Perturbative Renormalization and ϵ expansion

Max Dohse

Outline

1. s^4-model on a lattice
2. Spatial fluctuation variables
3. Averaging out high frequency fluctuations
   - Separating high and low frequencies
   - Perturbative analysis & diagrams
   - Recovering the original Hamiltonian
4. Calculation of critical exponent ν
   - Fixed points and critical points
   - Linearized recursion relations

Literature

- K. Wilson and J. Kogut: The renormalization group and the ϵ expansion
  (follow sections 4 and 3 of this main reference closely,
  also all figures are picked out therefrom)
- G. Parisi: Statistical field theory
- L. Kadanoff: Statistical physics - statics, dynamics and renormalization

Outline

- consider cubic lattice of spatial dimension d and lattice constant a, site on lattice denoted by $\underline{n} \in \mathbb{Z}^d$, thus dimensionless: $[\underline{n}] = 1$ and $[a] =$ length
- lattice site $\underline{n}$ located in space at position $\underline{x}_{\underline{n}} = a\underline{n}$, thus $[\underline{x}] =$ length^d

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- general definition of **partition function** depending on temperature T and coupling constants $\underline{g}$:

$$Z(T, \underline{g}) := \sum\!\!\!\!\!\!\!\int_{\text{states}} \mathrm{e}^{-\mathcal{H}[\text{state}, \underline{g}]/(k_b T)}$$

- Hamiltonian of **Gaussian model** in configuration space:

$$\mathcal{H}_{\mathsf{Gauss}[s,J,B]} = -J \sum_{\underline{n}} \sum_{i=1}^{d} s_{\underline{n}}\, s_{\underline{n}+\underline{e}_i} + \frac{B}{2} \sum_{\underline{n}} s_{\underline{n}}^2$$

- **Ising model**: dimensionless spins s live on discrete lattice and takes discrete values $s_{\underline{n}} = \pm 1$
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- **Ising model**: dimensionless spins s live on discrete lattice and takes discrete values $s_{\underline{n}} = \pm 1$
- Gaussian model: s on discrete lattice, but continuous $s_{\underline{n}} \in \mathbb{R}$
- redefine constants in order to save writing lots of $(k_B T)$'s: $j = J/(k_B T)$ and $b = B/(k_B T)$

$$\widetilde{\mathcal{H}}_{\mathsf{Gauss}[s,j,b]} := \frac{\mathcal{H}_{\mathsf{Gauss}[s]}}{k_B T} = -j \sum_{\underline{n}} \sum_{i=1}^{d} s_{\underline{n}}\, s_{\underline{n}+\underline{e}_i} + \frac{b}{2} \sum_{\underline{n}} s_{\underline{n}}^2$$

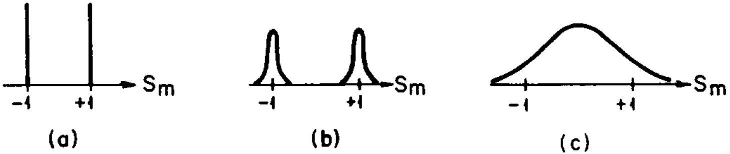

Fig. 3.1. The transition from the Ising to the Gaussian model.
The Ising model (a) has spin up or spin down at each lattice site.
Model (b) has spin variables which peak about the Ising values.
The Gaussian model (c) has spin variables at each site,
with smooth Gaussian distributions about zero

- Hamiltonian of s^4-**model** in configuration space:

$$\mathcal{H}_{s^4}[s,J,B,L] = \underbrace{-J \sum_{\underline{n}} \sum_{i=1}^{d} s_{\underline{n}}\, s_{\underline{n}+\underline{e}_i} + \frac{B}{2} \sum_{\underline{n}} s_{\underline{n}}^2}_{\mathcal{H}_{\mathsf{Gauss}}[s]} + \underbrace{L \sum_{\underline{n}} s_{\underline{n}}^4}_{\mathcal{H}_{\mathsf{Int}}[s]}$$

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- define $l = L/(k_B T)$

$$\widetilde{\mathcal{H}}_{s^4}[s,j,b,l] := \frac{\mathcal{H}_{s^4}[s]}{k_B T} = -j\sum_{\underline{n}}\sum_{i=1}^{d} s_{\underline{n}}\, s_{\underline{n}+\underline{e}_i} + \frac{b}{2}\sum_{\underline{n}} s_{\underline{n}}^2 + l\sum_{\underline{n}} s_{\underline{n}}^4$$

- s^4-model approaches Ising model for $l \to \infty$ and $b \to -\infty$ with fixed $b/l = -4$ if the lattice spins s are properly rescaled

- partition function becomes

$$Z(T,J,B,L) \;=\; Z(j,b,l) \;=\; \int \mathsf{d}\, s\ \exp \sum_{\underline{n}} \left(\sum_{i=1}^{d} j\, s_{\underline{n}}\, s_{\underline{n}+\underline{e}_i} - \frac{b}{2}\, s_{\underline{n}}^2 - l s_{\underline{n}}^4 \right)$$

$$\int \mathsf{d}\, s \;:= \prod_{\underline{n}} \int_{-\infty}^{+\infty} \mathsf{d}\, s_{\underline{n}}$$

Outline

- introduce dimensionless **spatial spin fluctuations** $\sigma[\underline{q},s]$
  with **fluctuation frequency (wave vector)** $\underline{q}$
  as discrete Fourier transform of lattice spins s:

$$\sigma[\underline{q},s] := \sum_{\underline{n}} \mathrm{e}^{-\mathrm{i}\underline{q}\underline{n}} \, s_{\underline{n}}$$

frequency has continuous values in first Brillouin zone: $\underline{q} \in [-\pi, +\pi]^d$,
and q^j denotes the j^{th} component of $\underline{q}$

$$\int \mathrm{d}^d\underline{q} := \prod_{j=1}^{d} \int_{-\pi}^{+\pi} \frac{\mathrm{d}\,q^j}{2\pi}$$

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$$\sigma[\underline{q},s] := \sum_{\underline{n}} \mathrm{e}^{-\mathrm{i}\underline{q}\underline{n}} \, s_{\underline{n}}$$

- can write Hamiltonian in terms of spin fluctuations with $\widetilde{r} := b - 2dj$:

$$\widetilde{\mathcal{H}}_{s^4}[s,\widetilde{r},j,l] = \frac{1}{2}\int \mathrm{d}^d q \;\; \sigma[\underline{q},s] \; \sigma[-\underline{q},s] \left(\widetilde{r} + j\sum_{k=1}^{d} \left| e^{\mathrm{i}q^k} - 1 \right|^2\right)$$
$$+ \, l \int \mathrm{d}^d q_1 \int \mathrm{d}^d q_2 \int \mathrm{d}^d q_3 \int \mathrm{d}^d q_4 \; \sigma[\underline{q}_1,s] \; \sigma[\underline{q}_2,s] \; \sigma[\underline{q}_3,s] \; \sigma[\underline{q}_4,s]$$
$$(2\pi)^d \, \delta^{(d)}(\underline{q}_1 + \underline{q}_2 + \underline{q}_3 + \underline{q}_4)$$

- now three changes will be applied in order to simplify our calculations
- first: replace $\sum_k |\exp(\mathrm{i}q^k) - 1|^2 = 2(1-\cos q^k)$ by its form for small $\underline{q}$, i.e.: by $\underline{q}^2$, no essential change of model, because our interest lies in long wavelengths behavior which comes from small fluctuation frequencies

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- second: rescale the spins such that $j = 1$, i.e.: $s_{\underline{n}} \rightarrow s_{\underline{n}}/\sqrt{j}$

- third: limit range of integration from $[-\pi,+\pi]^d$ to $|\underline{q}| \in [0,1]$,
  which for the same reason is no essential change in the model and from now on

$$\int \mathsf{d}^d_{\,} q \;:=\; \int\limits_{|\underline{q}|\leq 1} \prod_{k=1}^{d} \frac{dq^k}{2\pi}$$

