Asymptotic Safety for Gravity

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1 ERGE

2 Fixed Point

ERGE

We want to solve the ERGE equation

$$\partial_t \Gamma_k = \frac{1}{2} Tr \left[\left(\Gamma_k^2 + \mathcal{R}_k \right)^{-1} \partial_t \mathcal{R}_k \right] \tag{1}$$

for gravity. Where $t = \log(k/k_0)$ for some fixed k_0.

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Γ_k admits an expansion of the form

$$\Gamma_k[g_{\mu\nu}, g_i^{(2n)}] = \sum_{n=0}^{\infty} \sum_i g_i^{(2n)} \mathcal{P}_i^{(2n)}(g_{\mu\nu}) \tag{2}$$

where $g_i^{(2n)}$ are the coupling constants and $\mathcal{P}_i^{(2n)}(g_{\mu\nu})$ are all posible operators constructed from $g_{\mu\nu}$ and its n-derivates, which are compatible with the symmetry of the theory.

With the equations (1) and (2) we have

$$\frac{1}{2} Tr\left[\left(\Gamma_k^2 + \mathcal{R}_k\right)^{-1} \partial_t \mathcal{R}_k\right] = \sum_{n=0}^{\infty} \sum_i \beta_i^{(2n)} \mathcal{P}_i^{(2n)}(g_{\mu\nu}) \tag{3}$$

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To solve this equation we need an specific form of the cutoff $\mathcal{R}_k$ and Γ_k. For the scalar theory with $\mathbb{Z}_2$ symmetry we have

$$\Gamma_k^{(2)}(x,y) = (-\partial_x^2 + 2V_k' + 4\phi V_k'')\delta(x-y) \tag{4}$$

with $V_k = V_k(\phi^2)$ and primes denote ϕ^2 derivatives.

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Observe that we have an expresion of the form

$$\Gamma_k^{(2)}=-\partial^2+\mathbf{E} \tag{5}$$

Gravity

In general the inverse of $\Gamma^{(2)}$ is an operator of the form

$$\Delta = -\nabla^2 + \mathbf{E} \tag{6}$$

where $\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2$ is an operator acting on the fields. $\mathbf{E}_1$ doesn't contain couplings. $\mathbf{E}_2$ contain couplings.

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We consider that $\mathbf{E}_2 = 0$. This is called a cutoff of type II. Therefore the inverse propagator is

$$P_k(\Delta) = \Delta + R_k(\Delta) \tag{7}$$

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with $g^{(0)} = 2Z\Lambda$, $g^{(2)} = -Z$, $g_1^{(4)} = 1/(2\lambda)$, and $g_2^{(4)} = 1/\xi$.

$$\begin{array}{cc} \mathcal{M}^{(0)} = 1 & \mathcal{M}^{(2)} = R \\ \mathcal{M}_1^{(4)} = C^2 & \mathcal{M}_2^{(4)} = R^2 \end{array}$$

R is the scalar curvature and C^2 is the square of the Weyl tensor.

Therefore we get

$$\begin{aligned} \Gamma_k^{(n\leq 2)} &= \int d^4x\sqrt{g}\left[2Z\Lambda - ZR + \frac{1}{2\lambda}C^2 + \frac{1}{\xi}R^2\right] \\ &:= (\Gamma_k)_{grav} \end{aligned} \tag{8}$$

for pure gravity.

Now, consider n_s scalar fields, n_D Dirac fields and n_M Maxwell fields minimally coupled to gravity, the effective action is

$$\begin{aligned} \Gamma_k &= \int d^4x\sqrt{g}\left[\frac{1}{2}\nabla_\mu\phi\nabla^\mu\phi + \bar{\psi}\gamma^\mu\nabla_\mu\psi + \frac{1}{4}F^{\mu\nu}F_{\mu\nu}\right] \\ &:= (\Gamma_k)_{matt} \end{aligned} \tag{9}$$

where ∇ is the covariant derivative.

Hence, the effective action is given by

$$\Gamma_k(g_{\mu\nu}, \phi, \psi, A_\mu) = (\Gamma_k)_{grav} + (\Gamma_k)_{Matt} \tag{10}$$

Pag. [15]

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where

$$(\Gamma_k^{(2)})_{Matt}(x,y) = n_s \left(\frac{\delta^2 (\Gamma_k)_{Matt}}{\delta\phi(x)\delta\phi(y)} \right) + n_D \left(\frac{\delta^2 (\Gamma_k)_{Matt}}{\delta\bar{\psi}(x)\delta\psi(y)} \right) + n_M \left(\frac{\delta^2 (\Gamma_k)_{Matt}}{\delta A_\mu(x)\delta A_\nu(y)} \right)$$

