_1), (\overset{\Sigma}{\phi}_2, \overset{\Sigma}{\pi}_2)) \, \widehat{\mathbb{1}}$$

- using correspondence between phase space $\mathcal{P}$ and solution space $\mathcal{S}$ equivalently we can look for operators

$$\big[\hat{\Omega}_{\mathcal{S}}(\phi_1,\bullet),\ \hat{\Omega}_{\mathcal{S}}(\phi_2,\bullet)\big] \;=\; -i\hbar\,\Omega_{\mathcal{S}}(\phi_1,\phi_2)\,\hat{\mathbb{1}}$$

- Hilbert space in QFT on flat spacetime constructed using positive and negative frequency solutions, however in general curved spacetime there is no naturally preferred way of defining these
- now introduce compatible **complex structure** $\hat{J}$ on symplectic vector space $(\mathcal{S},\Omega_{\mathcal{S}})$, i.e., linear operator $\hat{J}: \;\; \mathcal{S} \;\to\; \mathcal{S}$ fulfilling $\hat{J}^2 \;=\; -\hat{\mathbb{1}}$ and $\Omega_{\mathcal{S}}(\phi_1,\phi_2) \;=\; \Omega_{\mathcal{S}}(\hat{J}\phi_1,\hat{J}\phi_2)$ for all $\phi_{1,2} \in \mathcal{S}$, and thus defining a positive definite metric μ on $\mathcal{S}$ via $\mu(\cdot,\cdot) \;=\; \Omega_{\mathcal{S}}(\hat{J}\,\cdot,\cdot)$
- this induces a Hermitian complex inner product on solution space $\mathcal{S}$:

$$\langle\,\cdot\,,\,\cdot\,\rangle_{\mathcal{S}} \;=\; \frac{1}{2\hbar}\mu(\cdot,\cdot) - \frac{i}{2\hbar}\Omega_{\mathcal{S}}(\cdot,\cdot) \;=\; \frac{1}{2\hbar}\Omega_{\mathcal{S}}\Big((\hat{J}-i\hat{\mathbb{1}})\cdot,\cdot\Big)$$

- complex structure $\hat{J}$ naturally splits complexification of solution space $\mathcal{S}_{\mathbb{C}} \;=\; \mathcal{S}\oplus i\mathcal{S}$ into positive and negative frequency subspaces:

$$\begin{aligned}\text{positive frequency vectors: } \phi^+ &\coloneqq \frac{1}{2}(\phi - i\hat{J}\phi) \;=\; \overline{\phi^-}\\ \text{negative frequency vectors: } \phi^- &\coloneqq \frac{1}{2}(\phi + i\hat{J}\phi) \;=\; \overline{\phi^+}\end{aligned}$$

- these subspaces are orthogonal $\langle\phi_1^+,\,\phi_2^-\rangle_{\mathcal{S}} \;=\; 0$ for all $\phi_{1,2} \in \mathcal{S}$, $\phi^+ \;=\; \overline{\phi^-}$ and complementary $\phi \;=\; \phi^+ + \phi^-$
- **one-particle Hilbert space** $\mathcal{H}_0$ of theory is positive frequency subspace of $\mathcal{S}_{\mathbb{C}}$ with inner product induced by the one of $\mathcal{S}$:

$$\langle\phi_1,\,\phi_2\rangle_{\mathcal{H}_0} \;=\; \langle\phi_1,\,\phi_2\rangle_{\mathcal{S}} \;=\; -\frac{i}{\hbar}\,\Omega_{\mathcal{S}}(\phi_1^-,\phi_2^+) \;=\; -\frac{i}{\hbar}\,\Omega_{\mathcal{S}}(\overline{\phi_1^+},\phi_2^+)$$