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  however conceptually there is a difficulty:
  with the new restricted range we can no longer relate the functional variable $\sigma[\underline{q},s]$
  to the ordinary variables $s_{\underline{n}}$,
  therefore consider the spin fluctuations as variables $\sigma(\underline{q})$ in their own right
  and change the definition of the partition function from ordinary integrals over the $s_{\underline{n}}$
  to functional integrals over $\sigma(\underline{q})$

- with $r := \widetilde{r}/j = \frac{b}{j} - 2d$ and $u := l/j^2$ after these three changes we get

$$\widetilde{\mathcal{H}}_{s^4}[\sigma,r,u] = \overbrace{\frac{1}{2}\int \mathrm{d}^d q \; \left(\underline{q}^2 + r\right)\sigma(\underline{q})\,\sigma(-\underline{q})}^{\widetilde{\mathcal{H}}_{\mathsf{Gauss}}[\sigma,r]}$$
$$+ \underbrace{u\int \mathrm{d}^d q_1 \int \mathrm{d}^d q_2 \int \mathrm{d}^d q_3 \int \mathrm{d}^d q_4 \;\; \sigma(\underline{q}_1)\,\sigma(\underline{q}_2)\,\sigma(\underline{q}_3)\,\sigma(\underline{q}_4)\;(2\pi)^d\;\delta^{(d)}(\underline{q}_1+\underline{q}_2+\underline{q}_3+\underline{q}_4)}_{\widetilde{\mathcal{H}}_{\mathsf{Int}}[\sigma,u]}$$

$$Z(r,u) := \int \mathcal{D}\sigma \;\; \mathrm{e}^{-\widetilde{\mathcal{H}}_{s^4}[\sigma,r,u]}$$

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  this will involve considerable simplifications/approximations
- in place of couplings r, u we will find new couplings $r'(r,u)$ and $u'(r,u)$,
  one of our aims is to find these **recursion relations**
  relating the new with the original couplings

- construction works as follows:
- integrals of spin fluctuations in original system are over frequencies $|\underline{q}| \in [0, 1]$, new "primed" system obtained by averaging about high frequency modes $|\underline{q}| \in [\frac{1}{2}, 1]$

- write function $\sigma(\underline{q})$ as sum over **partition of unity** p:

$$\sigma(\underline{q}) \;=\; \sigma_{<}(\underline{q}) + \sigma_{>}(\underline{q})$$

$$\sigma_{\lessgtr}(\underline{q}) \;:=\; p_{\lessgtr}(\underline{q})\,\sigma(\underline{q})$$

$$p_{\lessgtr}(\underline{q}) \in [0,1]$$
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- write function $\sigma(\underline{q})$ as sum over **partition of unity** p:

$$\sigma(\underline{q}) \;=\; \sigma_<(\underline{q}) + \sigma_>(\underline{q})$$

$$\sigma_{\lessgtr}(\underline{q}) \;:=\; p_{\lessgtr}(\underline{q})\,\sigma(\underline{q}) \qquad\qquad \begin{array}{l} p_{\lessgtr}(\underline{q}) \in [0{,}1] \\ p_<(\underline{q}) + p_>(\underline{q}) = 1 \;\; \forall \underline{q} \end{array}$$

- Wilson chooses discontinuous partition of unity:
Heaviside step functions $p_{\lessgtr}(q) \;=\; \theta\big(\mp(|\underline{q}| - \tfrac{1}{2})\big)$, thus

$$\sigma_<(\underline{q}) \;:=\; p_<(\underline{q})\,\sigma(\underline{q}) \;=\; \theta\big(\tfrac{1}{2} - |\underline{q}|\big)\;\sigma(\underline{q}) \;=\; \begin{cases} \sigma(\underline{q}) & |\underline{q}| < \frac{1}{2} \\ 0 & |\underline{q}| > \frac{1}{2} \end{cases}$$

$$\sigma_>(\underline{q}) \;:=\; p_>(\underline{q})\,\sigma(\underline{q}) \;=\; \theta\big(|\underline{q}| - \tfrac{1}{2}\big)\;\sigma(\underline{q}) \;=\; \begin{cases} 0 & |\underline{q}| < \frac{1}{2} \\ \sigma(\underline{q}) & |\underline{q}| > \frac{1}{2} \end{cases}$$

- in Gaussian Hamiltonian the high and low frequency modes decouple, there is NO interaction between fluctuations of high and low frequency:

$$
\begin{aligned}
\widetilde{\mathcal{H}}_{\mathsf{Gauss}[\sigma,r]} &= \frac{1}{2}\int \mathsf{d}^d q \;\left(\underline{q}^2 + r\right)\sigma(\underline{q})\,\sigma(-\underline{q}) \\
&= \frac{1}{2}\left(\int\limits_{|\underline{q}|<\frac{1}{2}} + \int\limits_{|\underline{q}|>\frac{1}{2}}^{|\underline{q}|<1}\right)\mathsf{d}^d q\;\left(\underline{q}^2 + r\right)\Big[\sigma_<(\underline{q})\,\sigma_<(-\underline{q}) + \overbrace{\sigma_<(\underline{q})\,\sigma_>(-\underline{q})}^{0} \\
&\qquad + \underbrace{\sigma_>(\underline{q})\,\sigma_<(-\underline{q})}_{0} + \sigma_>(\underline{q})\,\sigma_>(-\underline{q})\Big] \\
&= \underbrace{\frac{1}{2}\int\limits_{|\underline{q}|<\frac{1}{2}} \mathsf{d}^d q\;\left(\underline{q}^2 + r\right)\sigma_<(\underline{q})\,\sigma_<(-\underline{q})}_{\widetilde{\mathcal{H}}_{\mathsf{Gauss}[\sigma_<,r]}} + \underbrace{\frac{1}{2}\int\limits_{|\underline{q}|>\frac{1}{2}}^{|\underline{q}|<1} \mathsf{d}^d q\;\left(\underline{q}^2 + r\right)\sigma_>(\underline{q})\,\sigma_>(-\underline{q})}_{\widetilde{\mathcal{H}}_{\mathsf{Gauss}[\sigma_>,r]}}
\end{aligned}
$$

- however in σ^4-interaction no decoupling occurs, there IS an interaction between fluctuations of high and low frequencies: the Hamiltonian

$$\widetilde{\mathcal{H}}_{\mathsf{Int}[\sigma,u]} = u \int \mathsf{d}^d q_1 \int \mathsf{d}^d q_2 \int \mathsf{d}^d q_3 \int \mathsf{d}^d q_4 \; \sigma(\underline{q}_1)\, \sigma(\underline{q}_2)\, \sigma(\underline{q}_3)\, \sigma(\underline{q}_4)\; (2\pi)^d\, \delta^{(d)}(\underline{q}_1 + \underline{q}_2 + \underline{q}_3 + \underline{q}_4)$$

(in addition to the decoupling parts) also contains high-low frequency interactions like

$$u \int_{|\underline{q}_1|<\frac{1}{2}} \mathsf{d}^d q_1 \int_{|\underline{q}_2|<\frac{1}{2}} \mathsf{d}^d q_2 \int_{|\underline{q}_3|>\frac{1}{2}}^{|\underline{q}_4|<1} \mathsf{d}^d q_3 \int_{|\underline{q}_4|>\frac{1}{2}}^{|\underline{q}_4|<1} \mathsf{d}^d q_4 \;\; \sigma_<(\underline{q}_1)\, \sigma_<(\underline{q}_2)\, \sigma_>(\underline{q}_3)\, \sigma_>(\underline{q}_4)\; (2\pi)^d\; \delta^{(d)}(\underline{q}_1 + \underline{q}_2 + \underline{q}_3 + \underline{q}_4)$$

and therefore we have $\widetilde{\mathcal{H}}_{\mathsf{Int}[\sigma,u]} \neq \widetilde{\mathcal{H}}_{\mathsf{Int}[\sigma_<,u]} + \widetilde{\mathcal{H}}_{\mathsf{Int}[\sigma_>,u]}$

