Classical Scattering

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Mathematical Physics Seminar

Contents

Scattering Process

- One of the most important ways of obtaining information about the structure of small bodies (for example atomic nuclei) is to bombard them with particles and measure the number of particles scattered in various directions. The energy and angular distribution of scattered particles will depend on the shape of the target and on the nature of the forces between the particles and the target.
- Scattering of electromagnetic or acoustic waves is of widespread interest, because of the enormous number of technological applications.
- In radar the radio wave is transmitted toward an object and the scattered wave received by an antenna reveals the characteristics of the object, such as its position and motion.
- In biomedical applications, microwaves, optical waves, or acoustic waves are propagated through biological media and the scattering from various portions of a body is used to identify the objects for diagnostic purposes.

Classical particle scattering: Scattering cross sections

Consider the case where the target is a fixed, hard (that is, perfectly elastic) sphere of radius R, and a uniform, parallel beam of particles incident on it.

The force on the particles is an impulse central conservative force. Consequently, the kinetic energy and the angular momentum are conserved.

Let f be the particle flux in the beam = number of particles crossing unit area normal to the beam direction per unit time.

The number of particles which strike the target in unit time is

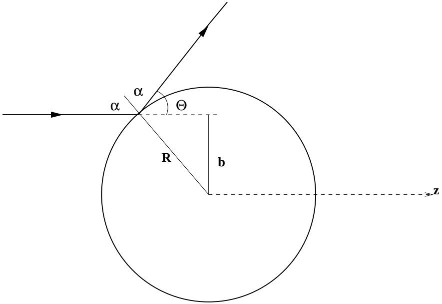
$$w = f\sigma, \tag{1}$$

where σ is the cross-sectional area presented by the target,

$$\sigma = \pi R^2. \tag{2}$$

Consider one of these particles incident with velocity v and impact parameter b.

- R: radius of the sphere
- α: incident angle
- b: impact parameter, $b = R \sin \alpha$



By the axial symmetry of the problem the particle must move in a plane $\varphi =$constant. Then the particle will bounce off the sphere at an angle to the normal equal to the incident angle α.

The particle is deflected through an angle $\theta = \pi - 2\alpha$, related to the impact parameter by

$$b = R \cos \frac{\theta}{2}. \tag{3}$$

We can calculate the number of particles scattered in a direction specified by the polar angles θ, φ, within angular ranges $\mathrm{d}\theta, \mathrm{d}\varphi$. The particles scattered through angles between θ and $\theta + \mathrm{d}\theta$ are those that came in with impact parameters between b and $b + \mathrm{d}b$,

$$\mathrm{d}b = -\frac{1}{2} R \sin\frac{\theta}{2}\, \mathrm{d}\theta. \tag{4}$$

Consider now a cross-section of the incoming beam. The particles we are interested in are those which cross a small element of area

$$\mathrm{d}\sigma = b\, |\mathrm{d}b|\, \mathrm{d}\varphi. \tag{5}$$

Inserting the values of b and $\mathrm{d}b$,

$$\mathrm{d}\sigma = \frac{1}{4} R^2 \sin\theta\, \mathrm{d}\theta\, \mathrm{d}\varphi. \tag{6}$$

The rate at which the particles cross this area, and therefore the rate at which they emerge in the given angular range, is

$$\mathrm{d}w = f \mathrm{d}\sigma. \tag{7}$$

To measure this rate we place a detector at a large distance from the target in the specified direction. We want to express our result in terms of the cross-sectional area $\mathrm{d}A$ of the detector, and its distance L form the target (we assume $L \gg R$). The element of area on a sphere of radius L is

$$\mathrm{d}A = L^2 \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\varphi. \tag{8}$$