We obtain the ERGE:

ERGE

$$\partial_t \Gamma_k = \frac{n_s}{2} Tr_{(S)}\left(\frac{\partial_t R_k(\Delta^{(s)})}{P_k(\Delta^{(s)})}\right) - \frac{n_D}{2} Tr_{(D)}\left(\frac{\partial_t R_k(\Delta^{(D)})}{P_k(\Delta^{(D)})}\right) + \frac{n_M}{2} Tr_{(M)}\left(\frac{\partial_t R_k(\Delta^{(M)})}{P_k(\Delta^{(M)})}\right) - \frac{n_M}{2} Tr_{(gh)}\left(\frac{\partial_t R_k(\Delta^{(gh)})}{P_k(\Delta^{(gh)})}\right) \quad (11)$$

with a type II cutoff $P_k(\Delta^{(A)}) = \Delta^{(A)} + R_k(\Delta^{(A)})$ we obtain for each case

scalar

$$\Delta^{(s)} = -\nabla^2 \tag{12}$$

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Ghost

$$\Delta^{(gh)} = -\nabla^2 \tag{15}$$

Hence

Trace

$$
\begin{aligned}
\partial_t \Gamma_k \;&=\; \frac{1}{2}\frac{1}{(4\pi)^2}\int d^4x\sqrt{g}\left[(n_s - 4n_D + 2n_M)Q_2\left(\frac{\partial_t R_k}{P_k}\right)\right. \\
&+\frac{1}{6}R(n_s - 2n_D - 4n_M)Q_1\left(\frac{\partial_t R_k}{P_k}\right) \\
&+\frac{1}{180}\left((3n_s + 18n_D + 36n_M)C^2 - (n_s + 11n_D + 62n_M)E\right. \\
&\left.\left.+5n_sR^2 + 12(n_s + n_D - 3n_M)\nabla^2 R\right) + \cdots\right] \qquad (16)
\end{aligned}
$$

C^2 is the square of the Weyl's tensor and
$E = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2$. *Pag. [15]*

Hence

Trace

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Remark

The coefficients of the 4-derivative terms are scheme independent.

The function Q is defined as

$$Q_n(W) = \frac{1}{\Gamma(n)} \int_0^\infty dz z^{n-1} W(z)$$

Fixed Point

In order to look for a fixed point we have to obtain the equations for the beta functions. This is done with (10) and (16). First we note that

$$
\begin{aligned}
\partial_t(\Gamma_k)_{grav} &= \int d^4x\sqrt{g}\left[\partial_t(2Z\Lambda) + \partial_t(-Z)R + \partial_t\left(\frac{1}{2\lambda}\right)C^2\right. \\
&\quad \left. + \partial_t\left(\frac{1}{\xi}\right)R^2\right] \\
&= \int d^4x\sqrt{g}\left[\partial_t(g^{(0)}) + \partial_t(g^{(2)})R + \partial_t(g_1^{(4)})C^2\right. \\
&\quad \left. + \partial_t(g_2^{(4)})R^2\right]
\end{aligned}
$$

Then

Equations

$$\partial_t g^{(0)} = \frac{1}{2(4\pi)^2}(n_s - 4n_D + 2n_M)Q_2\left(\frac{\partial_t R_k}{P_k}\right) \quad (17)$$

$$\partial_t g^{(2)} = \frac{1}{12(4\pi)^2}(n_s - 2n_D - 4n_M)Q_1\left(\frac{\partial_t R_k}{P_k}\right) \quad (18)$$

with the optimized cutoff

$$R_k(z) = (k^2 - z)\theta(k^2 - z)$$

The integrals Q are

$$Q_n\left(\frac{\partial_t R_k}{(P_k+q)^l}\right) = \frac{2}{n!}\frac{1}{(1+\tilde{q})^l}k^{2(n-l+1)} \tag{19}$$

with $\tilde{q} = k^{-2}q$.

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with $\tilde{q} = k^{-2}q$. Hence

$$Q_2\left(\frac{\partial_t R_k}{P_k}\right) = k^4 \tag{20}$$

$$Q_1\left(\frac{\partial_t R_k}{P_k}\right) = 2k^2 \tag{21}$$

Then

$$\partial_t g^{(0)} = a^{(0)} k^4 \tag{22}$$

$$\partial_t g^{(2)} = a^{(0)} k^2 \tag{23}$$

with

$$a^{(0)} = \frac{n_s - 4n_D + 2n_M}{2(4\pi)^2}$$

$$a^{(2)} = \frac{n_s - 2n_D - 4n_M}{6(4\pi)^2}$$

In general $\partial_t g^{(n)} = a^{(n)} k^{4-n}$.