- partition function can now be written as

$$
\begin{aligned}
Z_{(r,u)} &:= \int \mathcal{D}\sigma \ \ \mathsf{e}^{-\widetilde{\mathcal{H}}_{s^4}[\sigma,r,u]} \\
&= \int \mathcal{D}\sigma_< \ \int \mathcal{D}\sigma_> \ \ \mathsf{e}^{-\widetilde{\mathcal{H}}_{s^4}[\sigma_< + \sigma_>,r,u]}
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- integrating out high frequency modes $\sigma_>$ shall give us

$$Z_{(r,u)} = Z_{(r',u')} = \int \mathcal{D}\sigma' \;\; \mathrm{e}^{-\widetilde{\mathcal{H}}'_{s^4}[\sigma',r',u']}$$

with new effective couplings r', u' and rescaled fluctuations σ',

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and primed fluctuation modes are related to original long wavelength modes
by **scaling relations**

$$\sigma'(\underline{q}' := 2\underline{q}) = \zeta\,\sigma_<(\underline{q}) \qquad\qquad \forall\,|\underline{q}| \in [0,\, \tfrac{1}{2}]$$

- because of $\widetilde{\mathcal{H}}_{\mathsf{Gauss}}[\sigma,r] = \widetilde{\mathcal{H}}_{\mathsf{Gauss}}[\sigma_<,r] + \widetilde{\mathcal{H}}_{\mathsf{Gauss}}[\sigma_>,r]$, we can write

$$
\begin{aligned}
\mathrm{e}^{-\widetilde{\mathcal{H}}'_{s^4}[\sigma,r,u]} &= \int \mathcal{D}\sigma_> \; \mathrm{e}^{-\widetilde{\mathcal{H}}_{s^4}[\sigma,r,u]} = \int \mathcal{D}\sigma_> \; \mathrm{e}^{-\widetilde{\mathcal{H}}_{\mathsf{Gauss}}[\sigma,r]-\widetilde{\mathcal{H}}_{\mathsf{Int}}[\sigma,u]} \\
&= \mathrm{e}^{-\widetilde{\mathcal{H}}_{\mathsf{Gauss}}[\sigma_<,r]} \int \mathcal{D}\sigma_> \; \mathrm{e}^{-\widetilde{\mathcal{H}}_{\mathsf{Gauss}}[\sigma_>,r]-\widetilde{\mathcal{H}}_{\mathsf{Int}}[\sigma,u]}
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$$= \mathrm{e}^{-\widetilde{\mathcal{H}}_{\mathsf{Gauss}}[\sigma_<,r]} \int \mathcal{D}\sigma_> \ \mathrm{e}^{-\widetilde{\mathcal{H}}_{\mathsf{Gauss}}[\sigma_>,r]-\widetilde{\mathcal{H}}_{\mathsf{Int}}[\sigma,u]}$$

- factor $\exp -\widetilde{\mathcal{H}}_{\mathsf{Gauss}[\sigma_<,r]}$ easily expressed in primed spin fluctuations:

$$\widetilde{\mathcal{H}}_{\mathsf{Gauss}[\sigma_<,r]} = \frac{1}{2} \int\limits_{|\underline{q}|<\frac{1}{2}} \mathrm{d}^d q \ \left(\underline{q}^2 + r\right) \sigma_<(\underline{q})\, \sigma_<(-\underline{q})$$

$$= \frac{1}{2} \left(\zeta^2/2^{d+2}\right) \int\limits_{|\underline{q}|'<1} \mathrm{d}^d q' \left(\underline{q}'^2 + 4r\right) \sigma'(\underline{q}')\, \sigma'(-\underline{q}')$$

Outline

- now **perturbative part**: assuming coupling $u := L/(j^2 k_B T)$ to be small: $u \ll 1$, we can work out a precise relation between original and primed Hamiltonian

(a) 

(b) $$\exp -\widetilde{\mathcal{H}}_{\mathrm{Int}[\sigma,u]} = 1 - \widetilde{\mathcal{H}}_{\mathrm{Int}[\sigma,u]} + \tfrac{1}{2}\widetilde{\mathcal{H}}^2_{\mathrm{Int}[\sigma,u]} - \ldots$$

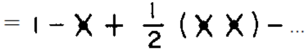

Fig. 4.1. (a) Graphical representation of $\mathcal{H}_{\mathrm{I}}$.
(b) Graphs for $\exp\{-\mathcal{H}_{\mathrm{I}}\}$.

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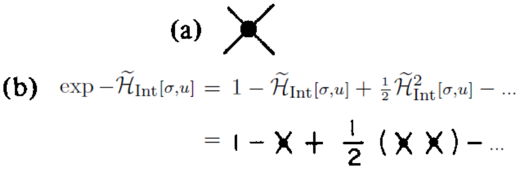

Fig. 4.1. (a) Graphical representation of $\mathcal{H}_{\mathbf{I}}$.
(b) Graphs for $\exp\{-\mathcal{H}_{\mathbf{I}}\}$.

- using this expansion we can now attack the functional integral

$$\int \mathcal{D}\sigma_> \; \mathrm{e}^{-\widetilde{\mathcal{H}}_{\mathsf{Gauss}}[\sigma_>,r]-\widetilde{\mathcal{H}}_{\mathsf{Int}}[\sigma,u]} = \int \mathcal{D}\sigma_> \; \mathrm{e}^{-\widetilde{\mathcal{H}}_{\mathsf{Gauss}}[\sigma_>,r]}\left(1 - \widetilde{\mathcal{H}}_{\mathsf{Int}}[\sigma,u] + \tfrac{1}{2}\widetilde{\mathcal{H}}^2_{\mathsf{Int}}[\sigma,u] - \ldots\right)$$

- in order to evaluate this integral, we need to calculate **Gaussian integrals** of the type

$$I(r, \underline{q}_{m_1}, \dots, \underline{q}_{m_k}) := \int \mathcal{D}\sigma_> \; \mathrm{e}^{-\widetilde{\mathcal{H}}_{\mathsf{Gauss}}[\sigma_>, r]} \, \sigma_>(\underline{q}_{m_1}) \dots \sigma_>(\underline{q}_{m_k})$$

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$$Z_{\mathsf{Gauss}}(r) := \int \mathcal{D}\sigma_> \ \mathsf{e}^{-\widetilde{\mathcal{H}}_{\mathsf{Gauss}}[\sigma_>, r]}$$

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- define (lousily LaTeXed) **contraction** $\overline{\sigma_> \, \sigma_>}$ of high frequency fluctuations as

$$\overline{\sigma_>(\underline{q}_{m_a}) \, \sigma_>(\underline{q}_{m_b})} := \theta\big(|\underline{q}_{m_a}| - \tfrac{1}{2}\big) \, (2\pi)^d \, \frac{\delta^{(d)}(\underline{q}_{m_a} + \underline{q}_{m_b})}{\underline{q}^2_{m_a} + r}$$

- evaluating Gaussian integrals $I(r, \underline{q}_{m_1}, \ldots, \underline{q}_{m_k})$ one finds result given by $Z_{\mathsf{Gauss}}(r)$ times sum of all possible ways of contracting the $\sigma_>(\underline{q}_{m_1}) \ldots \sigma_>(\underline{q}_{m_k})$ in pairs such that all of them are contained in some contraction:

$$I(r,\underline{q}_{m_1},\ldots,\underline{q}_{m_k}) = Z_{\mathsf{Gauss}}(r) \sum_{P(m_1,\ldots,m_k)} \overline{\sigma_>(\underline{q}_{P(m_1)})\, \sigma_>(\underline{q}_{P(m_2)})} \cdot \ldots \cdot \overline{\sigma_>(\underline{q}_{P(m_{k-1})})\, \sigma_>(\underline{q}_{P(m_k)})}$$

wherein the sum runs over all permutations P of the indices m_1, ..., m_k

- evaluating Gaussian integrals $I(r, \underline{q}_{m_1}, \ldots, \underline{q}_{m_k})$ one finds result given by $Z_{\text{Gauss}}(r)$ times sum of all possible ways of contracting the $\sigma_>(\underline{q}_{m_1}) \ldots \sigma_>(\underline{q}_{m_k})$ in pairs such that all of them are contained in some contraction:

$$I(r,\underline{q}_{m_1},\ldots,\underline{q}_{m_k}) = Z_{\text{Gauss}}(r) \sum_{P(m_1,\ldots,m_k)} \overline{\sigma_>(\underline{q}_{P(m_1)})\,\sigma_>(\underline{q}_{P(m_2)})} \cdot \ldots \cdot \overline{\sigma_>(\underline{q}_{P(m_{k-1})})\,\sigma_>(\underline{q}_{P(m_k)})}$$

wherein the sum runs over all permutations P of the indices m_1, ..., m_k

- in order to translate this result into diagrams, we remember $\sigma(\underline{q}) = \sigma_<(\underline{q}) + \sigma_>(\underline{q})$ and therefore have to consider two cases for each endpoint:

