

## 1 $s^4$ -model on a lattice

- consider cubic lattice of spatial dimension  $d$  and lattice constant  $a$ , site on lattice denoted by  $\underline{n} \in \mathbb{Z}^d$ , thus dimensionless:  $[\underline{n}] = 1$  and  $[a] = \text{length}$
- lattice site  $\underline{n}$  located in space at position  $\underline{x}_{\underline{n}} = a\underline{n}$ , thus  $[\underline{x}] = \text{length}^d$
- general definition of **partition function** depending on temperature  $T$  and coupling constants  $g$ :

$$Z(T, \underline{g}) := \sum_{\text{states}} e^{-\mathcal{H}[\text{state}, \underline{g}]/(k_B T)}$$

- Hamiltonian of **Gaussian model** in configuration space:

$$\mathcal{H}_{\text{Gauss}[s, J, B]} = -J \sum_{\underline{n}} \sum_{i=1}^d s_{\underline{n}} s_{\underline{n}+\underline{e}_i} + \frac{B}{2} \sum_{\underline{n}} s_{\underline{n}}^2$$

- **Ising model**: dimensionless spins  $s$  live on discrete lattice and takes discrete values  $s_{\underline{n}} = \pm 1$
- Gaussian model:  $s$  on discrete lattice, but continuous  $s_{\underline{n}} \in \mathbb{R}$
- redefine constants in order to save writing lots of  $(k_B T)$ 's:  $j = J/(k_B T)$  and  $b = B/(k_B T)$

$$\tilde{\mathcal{H}}_{\text{Gauss}[s, j, b]} := \frac{\mathcal{H}_{\text{Gauss}[s]}}{k_B T} = -j \sum_{\underline{n}} \sum_{i=1}^d s_{\underline{n}} s_{\underline{n}+\underline{e}_i} + \frac{b}{2} \sum_{\underline{n}} s_{\underline{n}}^2$$

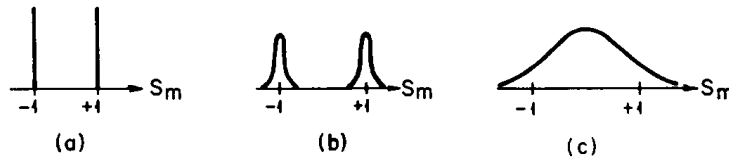


Fig. 3.1. The transition from the Ising to the Gaussian model.

The Ising model (a) has spin up or spin down at each lattice site.

Model (b) has spin variables which peak about the Ising values.

The Gaussian model (c) has spin variables at each site, with smooth Gaussian distributions about zero

- Hamiltonian of  $s^4$ -model in configuration space:

$$\mathcal{H}_{s^4[s, J, B, L]} = \underbrace{-J \sum_{\underline{n}} \sum_{i=1}^d s_{\underline{n}} s_{\underline{n}+\underline{e}_i}}_{\mathcal{H}_{\text{Gauss}[s]}} + \frac{B}{2} \sum_{\underline{n}} s_{\underline{n}}^2 + \underbrace{L \sum_{\underline{n}} s_{\underline{n}}^4}_{\mathcal{H}_{\text{Int}[s]}}$$

- define  $l = L/(k_B T)$

$$\tilde{\mathcal{H}}_{s^4[s,j,b,l]} := \frac{\mathcal{H}_{s^4[s]}}{k_B T} = -j \sum_{\underline{n}} \sum_{i=1}^d s_{\underline{n}} s_{\underline{n}+\underline{e}_i} + \frac{b}{2} \sum_{\underline{n}} s_{\underline{n}}^2 + l \sum_{\underline{n}} s_{\underline{n}}^4$$

- $s^4$ -model approaches Ising model for  $l \rightarrow \infty$  and  $b \rightarrow -\infty$  with fixed  $b/l = -4$  if the lattice spins  $s$  are properly rescaled
- partition function becomes

$$Z_{(T,J,B,L)} = Z_{(j,b,l)} = \int ds \exp \sum_{\underline{n}} \left( \sum_{i=1}^d j s_{\underline{n}} s_{\underline{n}+\underline{e}_i} - \frac{b}{2} s_{\underline{n}}^2 - l s_{\underline{n}}^4 \right)$$

$$\int ds := \prod_{\underline{n}} \int_{-\infty}^{+\infty} ds_{\underline{n}}$$

## 2 Spatial fluctuation variables

- introduce dimensionless **spatial spin fluctuations**  $\sigma_{[q,s]}$  with **fluctuation frequency (wave vector)**  $\underline{q}$  as discrete Fourier transform of lattice spins  $s$ :

$$\sigma_{[q,s]} := \sum_{\underline{n}} e^{-i\underline{q}\underline{n}} s_{\underline{n}}$$

frequency has continuous values in first Brillouin zone:  $\underline{q} \in [-\pi, +\pi]^d$ , and  $q^j$  denotes the  $j^{\text{th}}$  component of  $\underline{q}$

$$\int d^d q := \prod_{j=1}^d \int_{-\pi}^{+\pi} \frac{dq^j}{2\pi}$$

- can write Hamiltonian in terms of spin fluctuations with  $\tilde{r} := b - 2dj$ :

$$\tilde{\mathcal{H}}_{s^4[s,\tilde{r},j,l]} = \frac{1}{2} \int d^d q \sigma_{[q,s]} \sigma_{[-q,s]} \left( \tilde{r} + j \sum_{k=1}^d \left| e^{iq^k} - 1 \right|^2 \right)$$

$$+ l \int d^d q_1 \int d^d q_2 \int d^d q_3 \int d^d q_4 \sigma_{[q_1,s]} \sigma_{[q_2,s]} \sigma_{[q_3,s]} \sigma_{[q_4,s]} (2\pi)^d \delta^{(d)}(q_1+q_2+q_3+q_4)$$