  (respectively its Cauchy-completion with respect to norm induced by inner product)
- only ingredient needed to construct one-particle Hilbert space $\mathcal{H}_0$ from symplectic vector space of solutions $(\mathcal{S},\Omega_{\mathcal{S}})$ is complex structure $\hat{J}$
- the map $K: \;\; \mathcal{S} \;\to\; \mathcal{H}_0$ defined by $K\phi \;\coloneqq\; \phi^+$ is a linear bijection, its image is dense subspace of $\mathcal{H}_0$
- Hilbert space $\mathcal{H}$ of QFT is **symmetric Fock space**:

$$\mathcal{F}_{\text{sym}}(\mathcal{H}_0) \;\coloneqq\; \bigoplus_{n\,=\,0}^{\infty}\left(\bigotimes_{\text{sym}}^{n}\mathcal{H}_0\right)$$

- symmetrized tensor product $\otimes^n_{\text{sym}}\mathcal{H}_0$ of $\mathcal{H}_0$ is subspace of n-fold tensor product consisting of totally symmetric maps

$$\alpha : \bigoplus_1^n \overline{\mathcal{H}_0} \to \mathbb{C}$$

$$\sum_{j_1,...,j_n=1}^{n} |\alpha(\overline{e}_{j_1},...,\overline{e}_{j_n})|^2 < \infty$$

and $\otimes^0\mathcal{H}_0 \coloneqq \mathbb{C}$

- establish abstract index notation for Hilbert spaces: given some Hilbert space $\mathcal{H}$ we can construct its complex conjugate $\overline{\mathcal{H}}$, its dual space $\mathcal{H}^*$ and its complex conjugate dual $\overline{\mathcal{H}^*}$
- denote elements of these spaces by:

$\mathcal{H}$	$\overline{\mathcal{H}}$	$\mathcal{H}^*$	$\overline{\mathcal{H}^*}$
ϕ^A	$\phi^{A'} = \overline{\phi}^A$	ϕ_A	$\phi_{A'} = \overline{\phi}_A$

- **Riesz' Lemma:** for every element ϕ_A of $\mathcal{H}^*$ exists unique element $\phi^A \in \mathcal{H}$ such that

$$\phi_A(\bullet) = \langle \phi^A, \bullet \rangle_{\mathcal{H}}$$

i.e., every bounded linear map $\mathcal{H} \to \mathbb{C}$ is of form "take inner product with some fixed vector" Riesz' lemma provides antilinear bijection between Hilbert space and its dual

- by Riesz lemma we can identify $\overline{\mathcal{H}}$ with $\mathcal{H}^*$, i.e. $\phi^{A'} = \overline{\phi}^A$ with ϕ_A, and $\overline{\mathcal{H}^*}$ with $\mathcal{H}$, i.e. ϕ^A with $\phi_{A'} = \overline{\phi}_A$, therefore we do not need to use primed indices, can write inner product

$$\langle \psi^A, \phi^A \rangle_{\mathcal{H}} = \overline{\psi}_A \phi^A$$

- denote elements of $\otimes^n\mathcal{H}$ as $\phi^{A_1,...,A_n}$ and elements of $\otimes^n\mathcal{H}^*$ respectively $\overline{\mathcal{H}}$ as $\psi_{A_1,...,A_n}$
- elements of $\otimes^n_{\text{sym}}\mathcal{H}$ satisfy $\phi^{A_1,...,A_n} = \phi^{(A_1,...,A_n)}$
- vector in symmetric Fock $\mathcal{F}_{\text{sym}}(\mathcal{H}_0)$ space in abstract index notation can be written as

$$\Psi = (\psi, \psi^{A_1}, \psi^{A_1A_2}, ..., \psi^{A_1...A_n}, ...)$$

with norm given by

$$\| \Psi \|^2 \;=\; \langle \Psi, \Psi \rangle \;=\; \overline{\psi}\psi + \overline{\psi}_{A_1}\psi^{A_1} + \overline{\psi}_{A_1A_2}\psi^{A_1A_2} + \ldots < \infty$$