- evaluating Gaussian integrals $I(r, \underline{q}_{m_1}, ..., \underline{q}_{m_k})$ one finds result given by $Z_{\text{Gauss}}(r)$ times sum of all possible ways of contracting the $\sigma_>(\underline{q}_{m_1}) \ldots \sigma_>(\underline{q}_{m_k})$ in pairs such that all of them are contained in some contraction:

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  if spin fluctuation frequency $\underline{q}$ associated to endpoint has length $|\underline{q}| < \frac{1}{2}$,
  i.e.: $\sigma(\underline{q}) = \sigma_<(\underline{q})$, then it is within the long wavelength regime,
  which we don't integrate out, thus it's NOT contracted, just left uncontracted

- evaluating Gaussian integrals $I(r, \underline{q}_{m_1}, \ldots, \underline{q}_{m_k})$ one finds result given by $Z_{\mathsf{Gauss}}(r)$ times sum of all possible ways of contracting the $\sigma_>(\underline{q}_{m_1}) \ldots \sigma_>(\underline{q}_{m_k})$ in pairs such that all of them are contained in some contraction:

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  wherein the sum runs over all permutations P of the indices $m_1, \ldots, m_k$

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  which we don't integrate out, thus it's NOT contracted, just left uncontracted

  whereas if $|\underline{q}| > \frac{1}{2}$, i.e.: $\sigma(\underline{q}) = \sigma_>(\underline{q})$,
  then it IS integrated out, and must be contracted with another endpoint
  whose frequency also has length greater than one half

- thus for our functional integral we obtain

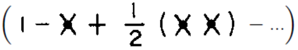
$$\begin{aligned}
\mathrm{e}^{-\widetilde{\mathcal{H}}'_{s4}[\sigma\ ,r\ ,u\ ]} &= \mathrm{e}^{-\widetilde{\mathcal{H}}_{\mathrm{Gauss}}[\sigma_<,r]} \int \mathcal{D}\sigma_> \ \mathrm{e}^{-\widetilde{\mathcal{H}}_{\mathrm{Gauss}}[\sigma_>,r]-\widetilde{\mathcal{H}}_{\mathrm{Int}}[\sigma,u]} \\
&= \mathrm{e}^{-\widetilde{\mathcal{H}}_{\mathrm{Gauss}}[\sigma_<,r]} \int \mathcal{D}\sigma_> \ \mathrm{e}^{-\widetilde{\mathcal{H}}_{\mathrm{Gauss}}[\sigma_>,r]} \left(1 - \widetilde{\mathcal{H}}_{\mathrm{Int}[\sigma,u]} + \tfrac{1}{2}\widetilde{\mathcal{H}}^2_{\mathrm{Int}}[\sigma,u] - \dots\right) \\
&= \mathrm{e}^{-\widetilde{\mathcal{H}}_{\mathrm{Gauss}}[\sigma_<,r]} \int \mathcal{D}\sigma_> \ \mathrm{e}^{-\widetilde{\mathcal{H}}_{\mathrm{Gauss}}[\sigma_>,r]} \left(1 - \times + \frac{1}{2}\,(\times\ \times) - \dots\right)
\end{aligned}$$

- thus for our functional integral we obtain

$$\mathrm{e}^{-\widetilde{\mathcal{H}}'_{s4}[\sigma\ ,r\ ,u\ ]} = \mathrm{e}^{-\widetilde{\mathcal{H}}_{\mathrm{Gauss}}[\sigma_<,r]} \int \mathcal{D}\sigma_> \ \mathrm{e}^{-\widetilde{\mathcal{H}}_{\mathrm{Gauss}}[\sigma_>,r]-\widetilde{\mathcal{H}}_{\mathrm{Int}}[\sigma,u]}$$

$$= \mathrm{e}^{-\widetilde{\mathcal{H}}_{\mathrm{Gauss}}[\sigma_<,r]} \int \mathcal{D}\sigma_> \ \mathrm{e}^{-\widetilde{\mathcal{H}}_{\mathrm{Gauss}}[\sigma_>,r]} \left(1 - \widetilde{\mathcal{H}}_{\mathrm{Int}}[\sigma,u] + \tfrac{1}{2}\widetilde{\mathcal{H}}^2_{\mathrm{Int}}[\sigma,u] - \ldots\right)$$

$$= \mathrm{e}^{-\widetilde{\mathcal{H}}_{\mathrm{Gauss}}[\sigma_<,r]} \int \mathcal{D}\sigma_> \ \mathrm{e}^{-\widetilde{\mathcal{H}}_{\mathrm{Gauss}}[\sigma_>,r]}$$



- order u^0

$$\int \mathcal{D}\sigma_> \ \mathrm{e}^{-\widetilde{\mathcal{H}}_{\mathsf{Gauss}}[\sigma_>,r]} \Big(1\Big) =: Z_{\mathsf{Gauss}}(r)$$

- order u^1:

$$\int \mathcal{D}\sigma_> \ \mathrm{e}^{-\widetilde{\mathcal{H}}_{\mathrm{Gauss}}[\sigma_>,r]} \Big(- $$  $$ \Big)$$

$$= \int \mathcal{D}\sigma_> \ \mathrm{e}^{-\widetilde{\mathcal{H}}_{\mathrm{Gauss}}[\sigma_>,r]} \Big(-u \int \mathrm{d}^d q_1 \int \mathrm{d}^d q_2 \int \mathrm{d}^d q_3 \int \mathrm{d}^d q_4$$
$$\sigma(\underline{q}_1)\ \sigma(\underline{q}_2)\ \sigma(\underline{q}_3)\ \sigma(\underline{q}_4)\ (2\pi)^d\, \delta^{(d)}(\underline{q}_1+\underline{q}_2+\underline{q}_3+\underline{q}_4) \Big)$$

$$= -u\, Z_{\mathrm{Gauss}}(r) \int \mathrm{d}^d q_1 \int \mathrm{d}^d q_2 \int \mathrm{d}^d q_3 \int \mathrm{d}^d q_4\ (2\pi)^d\, \delta^{(d)}(\underline{q}_1+\underline{q}_2+\underline{q}_3+\underline{q}_4)$$
$$\Big\{ \sigma_<(\underline{q}_1)\ \sigma_<(\underline{q}_2)\ \sigma_<(\underline{q}_3)\ \sigma_<(\underline{q}_4)$$
$$+ 6\ \overline{\sigma_>(\underline{q}_1)\ \sigma_>(\underline{q}_2)}\ \sigma_<(\underline{q}_3)\ \sigma_<(\underline{q}_4)$$
$$+ 3\ \overline{\sigma_>(\underline{q}_1)\ \sigma_>(\underline{q}_2)}\ \overline{\sigma_<(\underline{q}_3)\ \sigma_<(\underline{q}_4)} \Big\}$$