With $\tilde{g}^{(n)} = k^{n-4} g^{(n)}$

$$\partial_t \tilde{g}^{(n)} = (n-4)\tilde{g}^{(n)} + k^{n-4}\partial_t g^{(n)} = (n-4)\tilde{g}^{(n)} + a^{(n)}$$

and

$$Z = \frac{1}{16\pi G}$$

we have

$$\begin{aligned}
\partial_t \tilde{g}^{(0)} &= \partial_t(2k^{-4} Z\Lambda) = \partial_t\left(\frac{\tilde{\Lambda}}{8\pi \tilde{G}}\right) \\
&= \frac{\partial_t \tilde{\Lambda}}{8\pi \tilde{G}} - \frac{\tilde{\Lambda}}{8\pi \tilde{G}^2}\partial_t \tilde{G} \\
&= -\frac{\tilde{\Lambda}}{2\pi \tilde{G}} + a^{(0)}
\end{aligned}$$

With this we get

$$(\partial_t\tilde{\Lambda})\tilde{G}-(\partial_t\tilde{G})\tilde{\Lambda}=-4\tilde{G}\tilde{\Lambda}+8\pi\tilde{G}^2a^{(0)}$$

in the same way we obtain with $g^{(2)}$

$$\partial_t\tilde{G}=2\tilde{G}+16\pi\tilde{G}^2a^{(2)}$$

with these equations finally

$$\partial_t\tilde{\Lambda} = -2\tilde{\Lambda}+16\pi\tilde{G}\tilde{\Lambda}+8\pi\tilde{G}a^{(0)} \quad (24)$$
$$\partial_t\tilde{G} = 2\tilde{G}+16\pi\tilde{G}^2a^{(2)} \quad (25)$$

From (24) if

$$\partial_t \tilde{G} = 0 = 2\tilde{G}(1 + 8\pi \tilde{G} a^{(2)})$$

hence

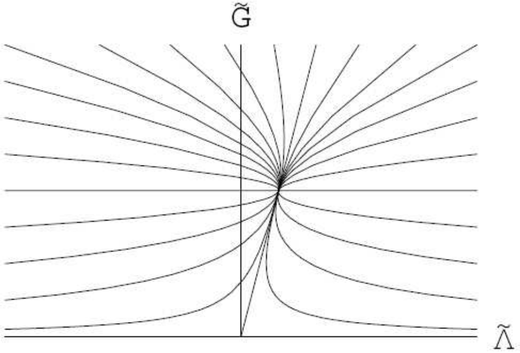
$$\begin{aligned} \tilde{G}_* &= \frac{1}{8\pi a^{(2)}} = \frac{1}{8\pi} \frac{6(4\pi)^2}{(n_s - 2n_D - 4n_M)} \\ \tilde{G}_* &= \frac{12\pi}{n_s - 2n_D - 4n_M} \end{aligned} \tag{26}$$

and

$$\tilde{\lambda}_* = -\frac{3}{4} \frac{n_s - 4n_D + 2n_M}{n_s - 2n_D - 4n_M} \tag{27}$$

There is a non trivial Fixed Point.

The flow is given by



For pure gravity, we consider the Einstein-Hilbert truncation

$$\Gamma_k^{(n\leq 2)} = \int d^4x\sqrt{g}\,[2Z\Lambda - ZR] + S_{GF} + S_{gh} \tag{28}$$

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the equations for the beta functions are

$$\partial_t\tilde{\Lambda} = \frac{-2(1-2\tilde{\Lambda})^2\tilde{\Lambda} + \frac{36-41\tilde{\Lambda}+42\tilde{\Lambda}^2-600\tilde{\Lambda}^3}{72\pi}\tilde{G} + \frac{467-572\tilde{\Lambda}}{288\pi^2}\tilde{G}^2}{(1-2\tilde{\Lambda})^2 - \frac{29-9\tilde{\Lambda}}{72\pi}\tilde{G}} \tag{29}$$

$$\partial_t\tilde{G} = \frac{2(1-2\tilde{\Lambda})^2\tilde{G} - \frac{373-654\tilde{\Lambda}+600\tilde{\Lambda}^2}{72\pi}\tilde{G}^2}{(1-2\tilde{\Lambda})^2 - \frac{29-9\tilde{\Lambda}}{72\pi}\tilde{G}} \tag{30}$$

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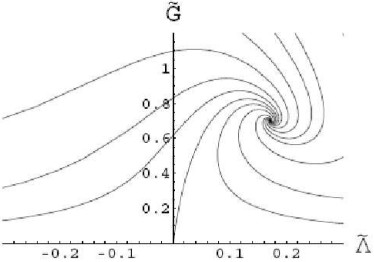

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