The solid angle subtended at the origin by the area $\mathrm{d}A$ is

$$\mathrm{d}\Omega = \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\varphi, \tag{9}$$

so that

$$\mathrm{d}A = L^2 \mathrm{d}\Omega. \tag{10}$$

(The total solid angle subtended by an entire sphere is 4π.) The important quantity is not the cross-sectional area $\mathrm{d}\sigma$ itself, but rather the ratio $\mathrm{d}\sigma/\mathrm{d}\Omega$, which is called the *differential cross-section*. The rate $\mathrm{d}w$ at which particles enter the detector is

$$\mathrm{d}w = f \, \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} \, \frac{\mathrm{d}A}{L^2}. \tag{11}$$

It is useful to note an alternative definition of the differential cross-section, which is applicable even if we cannot follow the trajectory of each individual particle. We *define* $\mathrm{d}\sigma/\mathrm{d}\Omega$ to be the ratio of scattered particles per unit solid angle to the number of incoming particles per unit area,

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\text{number of scattered particles per unit solid angle}}{\text{number of incoming particles per unit area}}.$$

Then the rate at which particles are detected is obtained by multiplying the differential cross-section by the flux of incoming particles, and by the solid angle subtended at the target by the detector, as in (11).
In the case of scattering from a hard sphere, the differential cross section is

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{4}R^2. \tag{12}$$

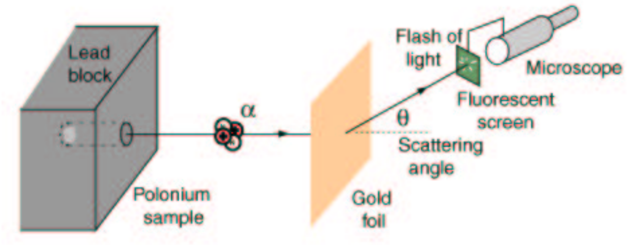
It has the special feature of being *isotropic*, or independent of the scattering angle. Thus the rate at which particles enter the detector is, in this case, independent of the direction in which it is placed.

Note that the total cross-section is correctly given by integrating (12) over all solid angles; in this case, we multiply by the total solid angle 4π.

Classical particle scattering: Rutherford scattering

Crucial importance in obtaining an understanding of the structure of the atom.

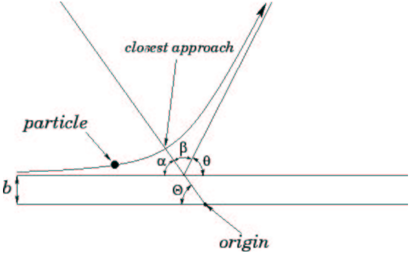
In a classic experiment, performed in 1911, Rutherford bombarded atoms with α-particles (helium nuclei). Because these particles are much heavier than electrons, they are deflected only very slightly by the electrons in the atom, and can therefore be used to study the heavy atomic nucleus. From observations of the angular distribution of the scattered α-particles, Rutherford was able to show that the law of force between α-particle and nucleus is the inverse square law down to very small distances. Thus he concluded that the positive charge is concentrated in a very small nuclear volume rather than being spread out over the whole volume of the atom.



Calculation of the differential cross-section for the scattering of a particle of charge q and mass m by a fixed point charge q'.

We look for the relation between the impact parameter b and the scattering angle $\theta = \pi - 2\Theta$ (see Figure 3). Change in angle for the particle of mass m and charge q,

$$\Theta(r) = \int_{r_{min}}^{r_{max}} \frac{J/r^2}{\sqrt{2m\left[E - U(r) - \frac{J^2}{2mr^2}\right]}} \, \mathrm{d}r,$$



where

- $r_{max} = \infty$ max distance between charges,
- r_{min} min distance between charges, $r_{min} = a + \sqrt{a^2 + b^2}$ where $a = \frac{q\,q'}{4\pi\epsilon_0\, m\, v^2}$, it can be derived from the radial energy equation: $\frac{1}{2}m(\dot{r}^2 + r^2\dot{\Theta}^2) - U(r) = E$
- the angular momentum $J = m\, r^2\, \dot{\Theta} = mvb = b\sqrt{2mT_0}$, (conserved quantity)
- T_0 initial kinetic energy,
- $E = T_0$ total energy, (conserved quantity)
- $U(r) = \frac{q\,q'}{4\pi\epsilon_0 r}$ Coulomb potential. (at infinity $U = 0 \Rightarrow E = T - U = T_0$).