- order u^1:

$$= -u\, Z_{\text{Gauss}(r)} \int \mathrm{d}^d q_1 \int \mathrm{d}^d q_2 \int \mathrm{d}^d q_3 \int \mathrm{d}^d q_4 \, (2\pi)^d \, \delta^{(d)}(\underline{q}_1+\underline{q}_2+\underline{q}_3+\underline{q}_4) \Big\{ 1$$  $$+6$$  $$+3$$  $$\Big\}$$

$$= -u\, Z_{\text{Gauss}(r)} \Big\{ 1 \int \mathrm{d}^d q_1 \int \mathrm{d}^d q_2 \int \mathrm{d}^d q_3 \, \sigma_<(\underline{q}_1)\, \sigma_<(\underline{q}_2)\, \sigma_<(\underline{q}_3)\, \sigma_<(-(\underline{q}_1+\underline{q}_2+\underline{q}_3))$$

$$+ 6 \int \mathrm{d}^d q_1 \int \mathrm{d}^d q_3 \, \frac{\sigma_<(\underline{q}_3)\, \sigma_<(-\underline{q}_3)}{(\underline{q}_1^2 + r)}$$

$$+ 3 \int \mathrm{d}^d q_1 \int \mathrm{d}^d q_3 \, \frac{(2\pi)^d \, \delta^{(d)}(0)}{(\underline{q}_1^2 + r)\,(\underline{q}_3^2 + r)} \Big\}$$

- order u^2: diagrams shown below, weights include $\frac{1}{2}$ in front of $\widetilde{\mathcal{H}}^2_{\mathsf{Int}}$

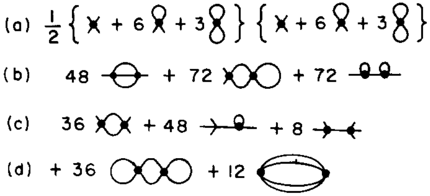


Fig. 4.3. Second order graphs after σ_1 integration.

- order u^2: diagrams shown below, weights include $\frac{1}{2}$ in front of $\widetilde{\mathcal{H}}^2_{\mathsf{Int}}$

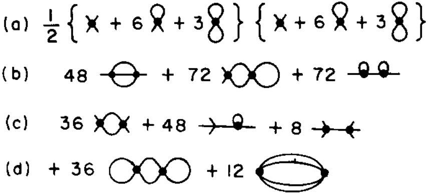


Fig. 4.3. Second order graphs after σ_1 integration.

- **any diagram computes according to following rules**:
  1. label frequencies/momenta in incoming sense at each vertex

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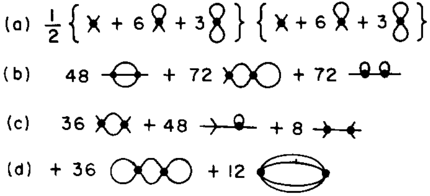

Fig. 4.3. Second order graphs after σ_1 integration.

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  2. external frequencies within $|\underline{q}| < \frac{1}{2}$, internal within $|\underline{q}| \in [\frac{1}{2}, 1]$

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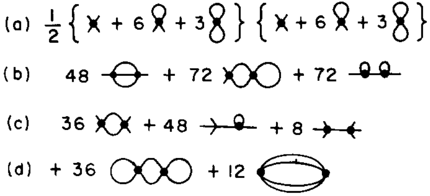

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  3. to each internal line connecting frequencies $\underline{q}_1$ and $\underline{q}_2$ associate propagator $(2\pi)^d \delta^{(d)}(\underline{q}_1 + \underline{q}_2)/(\underline{q}_1^2 + r)$

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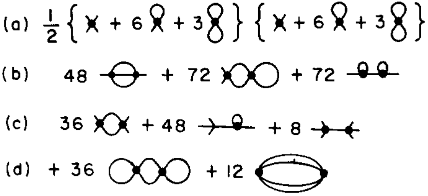

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  4. to each vertex a factor $u(2\pi)^d$ times delta function over sum of four incoming frequencies

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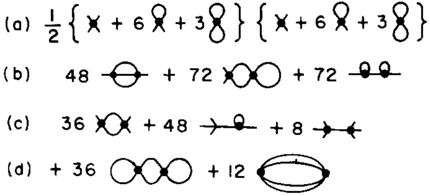

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  5. each external leg obtains spin fluctuation variable $\sigma_<(\underline{q}) = \zeta\sigma'(2\underline{q})$

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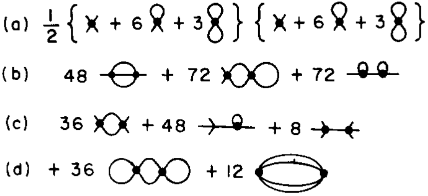

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  4. to each vertex a factor $u(2\pi)^d$ times delta function over sum of four incoming frequencies
  5. each external leg obtains spin fluctuation variable $\sigma_<(\underline{q}) = \zeta\sigma'(2\underline{q})$
  6. integrate over frequencies of internal and external lines according to rule 2.

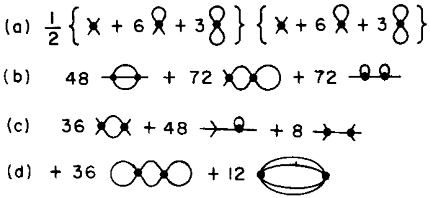
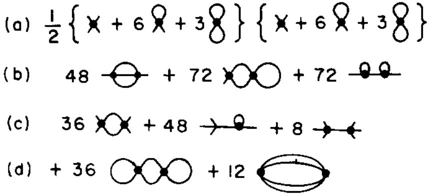


Fig. 4.3. Second order graphs after σ_1 integration.

- now achieved (up to certain order in u): calculate $\exp -\widetilde{\mathcal{H}}'_{s^4}[\sigma, r, u]$ in terms of $r, u, \sigma_<$



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- now achieved (up to certain order in u): calculate $\exp -\widetilde{\mathcal{H}}'_{s^4}[\sigma,r,u]$ in terms of $r, u, \sigma_<$
- really want: $\widetilde{\mathcal{H}}'_{s^4}[\sigma',r',u']$, thus remains to take logarithm, replace $\sigma_<(\underline{q})$ by $\sigma'(\underline{q}')$, and express r', u' in terms of r, u

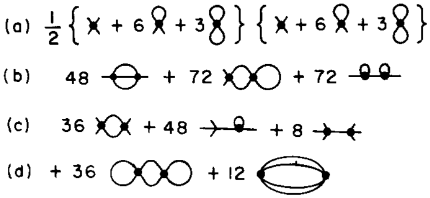

Fig. 4.3. Second order graphs after σ_1 integration.

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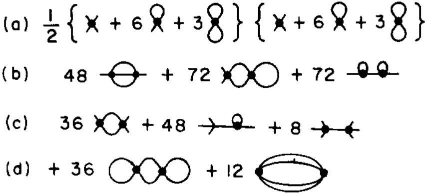

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- only study σ'-dependent terms, thus drop $Z_{\mathrm{Gauss}}(r)$ and all **diagrams without external lines**

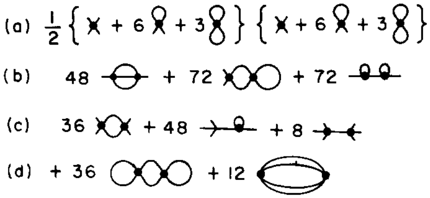

Fig. 4.3. Second order graphs after σ_1 integration.