- now three changes will be applied in order to simplify our calculations

- first: replace  $\sum_k |\exp(iq^k) - 1|^2 = 2(1 - \cos q^k)$  by its form for small  $q$ , i.e.: by  $q^2$ , which is no essential change of the model, because our interest lies in the long wavelengths behavior which comes from the small fluctuation frequencies
- second: rescale the spins such that  $j = 1$ , i.e.:  $s_n \rightarrow s_n/\sqrt{j}$
- third: limit range of integration from  $[-\pi, +\pi]^d$  to  $|q| \in [0, 1]$ , which for the same reason is no essential change in the model and from now on

$$\int d^d q := \int_{|q| \leq 1} \prod_{k=1}^d \frac{dq^k}{2\pi}$$

however conceptually there is a difficulty: with the new restricted range we can no longer relate the functional variable  $\sigma_{[q,s]}$  to the ordinary variables  $s_n$ , therefore we are forced to consider the spin fluctuations as variables  $\sigma_{(q)}$  in their own right and change the definition of the partition function from ordinary integrals over the  $s_n$  to functional integrals over  $\sigma_{(q)}$

- Wilson argues that this is not a problem for our purposes, because the method of translating the integration variable is valid for functional integrals as well as for ordinary integrals, because functional integrals are generally defined as limits of ordinary integrals, and each of these can be translated
- with  $r := \tilde{r}/j = \frac{b}{j} - 2d$  and  $u := l/j^2$  after these three changes we get

$$\begin{aligned} \tilde{\mathcal{H}}_{s^4}[\sigma, r, u] &= \frac{1}{2} \int d^d q \overbrace{\left( q^2 + r \right) \sigma_{(q)} \sigma_{(-q)}}^{\tilde{\mathcal{H}}_{\text{Gauss}}[\sigma, r]} \\ &\quad + u \underbrace{\int d^d q_1 \int d^d q_2 \int d^d q_3 \int d^d q_4 \sigma_{(q_1)} \sigma_{(q_2)} \sigma_{(q_3)} \sigma_{(q_4)} (2\pi)^d \delta^{(d)}(q_1 + q_2 + q_3 + q_4)}_{\tilde{\mathcal{H}}_{\text{Int}}[\sigma, u]} \\ Z_{(r, u)} &:= \int \mathcal{D}\sigma e^{-\tilde{\mathcal{H}}_{s^4}[\sigma, r, u]} \end{aligned}$$

### 3 Averaging out high frequency fluctuations

#### 3.1 Separating high and low frequencies

- now we attempt to define new physical system (denoted by a prime), in which high frequency modes of present system are **integrated out**, i.e., statistically averaged out
- **effective Hamiltonian** of new system will be designed as similar as possible to  $\tilde{\mathcal{H}}_{s^4}[\sigma, r, u]$ , this will involve considerable simplifications/approximations

- in place of couplings  $r, u$  we will find new couplings  $r'(r, u)$  and  $u'(r, u)$ , one of our aims is to find these **recursion relations** relating the new with the original couplings
- construction works as follows:
- integrals of spin fluctuations in original system are over frequencies  $|\underline{q}| \in [0, 1]$ , new "primed" system obtained by averaging about high frequency modes  $|\underline{q}| \in [\frac{1}{2}, 1]$
- this in analogy of **Kadanoff's block spins**: there one obtains an effective interaction, which involves only block spins, by averaging over all non-block-spin variables
- in our case we integrate out the rapid fluctuations (high frequencies), and the remaining long wavelength fluctuations correspond to the block spins
- write function  $\sigma(\underline{q})$  as sum over **partition of unity**  $p$ :

$$\sigma(\underline{q}) = \sigma_{<}(\underline{q}) + \sigma_{>}(\underline{q})$$

$$\sigma_{\leq}(\underline{q}) := p_{\leq}(\underline{q}) \sigma(\underline{q}) \quad \begin{array}{l} p_{\leq}(\underline{q}) \in [0, 1] \\ p_{<}(\underline{q}) + p_{>}(\underline{q}) = 1 \quad \forall \underline{q} \end{array}$$

- Wilson chooses discontinuous partition of unity: Heaviside step functions  $p_{\leq}(\underline{q}) = \theta(\mp(|\underline{q}| - \frac{1}{2}))$ , thus

$$\begin{aligned} \sigma_{<}(\underline{q}) &:= p_{<}(\underline{q}) \sigma(\underline{q}) = \theta\left(\frac{1}{2} - |\underline{q}|\right) \sigma(\underline{q}) = \begin{cases} \sigma(\underline{q}) & |\underline{q}| < \frac{1}{2} \\ 0 & |\underline{q}| > \frac{1}{2} \end{cases} \\ \sigma_{>}(\underline{q}) &:= p_{>}(\underline{q}) \sigma(\underline{q}) = \theta\left(|\underline{q}| - \frac{1}{2}\right) \sigma(\underline{q}) = \begin{cases} 0 & |\underline{q}| < \frac{1}{2} \\ \sigma(\underline{q}) & |\underline{q}| > \frac{1}{2} \end{cases} \end{aligned}$$