Integrating in $\mathrm{d}r$, we find

$$b = a \cot \frac{\theta}{2}.$$

Thus

$$\mathrm{d}b = -\frac{a\,\mathrm{d}\theta}{2\sin^2\frac{\theta}{2}},$$

so that, substituting in (5), we obtain

$$\mathrm{d}\sigma = \frac{a^2 \cos\frac{\theta}{2}\,\mathrm{d}\theta\,\mathrm{d}\varphi}{2\sin^3\frac{\theta}{2}}$$

Dividing by the solid angle (9), we find for the differential cross-section

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{a^2}{4\sin^4\frac{\theta}{2}}. \tag{13}$$

This is the *Rutherford scattering cross-section*.

Comments:

- This cross-section is strongly dependent both on the velocity of the incoming particle and on the scattering angle.
- It is independent of the signs of the charges, so that the form of the scattering distribution is the same for an attractive force as for a repulsive one.
- It also increases with increasing charge.
- To investigate the structure of the atom at small distances, we must use high-velocity particles, for which a is small, and examine large-angle scattering, corresponding to particles with small impact parameter b. The cross-section (13) is large for small values of the scattering angle, but physically it is the large angle scattering which is of most interest. For, the fact that particles can be scattered through large angles is an indication that there are very strong forces acting at very short distances. If the positive nuclear charge were spread out over a large volume, the force would be inverse-square-law only down to the radius of the charge distribution. Beyond that point, it would decrease as we go to even smaller distances. Consequently, the particles that penetrate to within this distance would be scattered through smaller angles.

- What about the total cross-section? A peculiar feature of the differential cross-section (13) is that the corresponding *total cross-section* is infinite. This is a consequence of the infinite range of the Coulomb force. However far away from the nucleus a particle may be, it still experiences some force, and is scattered through a non zero (though small) angle. Thus the total number of particles scattered through any angle, however small, is indeed infinite.
- We can calculate the number of particles scattered through any angle greater than some small lower limit θ_0. These are particles which had impact parameters b less than $b_0 = a \cot \frac{\theta_0}{2}$. The corresponding cross-section is

$$\sigma(\theta > \theta_0) = \pi b_0^2 = \pi a^2 \cot^2 \frac{\theta_0}{2}.$$

- A remarkable fact is that the quantum-mechanical treatment of Coulomb scattering leads to exactly the same result as does the classical derivation. (Why?)

Scattering of classical (electromagnetic or acoustic) waves

The scattering and diffraction of waves is universally and ubiquitously observable. It involves the transfer of energy and information without bulk motion; light, sound and elastic waves are important to us in natural and technological senses.

At a fundamental level Maxwell's equations or the scalar wave equation completely describe all electromagnetic or acoustic phenomena, when supplemented with appropriate boundary and radiation conditions. A variety of analytical and numerical techniques have been devised to solve these equations.

Acoustic waves

- Acoustics may be described as the theory of the propagation of small disturbances in fluids, liquid or gaseous. The propagation arises from rarefaction and compression of the fluid that causes a change in the density.
- Wave equation:

$$\nabla^2 \rho = \frac{1}{v^2} \frac{\partial^2 \rho}{\partial t^2},$$

  where ρ is the density perturbation and v the sound speed.
- When the density perturbation ρ varies harmonically in time,

$$\rho(t, x) = \rho(x) e^{-\mathrm{i}\omega t},$$

  then the wave equation reduces to the Helmholtz equation

$$\nabla^2 \rho(x) + k^2 \rho(x) = 0, \qquad k = \frac{\omega}{v}.$$

Electromagnetic waves

- The electromagnetic field is described by the Maxwell's equations

$$\begin{aligned} \mathrm{div}\vec{D} &= \rho, & \mathrm{curl}\vec{H} &= \frac{\partial \vec{D}}{\partial t} + \vec{J}, \\ \mathrm{div}\vec{B} &= 0, & \mathrm{curl}\vec{E} &= -\frac{\partial \vec{B}}{\partial t}. \end{aligned}$$