- really want: $\widetilde{\mathcal{H}}'_{s^4}[\sigma', r', u']$, thus remains to take logarithm, replace $\sigma_<(\underline{q})$ by $\sigma'(\underline{q}')$, and express r', u' in terms of r, u
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- only study σ'-dependent terms, thus drop $Z_{\mathrm{Gauss}}(r)$ and all **diagrams without external lines**
- replacing $\sigma_<(\underline{q})$ by $\sigma'(\underline{q}')$ by substituting $\zeta\sigma'(2\underline{q})$ for external line instead of $\sigma_<(\underline{q})$

Outline

- changing variables from $\underline{q}$ to $\underline{q}' = 2\underline{q}$ and using the results obtained up to now we arrive at our desired primed Hamiltonian $\widetilde{\mathcal{H}}'_{s^4[\sigma',r',u']}$ in the form

$$\widetilde{\mathcal{H}}'_{s^4}[\sigma',r,u] \;=\; \tfrac{1}{2}\int \mathrm{d}^d q\; u'_2(\underline{q},r,u)\;\sigma'(\underline{q})\,\sigma'(-\underline{q})$$
$$+\int \mathrm{d}^d q_1 \int \mathrm{d}^d q_2 \int \mathrm{d}^d q_3\;\; u'_4(\underline{q}_1,\underline{q}_2,\underline{q}_3,r,u)\;\; \sigma'(\underline{q}_1)\,\sigma'(\underline{q}_2)\,\sigma'(\underline{q}_3)\,\sigma'(-\underline{q}_1-\underline{q}_2-\underline{q}_3)$$
$$+\;\text{terms of order } (\sigma')^6 \text{ and higher } \ldots$$

$$\widetilde{\mathcal{H}}'_{s^4[\sigma', r, u]} = \frac{1}{2} \int \mathrm{d}^d q \; u'_{2(\underline{q}, r, u)} \, \sigma'_{(\underline{q})} \, \sigma'_{(-\underline{q})}$$

$$+ \int \mathrm{d}^d q_1 \int \mathrm{d}^d q_2 \int \mathrm{d}^d q_3 \;\; u'_{4(\underline{q}_1, \underline{q}_2, \underline{q}_3, r, u)} \;\; \sigma'_{(\underline{q}_1)} \, \sigma'_{(\underline{q}_2)} \, \sigma'_{(\underline{q}_3)} \, \sigma'_{(-\underline{q}_1 - \underline{q}_2 - \underline{q}_3)}$$

$$+ \text{ terms of order } (\sigma')^6 \text{ and higher } \ldots$$

$$u'_{2(\underline{q}, r, u)} = \zeta^2 / 2^d \left\{ \frac{1}{4} \underline{q}^2 + r + 12u \int \mathrm{d}^d p \;\; \frac{1}{(\underline{p}^2 + r)} \right.$$

$$- 96 u^2 \int \mathrm{d}^d p_1 \int \mathrm{d}^d p_2 \;\; \frac{1}{(\underline{p}_1^2 + r)} \;\; \frac{1}{(\underline{p}_2^2 + r)} \;\; \frac{1}{\left((\frac{1}{2}\underline{q} - \underline{p}_1 - \underline{p}_2)^2 + r \right)}$$

$$\left. + \text{ terms of order } u^3 \text{ and higher } \ldots \right\}$$

$$\widetilde{\mathcal{H}}'_{s^4[\sigma',r,u]} = \tfrac{1}{2}\int \mathrm{d}^d q \; u'_{2(\underline{q},r,u)}\,\sigma'(\underline{q})\,\sigma'(-\underline{q})$$

$$+ \int \mathrm{d}^d q_1 \int \mathrm{d}^d q_2 \int \mathrm{d}^d q_3 \;\; u'_{4(\underline{q}_1,\underline{q}_2,\underline{q}_3,r,u)}\;\; \sigma'(\underline{q}_1)\,\sigma'(\underline{q}_2)\,\sigma'(\underline{q}_3)\,\sigma'(-\underline{q}_1-\underline{q}_2-\underline{q}_3)$$

$$+ \text{ terms of order } (\sigma')^6 \text{ and higher } \dots$$

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$$+ \text{ terms of order } u^3 \text{ and higher } \dots \Bigg\}$$

$$u'_{4(\underline{q}_1,\underline{q}_2,\underline{q}_3,r,u)} = \zeta^4/2^{3d} \Bigg\{ u - 12u^2 \int \mathrm{d}^d p \;\; \frac{1}{(\underline{p}^2 + r)} \;\; \frac{1}{((\frac{1}{2}\underline{q}_1 + \frac{1}{2}\,\underline{q}_2 - \underline{p})^2 + r)}$$

$$- \text{ 2 permutations}$$

$$+ \text{ terms of order } u^3 \text{ and higher } \dots \Bigg\}$$

$$\widetilde{\mathcal{H}}'_{s^4[\sigma',r,u]} = \frac{1}{2}\int \mathrm{d}^d q\ u'_2(\underline{q},r,u)\ \sigma'(\underline{q})\,\sigma'(-\underline{q})$$
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$$+ \text{ terms of order } (\sigma')^6 \text{ and higher } \ldots$$

- again make approximations to cast new Hamiltonian $\widetilde{\mathcal{H}}'_{s^4[\sigma',r',u']}$ in same form as original Hamiltonian

$$\widetilde{\mathcal{H}}_{s^4[\sigma,r,u]} = \frac{1}{2}\int \mathrm{d}^d q\ \left(\underline{q}^2 + r\right)\sigma(\underline{q})\,\sigma(-\underline{q})$$
$$+ u\int \mathrm{d}^d q_1 \int \mathrm{d}^d q_2 \int \mathrm{d}^d q_3\ \sigma(\underline{q}_1)\ \sigma(\underline{q}_2)\ \sigma(\underline{q}_3)\ \sigma(-\underline{q}_1-\underline{q}_2-\underline{q}_3)$$

Wilson shows: these approximations are good for dimensions near $d = 4$

- first calculate $u'_{2(\underline{q},r,u)}$ only up to linear order in u, set

$$\zeta = 2^{1+d/2}$$

- first calculate $u'_{2}(\underline{q},r,u)$ only up to linear order in u, set

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- further approximate

$$\int\limits_{|\underline{p}|>\frac{1}{2}}^{|\underline{p}|<1} \mathsf{d}^{d}p \;\frac{1}{(\underline{p}^2+r)} \;\approx\; \frac{1}{(1+r)} \overbrace{\int\limits_{|\underline{p}|>\frac{1}{2}}^{|\underline{p}|<1} \mathsf{d}^{d}p}^{=:\,4c}$$

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$$\int_{|\underline{p}|>\frac{1}{2}}^{|\underline{p}|<1} \mathrm{d}^d p \; \frac{1}{(\underline{p}^2 + r)} \approx \frac{1}{(1+r)} \overbrace{\int_{|\underline{p}|>\frac{1}{2}}^{|\underline{p}|<1} \mathrm{d}^d p}^{=:\,4c}$$

$$\int_{|\underline{p}|>\frac{1}{2}}^{|\underline{p}|<1} \mathrm{d}^d p \; \frac{1}{(\underline{p}^2 + r)} \; \frac{1}{\left((\frac{1}{2}\underline{q}_1 + \frac{1}{2}\,\underline{q}_2 - \underline{p})^2 + r\right)} \approx \frac{1}{(1+r)^2} \underbrace{\int_{|\underline{p}|>\frac{1}{2}}^{|\underline{p}|<1} \mathrm{d}^d p}_{=:\,4c}$$

- then renaming $u'_4(\underline{q}_1, \underline{q}_2, \underline{q}_3, r, u)$ by u' we can write the **recursion relations** as

$$r'(r,u) = 4\Big(r + 3cu/(1+r)\Big) + \text{ higher orders in } u$$
$$u'(r,u) = 2^{4-d}\Big(u - 9cu^2/(1+r)^2\Big) + \text{ higher orders in } u$$

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- repeating many times the process of integrating out the high fluctuation frequencies, after starting at initial couplings r_0 and u_0, and using recursion relations

$$r_{k+1\,(r_k,u_k)} = 4\Big(r_k + 3cu_k/(1+r_k)\Big)$$
$$u_{k+1\,(r_k,u_k)} = 2^{4-d}\Big(u_k - 9cu_k^2/(1+r_k)^2\Big)$$

  after k steps we get effective Hamiltonian $\widetilde{\mathcal{H}}^{(k)}_{s^4}$
  with integrated out all fluctuation frequencies $|\underline{q}| \in [2^{-k}, 1]$,
  i.e.: describes behaviour of fluctuations with low frequencies $|\underline{q}| \in [0, 2^{-k}]$

Outline

Outline

- for temperatures T above but near the critical temperature T_{crit},
  the critical exponent ν describes power law behavior of the correlation length $\xi(T)$:

$$\xi(T) = (\underbrace{T - T_{\text{crit}}}_{\tau})^{-\nu} =: \tau^{-\nu}$$

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$$\xi(T) = (\underbrace{T - T_{\text{crit}}}_{\tau})^{-\nu} =: \tau^{-\nu}$$