- now for element $\xi^A \in \mathcal{H}_0$ and corresponding $\overline{\xi}_A \in \overline{\mathcal{H}_0}$ define creation $\hat{a}^\dagger$ and annihilation $\hat{a}$ operators: $\mathcal{F}_{\mathrm{sym}}(\mathcal{H}_0) \;\to\; \mathcal{F}_{\mathrm{sym}}(\mathcal{H}_0)$

$$\begin{aligned} \hat{a}^\dagger(\xi)\,\Psi \;&\coloneqq\; (0,\, \psi\xi^{A_1},\, \sqrt{2}\,\xi^{(A_1}\psi^{A_2)},\, \sqrt{3}\,\xi^{(A_1}\psi^{A_2A_3)},\, \ldots) \\ \hat{a}(\overline{\xi})\,\Psi \;&\coloneqq\; (\overline{\xi}_{A_1}\psi^{A_1},\, \sqrt{2}\,\overline{\xi}_{A_1}\psi^{A_1A_2},\, \sqrt{3}\,\overline{\xi}_{A_1}\psi^{A_1A_2A_3},\, \ldots) \end{aligned}$$

which satisfy commutation relations $[\hat{a}(\overline{\xi}), \hat{a}^\dagger(\eta)] \;=\; \overline{\xi}_A\eta^A\,\hat{\mathbb{1}}$

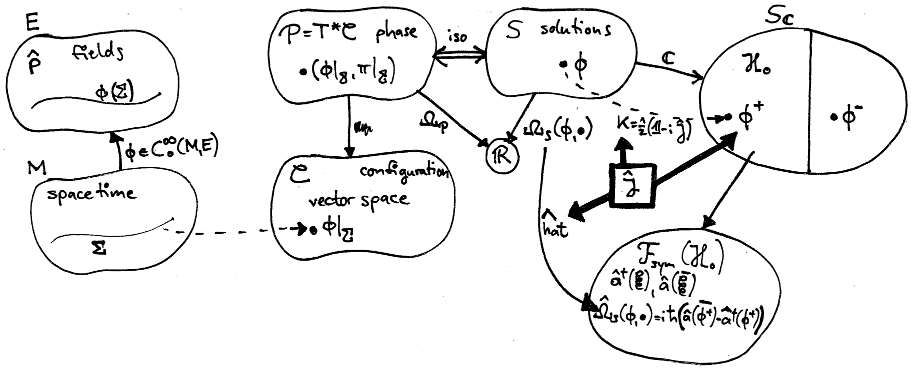
- represent linear classical observables $\Omega_{\mathcal{S}}(\phi, \bullet)$ by operator

$$\hat{\Omega}_{\mathcal{S}}(\phi, \bullet) \;\coloneqq\; i\hbar\Big(\hat{a}(\overline{K\phi}) - \hat{a}^\dagger(K\phi)\Big) \;=\; i\hbar\Big(\hat{a}(\overline{\phi^+}) - \hat{a}^\dagger(\phi^+)\Big)$$

then operators are self-adjoint and indeed satisfy desired commutation relations:

$$[\hat{\Omega}_{\mathcal{S}}(\eta, \bullet), \hat{\Omega}_{\mathcal{S}}(\xi, \bullet)] \;=\; -i\hbar\,\Omega_{\mathcal{S}}(\eta, \xi)\,\hat{\mathbb{1}}$$

- calculating these commutation relations we only need to use general properties of inner product, which is induced by complex structure $\hat{J}$, thus one gets representation of the CCR for any choice of complex structure, freedom to choose $\hat{J}$ represents freedom to chose quantum representation of the CCR



- we are free to choose complex structure $\hat{J}$, there is no naturally preferred $\hat{J}$, since complex structure exactly determines Fock space construction, which in turn defines notion of particles, thus there is no natural notion of particles in general, curved spacetime