- in Gaussian Hamiltonian the high and low frequency modes decouple, there is NO interaction between fluctuations of high and low frequency:

$$\begin{aligned} \tilde{\mathcal{H}}_{\text{Gauss}[\sigma, r]} &= \frac{1}{2} \int d^d \underline{q} \left( \underline{q}^2 + r \right) \sigma(\underline{q}) \sigma(-\underline{q}) \\ &= \frac{1}{2} \left( \int_{|\underline{q}| < \frac{1}{2}} + \int_{|\underline{q}| > \frac{1}{2}} \right) d^d \underline{q} \left( \underline{q}^2 + r \right) \left[ \sigma_{<}(\underline{q}) \sigma_{<}(-\underline{q}) + \overbrace{\sigma_{<}(\underline{q}) \sigma_{>}(-\underline{q})}^0 \right. \\ &\quad \left. + \underbrace{\sigma_{>}(\underline{q}) \sigma_{<}(-\underline{q})}_0 + \sigma_{>}(\underline{q}) \sigma_{>}(-\underline{q}) \right] \\ &= \underbrace{\frac{1}{2} \int_{|\underline{q}| < \frac{1}{2}} d^d \underline{q} \left( \underline{q}^2 + r \right) \sigma_{<}(\underline{q}) \sigma_{<}(-\underline{q})}_{\tilde{\mathcal{H}}_{\text{Gauss}[\sigma_{<}, r]}} + \underbrace{\frac{1}{2} \int_{|\underline{q}| > \frac{1}{2}} d^d \underline{q} \left( \underline{q}^2 + r \right) \sigma_{>}(\underline{q}) \sigma_{>}(-\underline{q})}_{\tilde{\mathcal{H}}_{\text{Gauss}[\sigma_{>}, r]}} \end{aligned}$$

- however in  $\sigma^4$ -interaction no decoupling occurs, there IS an interaction between fluctuations of high and low frequencies: the Hamiltonian

$$\tilde{\mathcal{H}}_{\text{Int}[\sigma,u]} = u \int d^d q_1 \int d^d q_2 \int d^d q_3 \int d^d q_4 \sigma_{(\underline{q}_1)} \sigma_{(\underline{q}_2)} \sigma_{(\underline{q}_3)} \sigma_{(\underline{q}_4)} (2\pi)^d \delta^{(d)}(\underline{q}_1 + \underline{q}_2 + \underline{q}_3 + \underline{q}_4)$$

(in addition to the decoupling parts) also contains high-low frequency interactions like

$$u \int d^d q_1 \int d^d q_2 \int d^d q_3 \int d^d q_4 \sigma_{<(\underline{q}_1)} \sigma_{<(\underline{q}_2)} \sigma_{>(\underline{q}_3)} \sigma_{>(\underline{q}_4)} (2\pi)^d \delta^{(d)}(\underline{q}_1 + \underline{q}_2 + \underline{q}_3 + \underline{q}_4)$$

$$|\underline{q}_1| < \frac{1}{2} \quad |\underline{q}_2| < \frac{1}{2} \quad |\underline{q}_3| > \frac{1}{2} \quad |\underline{q}_4| > \frac{1}{2}$$

and therefore we have  $\tilde{\mathcal{H}}_{\text{Int}[\sigma,u]} \neq \tilde{\mathcal{H}}_{\text{Int}[\sigma_{<},u]} + \tilde{\mathcal{H}}_{\text{Int}[\sigma_{>},u]}$

- partition function can now be written as

$$Z(r,u) := \int \mathcal{D}\sigma \, e^{-\tilde{\mathcal{H}}_{s^4}[\sigma,r,u]}$$

$$= \int \mathcal{D}\sigma_{<} \int \mathcal{D}\sigma_{>} \, e^{-\tilde{\mathcal{H}}_{s^4}[\sigma_{<} + \sigma_{>},r,u]}$$

- integrating out high frequency modes  $\sigma_{>}$  shall give us

$$Z(r,u) = Z(r',u') = \int \mathcal{D}\sigma' \, e^{-\tilde{\mathcal{H}}'_{s^4}[\sigma',r',u']}$$

with new effective couplings  $r', u'$  and rescaled fluctuations  $\sigma'$ , however at the moment the new primed Hamiltonian can only be expressed in terms of the original couplings:

$$e^{-\tilde{\mathcal{H}}'_{s^4}[\sigma,r,u]} = \int \mathcal{D}\sigma_{>} \, e^{-\tilde{\mathcal{H}}_{s^4}[\sigma_{<} + \sigma_{>},r,u]}$$

and the primed fluctuation modes are related to the original long wavelength modes by the **scaling relations**

$$\sigma'(\underline{q}' := 2\underline{q}) = \zeta \sigma_{<}(\underline{q}) \quad \forall |\underline{q}| \in [0, \frac{1}{2}]$$

i.e., the primed fluctuations are renormalized by the constant **scaling factor**  $\zeta$ , which remains to be determined, and depend on rescaled frequencies  $\underline{q}' := 2\underline{q} \in [0, 1]$