- we had sacrificed for simplicity the connection between spins $s_{\underline{n}}$ on lattice sites $\underline{n}$ and their discrete Fourier transform $\sigma[s, \underline{q}]$
- now we would like to have back something similar to the lattice spins, therefore introduce spin field:

$$s[\sigma, \underline{x}] := \int_{|\underline{q}|<1} \mathrm{d}^d q \;\; \mathrm{e}^{\mathrm{i}\underline{q}\underline{x}} \, \sigma(\underline{q})$$

- **2-point spin correlation**:

$$\Gamma(\underline{x},r,u) \;:=\; Z^{-1}(r,u) \int \mathcal{D}\sigma \;\; s(\underline{x})\, s(\underline{0}) \;\; \mathsf{e}^{-\widetilde{\mathcal{H}}_{s^4}[\sigma,r,u]}$$

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$$\Gamma(\underline{x},r,u) \; = \; \int \mathrm{d}^d q \;\; \mathrm{e}^{\mathrm{i}\underline{q}\underline{x}} \; \widetilde{\Gamma}(\underline{q},r,u)$$

- define:

$$\xi^2(r,u) \; := \; -\left[\frac{\mathrm{d}\widetilde{\Gamma}(\underline{q},r,u)/\mathrm{d}\underline{q}^2}{\widetilde{\Gamma}(\underline{q},r,u)} \right]_{\underline{q}=\underline{0}}$$

call so defined ξ **effective range of correlation**

- integrating out high frequencies we obtain a new 2-point spin correlation, which is related to the original one by the **scaling relation**

$$\widetilde{\Gamma}'(\underline{q}'{=}2\underline{q}, r', u') \;=\; 2^d/\zeta^2 \;\widetilde{\Gamma}(\underline{q}, r, u) \;=\; \tfrac{1}{4}\;\widetilde{\Gamma}(\underline{q}, r, u)$$

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- the **scaling relation** for effective correlation ranges is

$$\xi'(r',u') \;=\; \tfrac{1}{2}\,\xi(r,u)$$

- at a fixed point of the recursion relations we will have

$$r_{k+1}(r_k, u_k) = r_k$$
$$u_{k+1}(r_k, u_k) = u_k$$

which implies for the effective correlation range $\xi_{k+1}(r_{k+1}, u_{k+1}) = \xi_k(r_k, u_k)$

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  which implies for the effective correlation range $\xi_{k+1}(r_{k+1},u_{k+1}) = \xi_k(r_k,u_k)$
- together with $\xi_{k+1} = \frac{1}{2}\xi_k$ this implies
  that at fixed points we have either vanishing or infinite effective correlation range,
  but the vanishing case can be ruled out,
  thus fixpoints correspond to critical points of the system

$$r_{k+1}(r_k, u_k) = 4\Big(r_k + 3cu_k/(1+r_k)\Big)$$

$$u_{k+1}(r_k, u_k) = 2^{4-d}\Big(u_k - 9cu_k^2/(1+r_k)^2\Big)$$

- as in Gaussian model, recursion relations always have one **trivial fixpoint** $r^* = u^* = 0$, also called **Gaussian fixpoint**

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- as in Gaussian model, recursion relations always have one **trivial fixpoint** $r^* = u^* = 0$, also called **Gaussian fixpoint**
- thus for $d \geq 4$ recursion relations only have Gaussian fixpoint $r^* = u^* = 0$

$$
\begin{aligned}
r_{k+1\,(r_k,u_k)} &= 4\Big(r_k + 3cu_k/(1{+}r_k)\Big) \\
u_{k+1\,(r_k,u_k)} &= 2^{4-d}\Big(u_k - 9cu_k^2/(1{+}r_k)^2\Big)
\end{aligned}
$$

- $d < 4$: choose u_0 small, with increasing iterations u_k will increase until the second term in its recursion relation becomes comparable to the first, and a new non-Gaussian fixpoint is approximately given by

$$
\begin{aligned}
u^* &\approx \frac{2^{4-d} - 1}{9c} \\
r^* &\approx -4c\,u^* = -\frac{4}{9}\,(2^{4-d} - 1)
\end{aligned}
$$

$$u^* \approx \frac{2^{4-d} - 1}{9c}$$

$$r^* \approx -4c\, u^* = -\frac{4}{9}\,(2^{4-d} - 1)$$

- defining

$$\epsilon := 4 - d$$

and **expanding in** ϵ we find for small, positive ϵ that fixpoint is given by

$$u^* \approx \frac{\epsilon \ln 2}{9c} > 0$$

$$r^* \approx -\frac{4}{9}\,\epsilon \ln 2 < 0$$

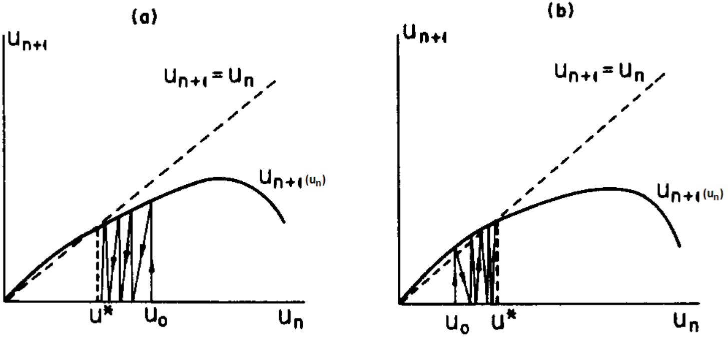

Fig. 4.4. The iteration formula for u_n.

(a) If $u_0 > u^*$, the iteration scheme leads u_n to u^* from above.

(b) If $u_0 < u^*$, u_n approaches u^* from below.

Outline

- recall definitions (after rescaling j to 1) of temperature-dependent couplings

$$r \equiv r_0 \; := \; \frac{b}{j} - 2d \; = \; \frac{B}{k_b T} - 2d$$
$$u \equiv u_0 \; := \; l/j^2 \; = \; L/(k_b T)$$

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- in non-Gaussian case $u \neq 0$ we can be at a critical temperature without having r_0 and u_0 at fixed point values, because for being at T_{crit} one only needs to adjust one parameter, but for being at fixed point one has to adjust two parameters

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- study theory for temperatures near T_{crit} to calculate critical exponent ν
- first step: study sequence $\{(r_{k(T)}, u_{k(T)})\}$ generated by repeatedly iterating recursion relations starting from $(r_{0(T)}, u_{0(T)})$

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$$r \equiv r_0 := \frac{b}{j} - 2d = \frac{B}{k_b T} - 2d$$
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- in non-Gaussian case $u \neq 0$ we can be at a critical temperature without having r_0 and u_0 at fixed point values, because for being at T_{crit} one only needs to adjust one parameter, but for being at fixed point one has to adjust two parameters
- study theory for temperatures near T_{crit} to calculate critical exponent ν
- first step: study sequence $\{(r_k(T), u_k(T))\}$ generated by repeatedly iterating recursion relations starting from $(r_0(T), u_0(T))$
- expect $r_k(T_{\text{crit}}) \stackrel{k\to\infty}{\longrightarrow} r^*$ and $u_k(T_{\text{crit}}) \stackrel{k\to\infty}{\longrightarrow} u^*$ as on line A in figure 4.5

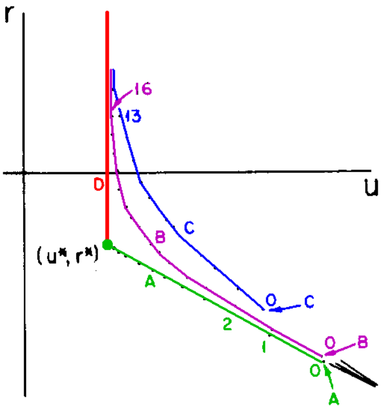

Fig. 4.5.
Plot of the iteration scheme for three different initial choices of parameters.
Sequence A(T=T_C) goes into the fixpoint (u^*, r^*).
Sequences B and C begin for choices of u and r slightly removed from criticality.
These sequences eventually deviate far from the fixed point but approach the unique curve D.