- only in stationary spacetimes symmetries or other structure can be exploited to naturally select some preferred complex structure
- if spacetime is nearly stationary, then there exists approximate notion of particles, it becomes ambiguous only for modes with frequency smaller than inverse timescale for change of metric, thus in laboratories on earth ther is no problem employing particle concept
- however in principle notion of particle in general curved spacetime is at best approximate
- **unitary map** $U: \mathcal{H} \to \mathcal{H}'$ is isomorphism between Hilbert spaces preserving the inner product: $\langle U\Psi_1, U\Psi_2\rangle_{\mathcal{H}'} = \langle\Psi_1, \Psi_2\rangle_{\mathcal{H}}$ for all $\Psi_{1,2} \in \mathcal{H}$
- quantum theory consisting of Hilbert space $\mathcal{H}$ together with set of operators $\hat{\mathcal{O}}_\alpha : \mathcal{H} \to \mathcal{H}$ is **unitary equivalent** to quantum theory $(\mathcal{H}', \hat{\mathcal{O}}'_\alpha)$ if there exists a unitary map $U: \mathcal{H} \to \mathcal{H}'$ such that $\hat{\mathcal{O}}_\alpha = U^{-1}\hat{\mathcal{O}}'_\alpha U$
- unitary equivalent quantum theories are physically equivalent in sense that state $\Psi \in \mathcal{H}$ in quantum theory $(\mathcal{H}, \hat{\mathcal{O}}_\alpha)$ has exactly same physical properties as state $U\Psi \in \mathcal{H}'$ in quantum theory $(\mathcal{H}', \hat{\mathcal{O}}'_\alpha)$ (i.e., both generate same matrix elements for operators)
- R is unitary representation of group G as operators on Hilbert space $\mathcal{H}$ if $R(g)$ is unitary operator on $\mathcal{H}$ for all $g \in G$
- working with relation

  $$\left[\hat{\Omega}_{\mathcal{S}}(\phi_1, \bullet),\ \hat{\Omega}_{\mathcal{S}}(\phi_2, \bullet)\right] = -i\hbar\,\Omega_{\mathcal{S}}(\phi_1, \phi_2)\,\hat{\mathbb{1}}$$

  technical difficulties arise because self-adjoint operators can be unbounded and thus not everywhere defined, therefore their composition and commutators need not be well defined
- more convenient to work with exponentiated version, write $W(\phi) = \exp i\Omega_{\mathcal{S}}(\phi, \bullet)$ and look for map turning $W(\phi)$ into operator $\widehat{W(\phi)}$ such that it is unitary, varies continuously with ϕ (in strong operator topology) and as equivalent of commutation relations satisfies **Weyl relations**:

  $$\begin{aligned} \widehat{W(0)} &= \mathbb{1} \\ \widehat{W(-\phi)} &= \widehat{W(\phi)}^\dagger \\ \widehat{W(\phi)}\,\widehat{W(\psi)} &= e^{i\omega(\phi,\psi)/2}\,\widehat{W(\varphi+\psi)} \end{aligned}$$

  since $\widehat{W(\phi)}$ unitary, its action is well defined on whole Hilbert space
- **Stone-von Neumann Theorem:** If $(\mathcal{S}, \Omega)$ is a finite-dimensional symplectic vector space and $(\mathcal{H}, \widehat{W(\phi)})$ and $(\mathcal{H}', \widehat{W'(\phi)})$ are strongly continuous, irreducible, unitary representations of the Weyl relations, then they are unitarily equivalent

- if Stone-von Neumann were valid also for QFT of infinite-dimensional phase space $\mathcal{P}$, then choice of complex structure $\widehat{J}$ would lead to unitarily equivalent theories and thus not affect physical predictions
- however if $\mathcal{P}$ is of infinite dimension, then different choices of $\widehat{J}$ can indeed yield unitarily inequivalent theories, Stone-von Neumann does not hold here
- therefore in order to uniquely define **the** QFT for a general, curved spacetime it is essential to find some preferred unique unitary equivalence class of complex structures $\widehat{J}$
- this seems possible only for spacetimes representing a closed universe, i.e., all Cauchy surfaces are compact, but for general curved spacetimes with noncompact Cauchy surfaces no criterion to single out unique preferred equivalence class of complex structures
- this problem is circumvented by algebraic approach to QFT which does not require specification of preferred unitary equivalence class of $\widehat{J}$'s