  $\vec{J}$: current density, ρ: charge density
  $\vec{E}, \vec{H}$: electric and magnetic field
  $\vec{B}, \vec{D}$: magnetic and electric flux density
- The macroscopic electromagnetic equations must be supplemented by the constitutive equations connecting field intensities with flux densities. In isotropic bodies the constitutive relations are

$$\vec{D} = \epsilon\vec{E}, \qquad \vec{B} = \mu\vec{H},$$

  ϵ is the medium permittivity, μ is the medium permeability

- For electromagnetic fields varying with time harmonic dependence $e^{-\mathrm{i}\omega t}$, the Maxwell's equations reduce to the wave equations

$$
\begin{aligned}
\nabla^2 \vec{E} + k^2 \vec{E} &= -\mathrm{i}\omega\mu \vec{J} + \frac{1}{\mathrm{i}\omega\epsilon} \text{grad div} \vec{J}, \\
\nabla^2 \vec{H} + k^2 \vec{H} &= -\text{curl} \vec{J},
\end{aligned}
$$

  where $k^2 = \omega^2 \epsilon\mu$.

- Alternative representation in terms of scalar and vector potentials

$$
\begin{aligned}
\nabla^2 \vec{A} + k^2 \vec{A} &= -\mu \vec{J}, \\
\nabla^2 \phi + k^2 \phi &= -\rho/\epsilon.
\end{aligned}
$$

We are interest in the interaction of traveling acoustic or electromagnetic waves with bodies of varying acoustic or electromagnetic properties and of varying shape.

The interaction between waves and obstacles (*scatterers*) causes disturbances to incident wave fields $\rightarrow$ diffraction phenomena.

Different situations depending on the ratio of the characteristic dimension l of a scatterer and the wavelength λ of the incident wave:

1. $\lambda/l \gg 1$, low-frequency scattering (that is a perturbation of the incident wave) occurs: known as *Rayleigh scattering*; it can be studied by various perturbation methods.
2. $\lambda \sim l$, one or several diffraction phenomena dominate; this region is called *resonance region*; there are no suitable approximation methods $\rightarrow$ need of rigorous approaches.
3. $\lambda/l \ll 1$, high-frequency region or quasi-optical region; it can be studied by well-developed high-frequency approximate techniques $\rightarrow$ geometrical optics.

Independently of the frequency regime and the specific scattering mechanisms, the general formulation of the diffraction problem is uniform.

The incident and total field satisfies the Helmholtz equation or Maxwell's equations as appropriate, and several conditions must be imposed to ensure that the total field exists and is unique.
These include:

- boundary conditions
- Sommerfeld's radiation conditions
- scatterers with sharp edges, boundedness condition on the scattered energy

1. Fields
   The total field is decomposed as a sum of the incident field and the scattered field,
   $$U = U^i + U^s,$$
   U must be solutions of the appropriate wave equation in the exterior of the scatterer. Furthermore, U and its partial derivatives are continuous everywhere in the space exterior to, and onto the surface of the scattered.
2. Boundary conditions
   - Acoustic case
     If the surface of the scattered is acoustically rigid or hard, the fluid velocity vanishes at the surface (*hard boundary condition*)
     $$\left.\frac{\partial U}{\partial n}\right|_S = 0,$$
     where $\vec{n}$ is the unit normal to the surface S of the scatterer.
     If the surface of the scattered is acoustically soft, the pressure vanishes on the surface (*soft boundary condition*)
     $$U|_S = 0.$$