- because of  $\tilde{\mathcal{H}}_{\text{Gauss}}[\sigma, r] = \tilde{\mathcal{H}}_{\text{Gauss}}[\sigma_{<, r}] + \tilde{\mathcal{H}}_{\text{Gauss}}[\sigma_{>, r}]$  we can write

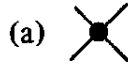
$$\begin{aligned} e^{-\tilde{\mathcal{H}}'_{s^4}[\sigma, r, u]} &= \int \mathcal{D}\sigma_{>} e^{-\tilde{\mathcal{H}}'_{s^4}[\sigma, r, u]} = \int \mathcal{D}\sigma_{>} e^{-\tilde{\mathcal{H}}_{\text{Gauss}}[\sigma, r] - \tilde{\mathcal{H}}_{\text{Int}}[\sigma, u]} \\ &= e^{-\tilde{\mathcal{H}}_{\text{Gauss}}[\sigma_{<, r}]} \int \mathcal{D}\sigma_{>} e^{-\tilde{\mathcal{H}}_{\text{Gauss}}[\sigma_{>, r}] - \tilde{\mathcal{H}}_{\text{Int}}[\sigma, u]} \end{aligned}$$

- the factor  $\exp -\tilde{\mathcal{H}}_{\text{Gauss}}[\sigma_{<, r}]$  can be easily expressed in the primed spin fluctuations:

$$\begin{aligned} \tilde{\mathcal{H}}_{\text{Gauss}}[\sigma_{<, r}] &= \frac{1}{2} \int_{|\underline{q}| < \frac{1}{2}} d^d \underline{q} \left( \underline{q}^2 + r \right) \sigma_{<(\underline{q})} \sigma_{<(-\underline{q})} \\ &= \frac{1}{2} \left( \zeta^2 / 2^{d+2} \right) \int_{|\underline{q}'| < 1} d^d \underline{q}' \left( \underline{q}'^2 + 4r \right) \sigma'_{(\underline{q}')} \sigma'_{(-\underline{q}')} \end{aligned}$$

### 3.2 Perturbative analysis & diagrams

- now comes the **perturbative part**: assuming our coupling  $u := L/(j^2 k_B T)$  to be small:  $u \ll 1$ , we can work out a precise relation between the original and the primed Hamiltonian
- because of the high-low frequency fluctuation interactions, the nontrivial physics is contained in terms depending on  $u$ , now in order to obtain these terms exploit assumption of small  $u$  and expand  $\exp -\tilde{\mathcal{H}}_{\text{Int}}[\sigma, u]$  in powers of  $u$ , we denote  $\tilde{\mathcal{H}}_{\text{Int}}[\sigma, u]$  by a cross (vertex), with each of the four endpoints representing one of the four  $\sigma$ 's in  $\tilde{\mathcal{H}}_{\text{Int}}$ , and can write in **diagrammatic notation**



(b) 
$$\begin{aligned} \exp -\tilde{\mathcal{H}}_{\text{Int}}[\sigma, u] &= 1 - \tilde{\mathcal{H}}_{\text{Int}}[\sigma, u] + \frac{1}{2} \tilde{\mathcal{H}}_{\text{Int}}^2[\sigma, u] - \dots \\ &= 1 - \text{X} + \frac{1}{2} (\text{X X}) - \dots \end{aligned}$$

**Fig. 4.1.** (a) Graphical representation of  $\mathcal{H}_{\text{I}}$ .  
(b) Graphs for  $\exp\{-\mathcal{H}_{\text{I}}\}$ .

- using this expansion we can now attack the functional integral

$$\int \mathcal{D}\sigma_{>} e^{-\tilde{\mathcal{H}}_{\text{Gauss}}[\sigma_{>, r}] - \tilde{\mathcal{H}}_{\text{Int}}[\sigma, u]} = \int \mathcal{D}\sigma_{>} e^{-\tilde{\mathcal{H}}_{\text{Gauss}}[\sigma_{>, r}]} \left( 1 - \tilde{\mathcal{H}}_{\text{Int}}[\sigma, u] + \frac{1}{2} \tilde{\mathcal{H}}_{\text{Int}}^2[\sigma, u] - \dots \right)$$

- in order to evaluate this integral, we need to calculate **Gaussian integrals** of the type

$$I(r, \underline{q}_{m_1}, \dots, \underline{q}_{m_k}) := \int \mathcal{D}\sigma_{>} e^{-\tilde{\mathcal{H}}_{\text{Gauss}}[\sigma_{>}, r]} \sigma_{>}(\underline{q}_{m_1}) \dots \sigma_{>}(\underline{q}_{m_k})$$

wherein the indices  $m_1, \dots, m_k$  are (not necessarily different) natural numbers

- define functional integral

$$Z_{\text{Gauss}}(r) := \int \mathcal{D}\sigma_{>} e^{-\tilde{\mathcal{H}}_{\text{Gauss}}[\sigma_{>}, r]}$$

$Z_{\text{Gauss}}(r)$  contributes only a  $\sigma'(\underline{q}')$ -independent constant factor to  $\exp -\tilde{\mathcal{H}}'_{s4}[\sigma, r, u]$

- define **contraction**  $\overline{\sigma_{>} \sigma_{>}}$  of high frequency fluctuations as

$$\overline{\sigma_{>}(\underline{q}_{m_a}) \sigma_{>}(\underline{q}_{m_b})} := \theta(|\underline{q}_{m_a}| - \frac{1}{2}) (2\pi)^d \frac{\delta^{(d)}(\underline{q}_{m_a} + \underline{q}_{m_b})}{q_{m_a}^2 + r}$$