2 Algebraic approach to quantization

- in usual approach to quantization, first states are constructed as vectors in some Hilbert space and then observables are defined as operators acting on these states
- algebraic approach reverses roles of states and observables in the sense that first here first observables are constructed as elements of an abstract algebra and then states are defined as objects acting on observables by assigning real number to each observable (in usual approach this corresponds to taking expectation values)
- advantage of this approach is that it allows to treat all states, also states arising in unitarily inequivalent QFTs, on equal footing, thereby it becomes possible to define theory without selecting preferred construction
- key observation in previous section which justifies algebraic approach is that algebra of observables is the same for all Fock constructions of a classical field theory, even for unitary inequivalent Fock constructions: even if $(\mathcal{F}_{\text{sym}}(\mathcal{H}_0^1), \{\widehat{\Omega}_{\mathcal{S}}^1(\phi, \bullet)\})$ and $(\mathcal{F}_{\text{sym}}(\mathcal{H}_0^2), \{\widehat{\Omega}_{\mathcal{S}}^2(\phi, \bullet)\})$ are unitarily inequivalent, the algebraic relations satisfied by observables $\{\widehat{\Omega}_{\mathcal{S}}^1(\phi, \bullet)\}$ are same as those of $\{\widehat{\Omega}_{\mathcal{S}}^2(\phi, \bullet)\}$
- now formulate this mathematically precise, therefor introduce several structures
- **$\mathbb{C}$-algebra** $\mathcal{A}$ = vector space over $\mathbb{C}$ with bilinear, associative vector multiplication: $\mathcal{A} \times \mathcal{A} \to \mathcal{A}, \quad (a, b) \mapsto ab$
- **C^*-algebra** is $\mathbb{C}$-algebra equipped with complete norm $\| \ \| : \mathcal{A} \to \mathbb{R}_0^+$ and antilinear star map $* : \mathcal{A} \to \mathcal{A}, \quad a \mapsto a^*$ fulfilling for all $a, b \in \mathcal{A}$:

$$a^{**} = a \qquad * \text{ is involution} \tag{2.1}$$
$$(ab)^* = b^* a^* \tag{2.2}$$
$$\| ab \| \leqslant \| a \| \| b \| \qquad \text{norm is submultiplicative} \tag{2.3}$$
$$\| a^* \| = \| a \| \qquad * \text{ is isometry} \tag{2.4}$$
$$\| a^* a \| = \| a \|^2 \qquad C^*\text{-property} \tag{2.5}$$

- C^***-subalgebra** is subset of C^*-algebra closed under all its operations: addition, scalar and vector multiplication and star map
- C^*-algebra has at most one unit $\mathbb{1}$, i.e. $a\mathbb{1} = \mathbb{1}a = a$ for all $a \in \mathcal{A}$ it fulfills $\mathbb{1}^* = \mathbb{1}$ and $\| \mathbb{1} \| = 1$
- in C^*-algebras all operations (addition, scalar and vector multiplication, inverse of vector multiplication, star map) are continuous on their domains of definition
- BLop($\mathscr{H}$) is C^*-algebra of **bounded linear operators** on some Hilbert space$\mathscr{H}$, star operation is taking adjoint of operator
- **Weyl system** $(\mathcal{A}, W)$ of symplectic vector space (V, ω) consists of C^*-algebra $\mathcal{A}$ with unit $\mathbb{1}$ and Weyl map $W : \quad V \rightarrow \mathcal{A}$ such that for all $\varphi, \psi \in V$ Weyl relations are fulfilled:

$$W(0) = \mathbb{1} \tag{2.6}$$
$$W(-\varphi) = W(\varphi)^* \tag{2.7}$$
$$W(\varphi)\, W(\psi) = e^{i\omega(\varphi,\psi)/2}\, W(\varphi + \psi) \tag{2.8}$$

  i.e. Weyl map W represents additive group V up to twisting factor $e^{i\omega(\varphi,\psi)/2}$,

  Weyl map W in general injective, but not continuous, $\mathcal{A}$ not separable, $W(\varphi)$ is unitary for all $\varphi \in V$, $\{W(\varphi)\}_{\varphi \in V}$ are linear independent

