CFT EXERCISES

1. Solve the one-dimensional Ising model and show that there is no phase transition.

2. Let $X^\mu(\sigma,\tau)$ denote the string coordinate. It describes the map from the string world sheet, with coordinates (σ,τ), to spacetime, with coordinates X^μ. Derive the Nambu-Goto action

$$S_{\rm NG} \;=\; -T\,(\text{world sheet area}) \;=\; -T\int d\tau d\sigma\,\sqrt{-(\partial_\tau X)^2(\partial_\sigma X)^2+(\partial_\tau X\cdot\partial_\sigma X)^2}\;.$$

Here T is the string tension.

3. Verify that the infinitesimal conformal transformations of $\mathbb{R}^d$ generate the Lie algebra $so(1,d+1)$.

4. Integrate the infinitesimal special conformal transformation

$$\delta x^\mu = b^\mu x^2 - 2x^\mu b\cdot x$$

to the finite transformation

$$x^\mu \to x'^{\,\mu} \;=\; \frac{x^\mu + b^\mu x^2}{1+2b\cdot x+b^2x^2}\quad .$$

5. Show that the decompositions

$$T(z) \;=\; \sum_{n\in\mathbb{Z}} z^{-2-n}L_n\ ,\quad \phi(z) \;=\; \sum_{n\in\mathbb{Z}} z^{-h-n}\phi_n\ ;$$

in the operator product expansion

$$T(z)\,\phi(z) \;\sim\; \frac{h}{(z-w)^2}\,\phi(w) \;+\; \frac{1}{z-w}\,\partial\phi(w)$$

yield the commutation relations

$$[L_m,\phi_n] \;=\; (m(h-1)-n)\,\phi_{m+n}\ .$$

6. Show that the Hermiticity of $T(z)$ implies that its "modes" obey

$$\big(L_n\big)^\dagger \;=\; L_{-n}\ .$$

What is the corresponding relation for a Hermitian primary field?

7. *Currents and affine Kac-Moody algebras*
Demonstrate that the commutation relations

$$[J^a_m,J^b_n] \;=\; \sum_c if^{ab}{}_c\,J^c_{m+n} \;+\; k\,m\,\delta^{ab}\,\delta_{m+n,0}$$

result from the following operator product of currents:

$$J^a(z)\,J^b(w) \;\sim\; \frac{k\,\delta^{ab}}{(z-w)^2} \;+\; \sum_c i f^{ab}{}_c\,\frac{J^c(w)}{z-w}\;.$$

Here, putting $m = n = 0$ in the first equation yields the horizontal subalgebra, isomorphic to a complex, simple Lie algebra (denote it by g) with structure constants $f^{ab}{}_c$. The commutation relations for all m, n realize an affine Kac-Moody algebra at fixed level k (denote it by g_k).

8. *Sugawara construction*
Why is Exercise 7 relevant to CFT? Show that

$$T(z) \;=\; \frac{1}{k+h^\vee}\sum_a \Big(J^a\,J^a\Big)(z)\;\;,$$

obeys the Virasoro operator product with a central charge

$$c \;=\; c(g_k) \;=\; \frac{k\;\mathrm{dim}(g)}{k+h^\vee}\;.$$

The brackets of $(J^a J^a)$ denote normal-ordering, and we use a basis for g in which the Killing form is orthonormal. $h^\vee$ indicates the dual Coxeter number of g, appearing in

$$\sum_{b,c} f_{abc}\,f_{dbc} \;=\; 2h^\vee\,\delta_{ad}\;.$$