- evaluating Gaussian integrals of type  $I(r, \underline{q}_{m_1}, \dots, \underline{q}_{m_k})$  one finds their value is given by  $Z_{\text{Gauss}}(r)$  times the sum of all possible ways of contracting the  $\sigma_{>}(\underline{q}_{m_1}) \dots \sigma_{>}(\underline{q}_{m_k})$  in pairs such that all of them are contained in some contraction:

$$I(r, \underline{q}_{m_1}, \dots, \underline{q}_{m_k}) = Z_{\text{Gauss}}(r) \sum_{P(m_1, \dots, m_k)} \overline{\sigma_{>}(\underline{q}_{P(m_1)}) \sigma_{>}(\underline{q}_{P(m_2)})} \cdot \dots \cdot \overline{\sigma_{>}(\underline{q}_{P(m_{k-1})}) \sigma_{>}(\underline{q}_{P(m_k)})}$$

wherein the sum runs over all permutations  $P$  of the indices  $m_1, \dots, m_k$

- in order to translate this result into diagrams, we remember  $\sigma(\underline{q}) = \sigma_{<}(\underline{q}) + \sigma_{>}(\underline{q})$  and therefore have to consider two cases for each endpoint:

if the spin fluctuation frequency  $\underline{q}$  associated to the endpoint has length  $|\underline{q}| < \frac{1}{2}$ , i.e.:  $\sigma(\underline{q}) = \sigma_{<}(\underline{q})$ , then it is within the long wavelength regime, which we do not integrate out, thus it is NOT contracted and just left uncontracted as it is

whereas if  $|\underline{q}| > \frac{1}{2}$ , i.e.:  $\sigma(\underline{q}) = \sigma_{>}(\underline{q})$ , then it IS integrated out, and must be contracted with another endpoint whose frequency also has length greater than one half

- in diagrams we symbolize contractions by drawing a line between the corresponding endpoints, this line is called **propagator** and in our case represents the function  $(\underline{q}^2 + r)^{-1}$

- call  $q$  **internal line/frequency/momentum** if it is sitting on a line connecting two vertices or one vertex with itself, and **external line/frequency/momentum** if it sits on a line attached to one vertex only
- thus for our functional integral we obtain

$$\begin{aligned}
e^{-\tilde{\mathcal{H}}'_s[\sigma, r, u]} &= e^{-\tilde{\mathcal{H}}_{\text{Gauss}}[\sigma_{<}, r]} \int \mathcal{D}\sigma_{>} e^{-\tilde{\mathcal{H}}_{\text{Gauss}}[\sigma_{>}, r] - \tilde{\mathcal{H}}_{\text{Int}}[\sigma, u]} \\
&= e^{-\tilde{\mathcal{H}}_{\text{Gauss}}[\sigma_{<}, r]} \int \mathcal{D}\sigma_{>} e^{-\tilde{\mathcal{H}}_{\text{Gauss}}[\sigma_{>}, r]} \left( 1 - \tilde{\mathcal{H}}_{\text{Int}}[\sigma, u] + \frac{1}{2} \tilde{\mathcal{H}}_{\text{Int}}^2[\sigma, u] - \dots \right) \\
&= e^{-\tilde{\mathcal{H}}_{\text{Gauss}}[\sigma_{<}, r]} \int \mathcal{D}\sigma_{>} e^{-\tilde{\mathcal{H}}_{\text{Gauss}}[\sigma_{>}, r]} \left( 1 - \mathbf{X} + \frac{1}{2} (\mathbf{X} \mathbf{X}) - \dots \right)
\end{aligned}$$

- order  $u^0$

$$\int \mathcal{D}\sigma_{>} e^{-\tilde{\mathcal{H}}_{\text{Gauss}}[\sigma_{>}, r]} (1) =: Z_{\text{Gauss}}(r)$$

- order  $u^1$ : functional integral over  $\sigma_{>}$  is obtained graphically by considering all ways of contracting some endpoints pairwise and leaving some endpoints uncontracted, we divide all graphs in classes according to their topology, and represent each topology class by one graph with the cardinality of the class as a **weight factor (multiplicity)**