- Electromagnetic case
  In general the boundary surface S separates two media with different electromagnetic parameters ϵ_1, μ_1 and ϵ_2, μ_2; denote the corresponding electromagnetic fields by $\vec{E}_1, \vec{H}_1$ and $\vec{E}_2, \vec{H}_2$, respectively. Assume that there are no charges and currents on S. The boundary conditions are

$$\begin{aligned} \epsilon_1 \vec{E}_1 \cdot \vec{n} &= \epsilon_2 \vec{E}_2 \cdot \vec{n}, \\ \vec{E}_1 \wedge \vec{n} &= \vec{E}_2 \wedge \vec{n}, \\ \mu_1 \vec{H}_1 \cdot \vec{n} &= \mu_2 \vec{H}_2 \cdot \vec{n}, \\ \vec{H}_1 \wedge \vec{n} &= \vec{H}_2 \wedge \vec{n}. \end{aligned}$$

  Tangential components continuous, normal components discontinuous.
  If a charge q and a current $\vec{J}$ are presented on S,

$$\begin{aligned} \vec{n} \wedge \vec{E}_1 &= 0, \\ \vec{E}_1 \cdot \vec{n} &= q, \\ \vec{n} \wedge \vec{H}_1 &= \vec{J}, \\ \vec{H}_1 \cdot \vec{n} &= 0, \end{aligned}$$

1. Sommerfeld radiation condition (physically reasonable solutions)
   - in 3d
     For any scattered scalar field U^s that satisfies the Helmholtz equation:
     $$|rU^s| < K, \qquad \forall r,$$
     for some constant K, and
     $$r\left(\frac{\partial U^s}{\partial r} - \mathrm{i}kU^s\right) \to 0, \qquad \text{as} \quad r \to \infty.$$
   - in 2d
     $$|\sqrt{r}\,U^s| < K \qquad \forall r,$$
     and
     $$\sqrt{r}\left(\frac{\partial U^s}{\partial r} - \mathrm{i}kU^s\right) \to 0, \qquad \text{as} \quad r \to \infty.$$
     These conditions mean that in 2d (3d) space the scattered field must behave as an outgoing cylindrical (resp., spherical) wave at very large distances from the scatterer. The minus sign in both formulae is replaced by a plus sign if the time harmonic dependence is changed from $\exp(-\mathrm{i}\omega t)$ to $\exp(+\mathrm{i}\omega t)$.

The corresponding conditions for the 3d electromagnetic case are

$$|r\vec{E}| < K, \quad |r\vec{H}| < K,$$

and

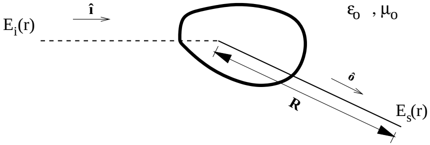
$$r\left(\vec{E} + \eta_0 \vec{r} \wedge \vec{H}\right) \to 0, \qquad r\left(\vec{H} - \eta_0^{-1} \vec{r} \wedge \vec{E}\right) \to 0,$$

as $r \to \infty$. $\eta_0 = \sqrt{\mu_0/\epsilon_0}$ is the characteristic impedance of the medium. In 2d the factor r is replaced by $\sqrt{r}$.

When an object is illuminated by a wave, a part of the incident power is scattered out and another part is absorbed by the object.

Consider a lin. polarized electromagnetic plane wave propagating in a medium with dielectric constant ϵ_0 and permeability μ_0 with the electric field given by

$$\vec{E}_i(\vec{r}) = \hat{e}_i \exp\left(-\mathrm{i}k\,\hat{\imath}\cdot\vec{r}\right)$$



Amplitude $|\vec{E}_i|$ chosen to be 1 (volt/m)
$k = \omega\sqrt{\mu_0\epsilon_0} = (2\pi)/\lambda$ is the wave number,
λ is a wavelength in the medium
$\hat{\imath}$ unit vector in the direction of wave prop.
$\hat{e}_i$ unit vector in the direction of its polariz.

The total field $\vec{E}$ at a distance R from a reference point in the object, in the direction of a unit vector ô, consists of the incident field $\vec{E}_i$ and the field $\vec{E}_s$ scattered by the object.