- $\langle W(V) \rangle \subset \mathcal{A}$ is complex linear span of all elements $\{W(\varphi\}_{\varphi \in V}$ is closed under vector multiplication and star/adjoint, completing it in norm of C^*-algebra $\mathcal{A}$ yields C^*-subalgebra: the **Weyl algebra** $\mathcal{W}_W(\mathcal{A})$ of $\mathcal{A}$ with respect to Weyl map W
- let $(\mathcal{S}, \Omega_{\mathcal{S}})$ be symplectic vector space with complex structure $\hat{J}$ inducing inner product as in previous section via

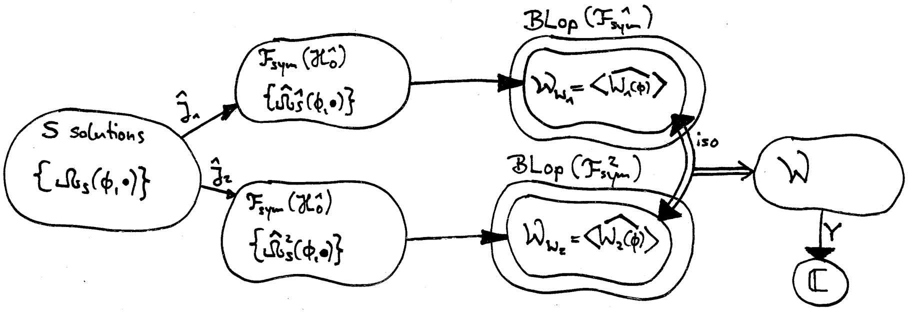
$$\langle \cdot\, ,\, \cdot \rangle_{\mathcal{S}} = \frac{1}{2\hbar} \Omega_{\mathcal{S}}\Big((\hat{J} - i\hat{\mathbb{1}})\cdot, \cdot \Big)$$

  now we can perform Fock construction of previous section and obtain Hilbert space $\mathcal{F}_{\text{sym}}(\mathcal{H}_0)$ and self-adjoint operators $\widehat{\Omega}_{\mathcal{S}}(\phi, \bullet)$

- Weyl map $W:\ \mathcal{S} \to \mathrm{BLop}(\mathcal{F}_{\mathrm{sym}}(\mathcal{H}_0))$ yielding unitary operators given by

$$\widehat{W(\phi)} \;=\; \exp i\widehat{\Omega}_{\mathcal{S}}(\phi, \bullet)$$

- key fact about this construction is that, although symmetric Fock space $\mathcal{F}_{\mathrm{sym}}(\mathcal{H}_0)$ and observables $\widehat{\Omega}_{\mathcal{S}}(\phi, \bullet)$ do depend on choice of complex structure $\widehat{J}$, the Weyl algebra $\mathcal{W}_W\Big(\mathrm{BLop}(\mathcal{F}_{\mathrm{sym}}(\mathcal{H}_0))\Big)$ does not! i.e. even if complex structures $\widehat{J}_1$ and $\widehat{J}_2$ define unitarily inequivalent QFTs, the induced Weyl algebras $\mathcal{W}_1$ and $\mathcal{W}_2$ are isomorphic



- this fact allows us to define fundamental observables for QFT in curved spacetime as elements of the Weyl algebra $\mathcal{W} = \mathcal{W}_W\Big(\mathrm{BLop}(\mathcal{F}_{\mathrm{sym}}(\mathcal{H}_0))\Big)$ constructed from symplectic vector space of solutions $\mathcal{S}$ using arbitrary complex structure $\widehat{J}$

- **algebraic state** of quantum field then defined as linear map $Y:\ \mathcal{W} \to \mathbb{C}$ satisfying

$$\begin{aligned} \text{positivity condition:} \quad & Y(w^* w) \geqslant 0 \qquad \forall w \in \mathcal{W} \\ \text{normalization:} \quad & Y(\mathbb{1}) \;=\; 1 \end{aligned}$$

- algebraic state Y is called **mixed** if it can be written as sum of states $Y_1 \neq Y_2$

$$Y \;=\; c_1 Y_1 + c_2 Y_2 \qquad c_{1,2} > 0$$

  else it is called **pure**

- $\mathcal{W}$ contains only fundamental (linear) observables, however in addition to these there are other physically relevant observables in theory, e.g., energy-momentum tensor T not represented as element of $\mathcal{W}$, thus we should view $\mathcal{W}$ as in some sense minimal collection of observables which is sufficiently large to formulate theory