- because recursion relations analytic, expect linear behavior for fixed k and T near T_{crit}:

$$
\begin{aligned}
r_k(T) &= r_k(T_{\text{crit}}) + \rho_k(\overbrace{T - T_{\text{crit}}}^{\tau}) \\
u_k(T) &= u_k(T_{\text{crit}}) + \mu_k(\underbrace{T - T_{\text{crit}}}_{\tau})
\end{aligned}
$$

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$$\begin{aligned} r_k(T) &= r_k(T_{\text{crit}}) + \rho_k(\overbrace{T - T_{\text{crit}}}^{\tau}) \\ u_k(T) &= u_k(T_{\text{crit}}) + \mu_k(\underbrace{T - T_{\text{crit}}}_{\tau}) \end{aligned}$$

- now if k is sufficiently large, then

$$\begin{aligned} r_k(T_{\text{crit}}) &\approx r^* & \Rightarrow \quad r_k(T) &\approx r^* \\ u_k(T_{\text{crit}}) &\approx u^* & \Rightarrow \quad u_k(T) &\approx u^* \end{aligned}$$

- in matrix form this reads

$$\begin{pmatrix} r_{k+1} - r^* \\ u_{k+1} - u^* \end{pmatrix} \approx M \begin{pmatrix} r_k - r^* \\ u_k - u^* \end{pmatrix}$$

$$M = \begin{pmatrix} 4 - \frac{12cu^*}{(1+r^*)^2} & \frac{12c}{(1+r^*)} \\ \frac{2^\epsilon 18cu^{*2}}{(1+r^*)^3} & 2^\epsilon - \frac{2\epsilon 18cu^*}{(1+r^*)^2} \end{pmatrix}$$

- in matrix form this reads

$$\begin{pmatrix} r_{k+1} - r^* \\ u_{k+1} - u^* \end{pmatrix} \approx M \begin{pmatrix} r_k - r^* \\ u_k - u^* \end{pmatrix}$$

- after iterating the linearized recursion relations many times the relation reads

$$\begin{pmatrix} r_{k+n} - r^* \\ u_{k+n} - u^* \end{pmatrix} \approx M^n \begin{pmatrix} r_k - r^* \\ u_k - u^* \end{pmatrix}$$

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$$\begin{pmatrix} r_{k+n} - r^* \\ u_{k+n} - u^* \end{pmatrix} \approx M^n \begin{pmatrix} r_k - r^* \\ u_k - u^* \end{pmatrix}$$

- by diagonalizing M explicit form of M^n neglecting eigenvalue 1 can be obtained:

$$M^n = \lambda^n \begin{pmatrix} 1 & v \\ w & vw \end{pmatrix} \tag{1}$$

$$\begin{aligned} \lambda &= 4\left(1 - \tfrac{1}{3}\epsilon \ln 2\right) \\ v &= +4c\left(1 + \tfrac{5}{9}\epsilon \ln 2\right) \\ w &= -4c\left(1 + \tfrac{5}{9}\epsilon \ln 2\right) \end{aligned}$$

- because we are considering $T \approx T_{\text{crit}}$, for fixed k we have

$$r_k(T) - r_k(T_{\text{crit}}) \sim (T - T_{\text{crit}}) = \tau$$
$$u_k(T) - u_k(T_{\text{crit}}) \sim (T - T_{\text{crit}}) = \tau$$

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$$u_{k(T)} - u_{k(T_{\text{crit}})} \sim (T - T_{\text{crit}}) = \tau$$

so that for sufficiently large k

$$(r_{k(T)} - r^*) + v(u_{k(T)} - u^*) = c_k(T - T_{\text{crit}}) = c_k \, \tau$$

- because we are considering $T \approx T_{\text{crit}}$, for fixed k we have

$$
\begin{aligned}
r_{k(T)} - r_{k(T_{\text{crit}})} &\sim (T - T_{\text{crit}}) = \tau \\
u_{k(T)} - u_{k(T_{\text{crit}})} &\sim (T - T_{\text{crit}}) = \tau
\end{aligned}
$$

so that for sufficiently large k

$$
(r_{k(T)} - r^*) + v(u_{k(T)} - u^*) = c_k(T - T_{\text{crit}}) = c_k \, \tau
$$

and thus from the matrix equation (1) we deduce

$$
\begin{aligned}
r_{k+n} - r^* &= \lambda^n \, c_k \, (T - T_{\text{crit}}) \\
u_{k+n} - u^* &= \lambda^n \, w \, c_k \, (T - T_{\text{crit}})
\end{aligned}
\tag{2}
$$

- now we can calculate ν, we had seen above that $\xi_{(r,u)}$ defined for any r, u, since these depend on T we have $\xi = \xi_{(T)}$
- we had also seen the scaling relation

$$\xi_{(r_{k+n}, u_{k+n})} = 2^{-(k+n)} \xi_{(r_0, u_0)}$$

- from (2) we read off the following:

$$
\begin{aligned}
\Big[r_{k+n+1}(T) - r^*\Big]_{T-T_{\text{crit}}=\tau/\lambda} &= \Big[r_{k+n}(T) - r^*\Big]_{T-T_{\text{crit}}=\tau} \\
\Big[u_{k+n+1}(T) - u^*\Big]_{T-T_{\text{crit}}=\tau/\lambda} &= \Big[u_{k+n}(T) - u^*\Big]_{T-T_{\text{crit}}=\tau}
\end{aligned}
$$

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$$\left[u_{k+n+1}(T) - u^*\right]_{T-T_{\mathsf{crit}}=\tau/\lambda} = \left[u_{k+n}(T) - u^*\right]_{T-T_{\mathsf{crit}}=\tau}$$

or written more clearly

$$r_{k+n+1}(T_{\mathsf{crit}}+\tau/\lambda) = r_{k+n}(T_{\mathsf{crit}}+\tau)$$
$$u_{k+n+1}(T_{\mathsf{crit}}+\tau/\lambda) = u_{k+n}(T_{\mathsf{crit}}+\tau)$$

or written more clearly

$$r_{k+n+1}(T_{\mathsf{crit}}+\tau/\lambda) \;=\; r_{k+n}(T_{\mathsf{crit}}+\tau)$$
$$u_{k+n+1}(T_{\mathsf{crit}}+\tau/\lambda) \;=\; u_{k+n}(T_{\mathsf{crit}}+\tau)$$

thus

$$\xi_{(r_{k+n+1},u_{k+n+1})_{T=T_{\mathsf{crit}}+\tau/\lambda}} \;=\; \xi_{(r_{k+n},u_{k+n})_{T=T_{\mathsf{crit}}+\tau}}$$

- thus

$$\xi(r_{k+n+1}, u_{k+n+1})_{T=T_{\text{crit}}+\tau/\lambda} = \xi(r_{k+n}, u_{k+n})_{T=T_{\text{crit}}+\tau}$$

which with the scaling relation implies

$$2^{-(k+n+1)}\xi(T_{\text{crit}}+\tau/\lambda) = 2^{-(k+n)}\xi(T_{\text{crit}}+\tau) \qquad (3)$$

which with the scaling relation implies

$$2^{-(k+n+1)}\xi_{(T_{\mathsf{crit}}+\tau/\lambda)} = 2^{-(k+n)}\xi_{(T_{\mathsf{crit}}+\tau)} \tag{3}$$

thus assuming power law behavior for the effective correlation range

$$\xi_{(T_{\mathsf{crit}}+\tau)} \sim \tau^{-\nu}$$

(3) for arbitrarily small τ gives us

$$(\tau/\lambda)^{-\nu} = 2\tau^{-\nu}$$

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thus

$$\nu = \frac{\ln 2}{\ln \lambda} = \frac{1}{2 - \epsilon/3}$$

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- for nonzero ϵ our ν differs from value $\frac{1}{2}$ obtained in Gaussian and mean field model,
  experimentally (d=3) $\nu \approx 0.6...0.7$,
  in 3-dim. (2-dim.) Ising model one has $\nu = 0.64\ (1.0)$,
  here obtained for $(d = 3), (\epsilon = 1)$ that $\nu \approx 0.58$,
  for $(d = 2), (\epsilon = 2)$ that $\nu \approx \frac{2}{3}$