- Within a distance $R < D^2/\lambda$ (where D is a typical dimension of the object, such as its diameter), the field $\vec{E}_s$ has a complicated amplitude and phase variations because of the interference between contributions from different parts of the object and the observation point $\vec{r}$ is said to be in the near field of the object.
- When $R > D^2/\lambda$ the scattered field $\vec{E}_s$ behaves as a spherical wave

$$\vec{E}_s(\vec{r}) = \vec{f}(\hat{o}, \hat{\imath}) \frac{e^{-\mathrm{i}kR}}{R}, \qquad \text{for} \quad R > \frac{D^2}{\lambda}$$

$\vec{f}(\hat{o}, \hat{\imath})$ represents the amplitude, phase, and polarization of the scattered wave in the far field in the direction $\hat{o}$ when the object is illuminated by a plane wave propagating in the direction $\hat{\imath}$ with unit amplitude.

$\vec{f}(\hat{o}, \hat{\imath})$ is called the *scattering amplitude*

Consider the scattered power flux density S_s at a distance R in the direction ô, caused by an incident power flux density S_i.

The *differential scattering cross-section* is

$$\frac{\mathrm{d}\sigma_s}{\mathrm{d}\Omega} = \lim_{R\to\infty} \frac{R^2 S_i}{S_s} = |\vec{f}(\hat{\mathsf{o}}, \hat{\mathsf{\imath}})|^2$$

where S_i and S_s are the magnitudes of the incident and the scattering power flux density vectors

$$\vec{S}_i = \frac{1}{2}(\vec{E}_i \wedge \vec{H}_i^*) = \frac{|\vec{E}_i|^2}{2\eta_0}\,\hat{\mathsf{\imath}}, \qquad \vec{S}_s = \frac{1}{2}(\vec{E}_s \wedge \vec{H}_s^*) = \frac{|\vec{E}_s|^2}{2\eta_0}\,\hat{\mathsf{o}},$$

η_0 is the characteristic impedance of the medium.

Cross sections and scattering amplitude

Total observed scattered power at all angles surrounding the object: *scattering cross-section*,

$$\sigma_s = \int \mathrm{d}\sigma_s = \int |\vec{f}(\hat{\mathbf{o}}, \hat{\mathbf{i}})|^2 \mathrm{d}\Omega$$

Alternatively, σ_s can be written as

$$\sigma_s = \frac{\int_{S_0} \mathrm{Re}\left(\frac{1}{2}\vec{E}_s \wedge \vec{H}_s^*\right) \mathrm{d}\vec{A}}{|\vec{S}_i|}$$

where S_0 is an arbitrary surface enclosing the object and $\mathrm{d}\vec{A}$ is the differential surface area directed outward.
Total power absorbed by the object: *absorption cross-section*,

$$\sigma_a = -\frac{\int_{S_0} \mathrm{Re}\left(\frac{1}{2}\vec{E} \wedge \vec{H}^*\right) \mathrm{d}\vec{A}}{|\vec{S}_i|}$$

where $\vec{E} = \vec{E}_i + \vec{E}_s$ and $\vec{H} = \vec{H}_i + \vec{H}_s$ are the total fields.
Finally the *total cross-section* is

$$\sigma = \sigma_s + \sigma_a$$

In SR the "length" of a 4-vector is invariant under Lorentz transformation,

$$x^\mu = (ct, x, y, z) \quad \rightarrow \quad x_\mu x^\mu = c^2 t^2 - x^2 - y^2 - z^2.$$

The relativistic momentum and the relativistic energy are

$$\vec{p} = \gamma\, m\, \vec{v}, \qquad E = \gamma\, m\, c^2,$$

where m is the rest mass, and the factor γ is

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}, \qquad \text{with} \qquad \beta = \frac{v}{c} \tag{14}$$

(Newton $\gamma \sim 1$, elementary particles $\gamma \sim 10^4$, some cosmic-ray protons $\gamma \sim 10^{11}$)
The relativistic momentum and the relativistic energy are components of the energy-momentum 4-vector,

$$P = (E/c, p_x, p_y, p_z).$$

We have the following relativistic energy momentum relation,

$$E^2 = m^2 c^4 + p^2 c^2 \quad \xrightarrow{c=1} \quad E^2 = m^2 + p^2.$$

- Kinematic variables: masses, energies, momenta, angles,...
  - Not all are independent!
- Conservation laws:
  - Energy and momentum conservation laws (set of 4 eqs valid in all inertial frames)
- Physical quantities (like σ) may have simple form in specific reference frames:
  - Transformations between different coordinate systems (lab and c.o.m.)
  - Quantities conserved during the transformation (invariant variables)

2-body → 2-body process

We consider the scattering process of the form $A + B \to C + D$, see Figure 5.