- in order to get additional observables, later enlarge Weyl algebra and/or restrict abstract notion of state, e.g., restricting to states satifying Hadamard condition appears necessary for definition of energy-momentum tensor
- even though it has simple form, checking positivity condition for a possible state is nontrivial task, not sufficient to check it for basis of $\mathcal{W}$ because positivity condition is nonlinear in w
- if system is in mixed state with each pure state $|\psi_k\rangle$ occurring with probability p_k then mixed state represented by **density operator = density matrix** written

$$\hat{\rho} = \sum_k p_k |\psi_k\rangle\langle\psi_k|$$
$$\mathrm{tr}\,\hat{\rho} = \sum_k p_k = 1$$

expectation value: $\langle\hat{O}\rangle_\psi = \langle\psi|\hat{O}|\psi\rangle = \sum_k p_k \langle\psi_k|\hat{O}|\psi_k\rangle = \mathrm{tr}\,(\hat{\rho}\,\hat{O})$

- consider relations between algebraic notion of states and notion of states in Fock construction
- given any Hilbert space $\mathcal{F}_{\mathrm{sym}}(\mathcal{H}_0)$ carrying a representation $R: \mathcal{W} \to \mathrm{BLop}(\mathcal{F}_{\mathrm{sym}}(\mathcal{H}_0))$ of the considered Weyl algebra $\mathcal{W}$, for any (mixed or pure) state in $\mathcal{F}_{\mathrm{sym}}(\mathcal{H}_0)$ there is unique density matrix $\hat{\rho}: \mathcal{F}_{\mathrm{sym}}(\mathcal{H}_0) \to \mathcal{F}_{\mathrm{sym}}(\mathcal{H}_0)$
- obtain algebraic state $Y_{\hat{\rho}}: \mathcal{W} \to \mathbb{C}$ by

$$Y_{\hat{\rho}}(w) \coloneqq \mathrm{tr}\,\big(\hat{\rho}\,R(w)\big)$$

thus for each state in each possible Fock construction there is a corresponding algebraic state

- converse of this result also holds:
- **GNS construction:** (Gelfand-Naimark-Segal) let $\mathcal{W}$ be C^*-algebra with unit and a state $Y: \mathcal{W} \to \mathbb{C}$, then there exist a Hilbert space $\mathcal{H}$, a representation $R: \mathcal{W} \to \mathrm{BLop}(\mathcal{H})$ and a vector $|\psi_Y\rangle \in \mathcal{H}$ such that

$$Y(w) = \langle\psi_Y|R(w)|\psi_Y\rangle \qquad \forall w \in \mathcal{W}$$

Hilbert space, representation and vector are unique up to unitary equivalence, additional property: $|\psi_Y\rangle$ is **cyclic**, i.e., vectors $\{R(w)|\psi_Y\rangle\}_{w \in \mathcal{W}}$ are dense subspace of $\mathcal{H}$

- sketch of GNS construction: first use state Y to define non-negative, bilinear map

$$\begin{aligned} \langle \cdot\,,\ \cdot \rangle_{\mathcal{W}} :\ \mathcal{W} \times \mathcal{W} &\to \mathbb{C} \\ \langle v,\ w \rangle_{\mathcal{W}} &\coloneqq Y(v^* w) \end{aligned}$$

  (after factoring out kernel of Y) this defines positive definite inner product on (quotient space of) Weyl algebra $\mathcal{W}$