$$\begin{aligned}
& \int \mathcal{D}\sigma_{>} e^{-\tilde{\mathcal{H}}_{\text{Gauss}}[\sigma_{>}, r]} \left( - \text{✖} \right) \\
&= \int \mathcal{D}\sigma_{>} e^{-\tilde{\mathcal{H}}_{\text{Gauss}}[\sigma_{>}, r]} \left( -u \int d^d q_1 \int d^d q_2 \int d^d q_3 \int d^d q_4 \right. \\
&\quad \left. \sigma_{(\underline{q}_1)} \sigma_{(\underline{q}_2)} \sigma_{(\underline{q}_3)} \sigma_{(\underline{q}_4)} (2\pi)^d \delta^{(d)}(\underline{q}_1 + \underline{q}_2 + \underline{q}_3 + \underline{q}_4) \right) \\
&= -u Z_{\text{Gauss}}(r) \int d^d q_1 \int d^d q_2 \int d^d q_3 \int d^d q_4 (2\pi)^d \delta^{(d)}(\underline{q}_1 + \underline{q}_2 + \underline{q}_3 + \underline{q}_4) \\
&\quad \left\{ \sigma_{<}(\underline{q}_1) \sigma_{<}(\underline{q}_2) \sigma_{<}(\underline{q}_3) \sigma_{<}(\underline{q}_4) \right. \\
&\quad \left. + 6 \overbrace{\sigma_{>}(\underline{q}_1) \sigma_{>}(\underline{q}_2)} \sigma_{<}(\underline{q}_3) \sigma_{<}(\underline{q}_4) \right. \\
&\quad \left. + 3 \overbrace{\sigma_{>}(\underline{q}_1) \sigma_{>}(\underline{q}_2)} \overbrace{\sigma_{<}(\underline{q}_3) \sigma_{<}(\underline{q}_4)} \right\} \\
&= -u Z_{\text{Gauss}}(r) \int d^d q_1 \int d^d q_2 \int d^d q_3 \int d^d q_4 (2\pi)^d \delta^{(d)}(\underline{q}_1 + \underline{q}_2 + \underline{q}_3 + \underline{q}_4) \left\{ 1 \text{✖} + 6 \text{⦿} + 3 \text{⦿} \right\} \\
&= -u Z_{\text{Gauss}}(r) \left\{ 1 \int d^d q_1 \int d^d q_2 \int d^d q_3 \sigma_{<}(\underline{q}_1) \sigma_{<}(\underline{q}_2) \sigma_{<}(\underline{q}_3) \sigma_{<}(-(\underline{q}_1 + \underline{q}_2 + \underline{q}_3)) \right. \\
&\quad \left. + 6 \int d^d q_1 \int d^d q_3 \frac{\sigma_{<}(\underline{q}_3) \sigma_{<}(-\underline{q}_3)}{(q_1^2 + r)} \right. \\
&\quad \left. + 3 \int d^d q_1 \int d^d q_3 \frac{(2\pi)^d \delta^{(d)}(0)}{(\underline{q}_1^2 + r)(\underline{q}_3^2 + r)} \right\}
\end{aligned}$$

in the first integral we integrate over all frequencies fullfilling

$$|\underline{q}_1|, |\underline{q}_2|, |\underline{q}_3|, |-\underline{q}_1 - \underline{q}_2 - \underline{q}_3| < \frac{1}{2}$$

in the second over

$$|\underline{q}_1| > \frac{1}{2} \quad |\underline{q}_3| < \frac{1}{2}$$

and in the third over

$$|\underline{q}_1|, |\underline{q}_3| > \frac{1}{2}$$

- other diagrams are calculated similarly
- order  $u^2$ : diagrams shown below, weights include  $\frac{1}{2}$  in front of  $\tilde{\mathcal{H}}_{\text{Int}}^2$

$$\begin{aligned}
\text{(a)} \quad & \frac{1}{2} \left\{ \text{X} + 6 \text{X} + 3 \text{X} \right\} \left\{ \text{X} + 6 \text{X} + 3 \text{X} \right\} \\
\text{(b)} \quad & 48 \text{---} \text{---} + 72 \text{---} \text{---} + 72 \text{---} \text{---} \\
\text{(c)} \quad & 36 \text{---} \text{---} + 48 \text{---} \text{---} + 8 \text{---} \text{---} \\
\text{(d)} \quad & + 36 \text{---} \text{---} + 12 \text{---} \text{---}
\end{aligned}$$

Fig. 4.3. Second order graphs after  $\sigma_1$  integration.

- **any diagram computes according to following rules:**
  1. label frequencies/momenta in incoming sense at each vertex
  2. external frequencies range within  $|\underline{q}| < \frac{1}{2}$ , internal within  $|\underline{q}| \in [\frac{1}{2}, 1]$
  3. to each internal line connecting frequencies  $\underline{q}_1$  and  $\underline{q}_2$  associate propagator  $(2\pi)^d \delta^{(d)}(\underline{q}_1 + \underline{q}_2) / (q_1^2 + r)$
  4. to each four-point vertex a factor  $u(2\pi)^d$  times a delta function over the sum of the four incoming frequencies
  5. each external leg obtains spin fluctuation variable  $\sigma_{<}(\underline{q}) = \zeta \sigma'(2\underline{q})$
  6. integrate over frequencies of internal and external lines according to rule 2.
- what we have now achieved, up to a certain order in  $u$ , is calculating  $\exp -\tilde{\mathcal{H}}'_{s^4}[\sigma, r, u]$  in terms of  $r, u$  and  $\sigma_{<}$
- however what we really want is  $\tilde{\mathcal{H}}'_{s^4}[\sigma', r', u']$ , thus it remains to take the logarithm of the expression, replace  $\sigma_{<}(\underline{q})$  by  $\sigma'(\underline{q}')$ , and express  $r'$  and  $u'$  in terms of  $r$  and  $u$
- it can be shown that if and only if  $\tilde{\mathcal{H}}'_{\text{Int}}$  is the sum of all connected graphs to all orders in  $u$ , then  $\exp -\tilde{\mathcal{H}}'_{\text{Int}}$  is the sum of all connected and disconnected graphs to all orders in  $u$ , thus taking the logarithm is equivalent to **removing all disconnected diagrams**
- up to order  $u^2$  the disconnected diagrams are those in figure 4.3(a)
- moreover we only study  $\sigma'$ -dependent terms, thus we can drop  $Z_{\text{Gauss}}(r)$  and all **diagrams without external lines** (those of figure 4.3(d) and the 8-shaped diagram of first order in  $u$ )
- replacing  $\sigma_{<}(\underline{q})$  by  $\sigma'(\underline{q}')$  simply happens by substituting  $\zeta \sigma'(2\underline{q})$  for an external line instead of  $\sigma_{<}(\underline{q})$