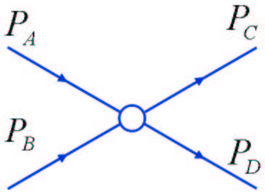
We have two conservation laws:

1. energy conservation,

$$E_A + E_B = E_C + E_D$$

2. 3-momentum conservation,

$$\vec{p}_A + \vec{p}_B = \vec{p}_C + \vec{p}_D$$



Conservation of the 4-momenta:

$$P_A + P_B = P_C + P_D$$

For I initial particles and F final particles:

$$\sum_{i=1}^{I} P_i = \sum_{f=1}^{F} P_f$$

The *Mandelstam variables* which are invariant under Lorentz transformations, are defined as

$$\begin{aligned} s &= (P_A + P_B)^2 = (P_C + P_D)^2 \\ t &= (P_A - P_C)^2 = (P_B - P_D)^2 \\ u &= (P_A - P_D)^2 = (P_B - P_C)^2 \end{aligned} \tag{15}$$

By adding up the terms on the right-hand side of Eqs. (15), and applying momentum conservation in the form $(P_A + P_B - P_C - P_D)^2 = 0$, we arrive at the relation,

$$s + t + u = m_A^2 + m_B^2 + m_C^2 + m_D^2 = constant$$

- For any process, s is the square of the total initial 4-momentum. Then in the center-of-mass frame, defined by $\vec{p}_A = -\vec{p}_B$ (3-vectors), s has the form,

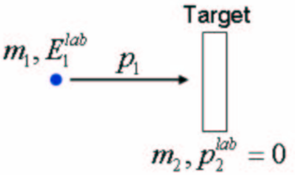
$$\begin{aligned} s &= (P_A + P_B)^2 \\ &= m_A^2 + m_B^2 + 2E_A E_B - 2\vec{p}_A \cdot \vec{p}_B \\ &= E_A^2 + E_B^2 + 2E_A E_B \\ &= (E_A + E_B)^2 \\ &= E_{cm}^2 \end{aligned}$$

  $\sqrt{s}$ is the total energy in the center-of-mass frame.
- The definitions of t and u appear to be interchangeable (by renaming $P_C \rightarrow P_D$).
- $t = (P_A - P_C)^2$ is the momentum transfer (square difference of the initial and final 4-momenta).

- Collision of particles, (m_1, E_1^{lab}), with fixed target $(m_2, E_2^{lab} = 0)$

  In the laboratory system we have,

$$\begin{aligned} s &= E_{cm}^2, \\ &= m_1^2 + m_2^2 + 2E_1^{lab} m_2, \\ &= 2E_1^{lab} m + 2m^2, \quad \text{for} \quad m_1 = m_2 = m. \end{aligned}$$



  Hence

$$E_1^{lab} = \frac{E_{cm}^2}{2m} - m.$$

- Threshold energy: minimum energy to create new particles.

$$P_1 + P_2 = \sum P_f$$

  where the final 4-momenta are $P_f = (E_f = m_f, 0, 0, 0)$. Squaring

$$m_1^2 + m_2^2 + 2m_2 E_1^{lab} = \left(\sum m_f\right)^2 \quad \Rightarrow \quad E_1^{lab} = \frac{1}{2m_2}\left[\left(\sum m_f\right)^2 - m_1^2 - m_2^2\right]$$

- Collision of two particles with (m_1, E_1^{lab}) and (m_2, E_2^{lab})