- complete (quotient space of) $\mathcal{W}$ in norm induced by inner product, thereby get GNS Hilbert space $\mathcal{H} = \overline{\mathcal{W}}$
- letting (quotient space of) $\mathcal{W}$ act upon itself by vector multiplication and extending this action continuously to $\mathcal{H}$ we get representation $R:\ \mathcal{W} \to \mathrm{BLop}(\mathcal{H})$ by $R(w) = w$ for all $w \in \overline{\mathcal{W}}$
- cyclic vector $|\psi_Y\rangle \in \mathcal{H}$ is unit $\mathbb{1}$ of $\mathcal{W}$
- GNS construction expresses both pure and mixed algebraic states as pure states in GNS Hilbert space, however the GNS representation of $\mathcal{W}$ is irreducible if and only if algebraic state is pure
- in usual Hilbert space approach observable represented by self-adjoint operator $\widehat{A}:\ \mathcal{H} \to \mathcal{H}$, with real eigenvalues α_k and eigenvectors $|a_k\rangle$, by spectral theorem it has associated family of projection operators $\widehat{P}_k:\ \mathcal{H} \to V_k$ projecting onto the eigenspace $V_k \subset \mathcal{H}$ spanned by eigenvectors of eigenvalue α_k
- if system is in normalized state $|\psi\rangle$, then probability that measurement of $\widehat{A}$ yields value in interval $I \subset \mathbb{R}$ is given by $\| \widehat{P}_I |\psi\rangle \|^2$ where $\widehat{P}_I$ is projection operator of $\widehat{A}$ for interval I:

$$\widehat{P}_I = \sum_k^{\alpha_k \in I} \widehat{P}_k$$

- more general: in Heisenberg representation, let state be represented by its density matrix $\hat{\rho}$, normalized: $\mathrm{tr}\,\hat{\rho} = 1$, then probability that measurement of self-adjoint observables $\widehat{A}_1, ..., \widehat{A}_n$ made at times $t_1 < ... < t_n$ will yield results lying in intervals $I_1, ..., I_n$ is given by

$$\mathrm{tr}\,(\widehat{P}_n ... \widehat{P}_1\ \hat{\rho}\ \widehat{P}_1 ... \widehat{P}_n)$$

  with $\widehat{P}_k$ denoting here the projection operator of $\widehat{A}_k(t_k)$ on interval I_k, this equation contains all available information in standard quantum mechanical measurement theory, in order to have complete formulation of quantum theory we must provide some analog of this equation

- in algebraic approach for arbitrary normalized state Y probability that measurement of self-adjoint observables $\widehat{A}_1, ..., \widehat{A}_n$ made at times $t_1 < ... < t_n$ will yield results lying in intervals $I_1, ..., I_n$ can be defined by

$$\lim_{j_1,...,j_n \to \infty} Y\Big((Q_1)_{j_1(\hat{A}_1)}...(Q_n)_{j_n(\hat{A}_n)}(Q_n)_{j_n(\hat{A}_n)}...(Q_1)_{j_1(\hat{A}_1)}\Big)$$

wherein $\{(Q_k)_{j_k(\hat{A}_k)}\}_{j_k \in \mathbb{N}}$ is any sequence of polynomials in $\widehat{A}_k$ such that polynomials $\{(Q_k)_{j_k(x)}\}$ are uniformly bounded on spectrum of $\widehat{A}_k$ and converge on spectrum of $\widehat{A}_k$ to characteristic function 1_{I_k} of interval I_k

$$1_{I(x)} = \begin{cases} 1: & x \in I \\ 0: & x \notin I \end{cases}$$

- evaluating this definition of probability in GNS representation shows that the limit exists and equals what would be obtained from usual QM formula in GNS representation, or in any other representation of Weyl algebra $\mathcal{W}$ in which algebraic state Y can be realized as density matrix
- thus algebraic definition of probability is equivalent to putting observables into any representation and use standard Hilbert space rule
- however the algebraic definition of measurement probability ensures independence from representation
- thus probabilities for outcomes of any sequence of measurements of observables in $\mathcal{W}$ well defined in algebraic approach

3 References

- R. Wald: QFT in curved spacetime and black hole thermodynamics, university of Chicago Press, 1994
- A. Corichi, J. Cortez, H. Quevedo: Schrödinger and Fock representation for a field theory on curved spacetime, Annals of Physics 313 (2004) 446-478, [arXiv:hep-th/0202070]
- C. Bär, N. Ginoux, F. Pfäffle: Wave equations on Lorentzian manifolds and Quantization, EMS Publishing House, 2007

  downloadable online [http://users.math.uni-potsdam.de/ baer/homepage.html]