### 3.3 Recovering the original Hamiltonian

- changing variables from  $\underline{q}$  to  $\underline{q}' = 2\underline{q}$  and using the results obtained up to now we arrive at our desired primed Hamiltonian  $\tilde{\mathcal{H}}'_{s^4}[\sigma', r', u']$  in the form

$$\begin{aligned}\tilde{\mathcal{H}}'_{s^4[\sigma',r,u]} &= \frac{1}{2} \int d^d q u'_2(\underline{q},r,u) \sigma'(\underline{q}) \sigma'(-\underline{q}) \\ &\quad + \int d^d q_1 \int d^d q_2 \int d^d q_3 u'_4(\underline{q}_1,\underline{q}_2,\underline{q}_3,r,u) \sigma'(\underline{q}_1) \sigma'(\underline{q}_2) \sigma'(\underline{q}_3) \sigma'(-\underline{q}_1-\underline{q}_2-\underline{q}_3) \\ &\quad + \text{terms of order } (\sigma')^6 \text{ and higher ...}\end{aligned}$$

with the functions

$$\begin{aligned}u'_2(\underline{q},r,u) &= \zeta^2/2^d \left\{ \frac{1}{4} \underline{q}^2 + r + 12u \int d^d p \frac{1}{(\underline{p}^2 + r)} \right. \\ &\quad \left. - 96u^2 \int d^d p_1 \int d^d p_2 \frac{1}{(\underline{p}_1^2 + r)} \frac{1}{(\underline{p}_2^2 + r)} \frac{1}{((\frac{1}{2}\underline{q} - \underline{p}_1 - \underline{p}_2)^2 + r)} \right. \\ &\quad \left. + \text{terms of order } u^3 \text{ and higher ...} \right\}\end{aligned}$$

$$\begin{aligned}u'_4(\underline{q}_1,\underline{q}_2,\underline{q}_3,r,u) &= \zeta^4/2^{3d} \left\{ u - 12u^2 \int d^d p \frac{1}{(\underline{p}^2 + r)} \frac{1}{((\frac{1}{2}\underline{q}_1 + \frac{1}{2}\underline{q}_2 - \underline{p})^2 + r)} \right. \\ &\quad \left. - 2 \text{ permutations} \right. \\ &\quad \left. + \text{terms of order } u^3 \text{ and higher ...} \right\}\end{aligned}$$

wherein we integrate about

$$|\underline{p}|, |\underline{p}_1|, |\underline{p}_2|, |\frac{1}{2}\underline{q} - \underline{p}_1 - \underline{p}_2|, |\frac{1}{2}\underline{q}_1 + \frac{1}{2}\underline{q}_2 - \underline{p}| \in [\frac{1}{2}, 1]$$

- now again we make some approximations in order to cast our new Hamiltonian  $\tilde{\mathcal{H}}'_{s^4[\sigma',r,u]}$  into the same form as the original Hamiltonian

$$\begin{aligned}\tilde{\mathcal{H}}_{s^4[\sigma,r,u]} &= \frac{1}{2} \int d^d q \left( \underline{q}^2 + r \right) \sigma(\underline{q}) \sigma(-\underline{q}) \\ &\quad + u \int d^d q_1 \int d^d q_2 \int d^d q_3 \sigma(\underline{q}_1) \sigma(\underline{q}_2) \sigma(\underline{q}_3) \sigma(-\underline{q}_1-\underline{q}_2-\underline{q}_3)\end{aligned}$$

Wilson shows: these approximations are good for dimensions near  $d = 4$

- first we calculate  $u'_2(\underline{q},r,u)$  only up to linear order in  $u$ , set

$$\zeta = 2^{1+d/2}$$

and further approximate

$$\int_{|p|>\frac{1}{2}}^{|p|<1} d^d p \frac{1}{(p^2+r)} \approx \frac{1}{(1+r)} \int_{|p|>\frac{1}{2}}^{|p|<1} d^d p \stackrel{=: 4c}{\approx}$$

$$\int_{|p|>\frac{1}{2}}^{|p|<1} d^d p \frac{1}{(p^2+r)} \frac{1}{((\frac{1}{2}q_1 + \frac{1}{2}q_2 - p)^2+r)} \approx \frac{1}{(1+r)^2} \int_{|p|>\frac{1}{2}}^{|p|<1} d^d p \stackrel{=: 4c}{\approx}$$

- then renaming  $u'_4(q_1, q_2, q_3, r, u)$  by  $u'$  we can write the **recursion relations** as

$$r'(r, u) = 4\left(r + 3cu/(1+r)\right) + \text{higher orders in } u$$

$$u'(r, u) = 2^{4-d}\left(u - 9cu^2/(1+r)^2\right) + \text{higher orders in } u$$