  In the laboratory system we have,

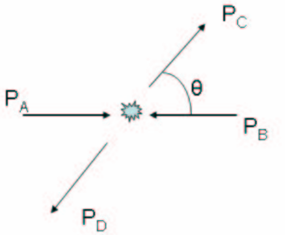
$$
\begin{aligned}
s &= m_1^2 + m_2^2 - 2E_1^{lab}E_2^{lab} + 2p_1^{lab}p_2^{lab}, \\
&= 4(E_1^{lab})^2, \quad \text{for} \quad \vec{p}_1 = -\vec{p}_2,\ m_1 = m_2.
\end{aligned}
$$

  Hence

$$E_1^{lab} = \frac{E_{cm}}{2}$$

Consider the 2-body $\to$ 2-body process in the c.o.m. frame for particles all of mass m.

In the center-of-mass frame we have:
$|\vec{p}_A| = |\vec{p}_B| = |\vec{p}_C| = |\vec{p}_D| = p,$
and
$E_A = E_B = E_C = E_D = E.$



The Mandelstam variables are

$$
\begin{aligned}
s &= (P_A + P_B)^2 = (2E)^2 = E_{cm}^2, \\
t &= (P_A - P_C)^2 \\
&= 2m^2 - 2E^2 + 2p^2 \cos\theta = -2p^2(1 - \cos\theta), \\
u &= (P_A - P_D)^2 = 2m^2 - 2E^2 + 2p^2 \cos(\pi - \theta) = -2p^2(1 + \cos\theta).
\end{aligned}
$$

We see that

- because $-1 \leq \cos\theta \leq 1$ it is $t < 0$ and $u < 0$,
- $t \to 0$ as $\theta \to 0$,
- $u \to 0$ as $\theta \to \pi$.

The sum of the Mandelstam variables is: $s + t + u = 4E^2 - 4p^2 = 4m^2$.

The relation between β and γ given in (14) and the ranges $0 \leq \beta \leq 1$, $1 \leq \gamma \leq \infty$ allow the alternative parametrization

$$\beta = \tanh \eta$$

and so

$$\begin{aligned} \gamma &= \cosh \eta \\ \gamma\beta &= \sinh \eta \end{aligned}$$

where η is known as the *boost parameter* or *rapidity*.

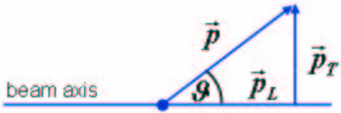
In term of η the Lorentz transformations between coordinate systems moving with velocity v along the z direction become

$$c\,t' = c\,t\,\cosh\eta - z\,\sinh\eta, \tag{16}$$

$$z' = -c\,t\,\sinh\eta + z\,\cosh\eta. \tag{17}$$

Suppose that a particle has momentum $\vec{p}$ in a frame K, with transverse momentum $\vec{p}_T$ and a z (beam axis) component p_L.

There is a unique Lorentz transformation in the z direction to a frame K' where the particle has no z component of momentum.
In K' the particle has momentum and energy



$$\vec{p}\,' = \vec{p}_T, \qquad \frac{E'}{c} = m_T = \sqrt{p_T^2 + m^2c^2}$$

In terms of the rapidity parameter, the momentum components and energy of the particle in the original frame K can be written,

$$\vec{p}_T, \qquad p_L = m_T \sinh\eta, \qquad \frac{E}{c} = m_T \cosh\eta$$

If a particle is at rest in frame K', that is $\vec{p}_T = 0$, then the above expressions become

$$p = m\,c\,\sinh\eta, \qquad E = m\,c^2 \cosh\eta$$

and the rapidity can be expressed as

$$\eta = \frac{1}{2}\ln\frac{E/c + p_L}{E/c - p_L}$$

Rapidity III

- The convenience of $\vec{p}_\perp$ and η as kinematic variables is that a Lorentz transformation in the z direction shifts the rapidity by a constant amount, $\eta \to \eta - Z$, where Z is the rapidity parameter of the transformation.
- With these variables, the configuration of particles in a collision process viewed in the laboratory frame differs only by a trivial shift of the origin of the rapidity from the same process viewed in the center of mass frame.