- thus we are ignoring effects of order  $(\sigma')^6$  and  $u^2$  or higher
- repeating many times the process of integrating out the high fluctuation frequencies, after starting at initial interaction with couplings  $r_0$  and  $u_0$  and using our recursion relations

$$r_{k+1}(r_k, u_k) = 4\left(r_k + 3cu_k/(1+r_k)\right)$$

$$u_{k+1}(r_k, u_k) = 2^{4-d}\left(u_k - 9cu_k^2/(1+r_k)^2\right)$$

(each time neglecting higher order in  $u_k$ ) after  $k$  steps we get an effective Hamiltonian  $\tilde{\mathcal{H}}_{s^d}^{(k)}$  in which we have integrated out all fluctuation frequencies  $|q| \in [2^{-k}, 1]$ , i.e., this effective Hamiltonian describes the behaviour of fluctuations with low frequencies  $|q| \in [0, 2^{-k}]$

## 4 Calculation of critical exponent $\nu$

### 4.1 Fixed points and critical points

- for temperatures  $T$  above but near the critical temperature  $T_{\text{crit}}$ , the critical exponent  $\nu$  describes power law behavior of the correlation length  $\xi(T)$ :

$$\xi(T) = \underbrace{(T - T_{\text{crit}})}_{\tau}^{-\nu} =: \tau^{-\nu}$$

- we remember having giving up for the sake of simplicity the connection between the spin variables  $s_{\underline{n}}$  living on the lattice sites  $\underline{n}$  and its discrete Fourier transform  $\sigma_{[s,q]}$
- now we would like to have back something similar to the lattice spins, therefore we introduce a spin field depending on a (dimensionless) continuous position variable  $\underline{x}$ :

$$s_{[\sigma,\underline{x}]} := \int_{|q|<1} d^d q e^{iq\underline{x}} \sigma(q)$$

- then we have the **2-point spin correlation**

$$\Gamma_{(\underline{x},r,u)} := Z^{-1}_{(r,u)} \int \mathcal{D}\sigma s(\underline{x}) s(\underline{0}) e^{-\tilde{\mathcal{H}}_{s^4}[\sigma,r,u]}$$

which can be written

$$\Gamma_{(\underline{x},r,u)} = \int d^d q e^{iq\underline{x}} \tilde{\Gamma}_{(q,r,u)}$$

- defining the **correlation length**  $\xi$  in terms of the behavior of the 2-point spin correlation for large distances  $\underline{x}$  leads to technical problems, thus we use a convenient alternative definition:

$$\xi^2_{(r,u)} := - \left[ \frac{d\tilde{\Gamma}_{(q,r,u)}/dq^2}{\tilde{\Gamma}_{(q,r,u)}} \right]_{q=0}$$

and call the so defined  $\xi$  **effective range of correlation**

- **critical points** correspond to infinite effective correlation range
- integrating out high frequencies we obtain a new 2-point spin correlation, which is related to the original one by the **scaling relation**

$$\tilde{\Gamma}'_{(q'=2q,r',u')} = 2^d/\zeta^2 \tilde{\Gamma}_{(q,r,u)} = \frac{1}{4} \tilde{\Gamma}_{(q,r,u)}$$

- the **scaling relation** for effective correlation ranges is

$$\xi'_{(r',u')} = \frac{1}{2} \xi_{(r,u)}$$

- these two scaling relations say that the new system is less correlated than the original one, or in other words, correlation decreases on larger scales

- now we will look for **fixed points**  $(r^*, u^*)$  of the recursion relations
- at a fixed point of the recursion relations we will have

$$\begin{aligned} r_{k+1}(r_k, u_k) &= r_k \\ u_{k+1}(r_k, u_k) &= u_k \end{aligned}$$

which implies for the effective correlation range  $\xi_{k+1}(r_{k+1}, u_{k+1}) = \xi_k(r_k, u_k)$

- together with  $\xi_{k+1} = \frac{1}{2}\xi_k$  this implies that at fixed points we have either vanishing or infinite effective correlation range, but the vanishing case can be ruled out, thus fixpoints correspond to critical points of the system
- as in the Gaussian model, our recursion relations always have one **trivial fixpoint**  $r^* = u^* = 0$ , also called **Gaussian fixpoint**
- in case of dimension  $d > 4$ : iterating the recursion relations many times  $u_k$  approaches zero because of the factor  $2^{4-d}$ , and with  $u_k$  zero the recursion relation for  $r_k$  has zero as only fixpoint
- in case  $d = 4$ : the recursion relation for  $u_k$  has zero as only fixpoint, thus again  $r_k$  goes to zero, too
- thus in case  $d \geq 4$  our recursion relations only have the Gaussian fixpoint  $r^* = u^* = 0$
- in case  $d < 4$ : choose  $u_0$  small, then with increasing iterations  $u_k$  will increase until the second term in its recursion relation becomes comparable to the first, and a new non-Gaussian fixpoint is approximately given by

$$\begin{aligned} u^* &\approx \frac{2^{4-d} - 1}{9c} \\ r^* &\approx -4cu^* = -\frac{4}{9}(2^{4-d} - 1) \end{aligned}$$

- Wilson shows that these approximations are reasonable for  $d \lesssim 4$ , because then both  $u^*$  and  $r^*$  are small
- defining

$$\epsilon := 4 - d$$

and **expanding in**  $\epsilon$  we find for small, positive  $\epsilon$  that the fixpoint is given by

$$\begin{aligned} u^* &\approx \frac{\epsilon \ln 2}{9c} > 0 \\ r^* &\approx -\frac{4}{9} \epsilon \ln 2 < 0 \end{aligned}$$

with both  $|r^*|, |u^*| \ll 1$

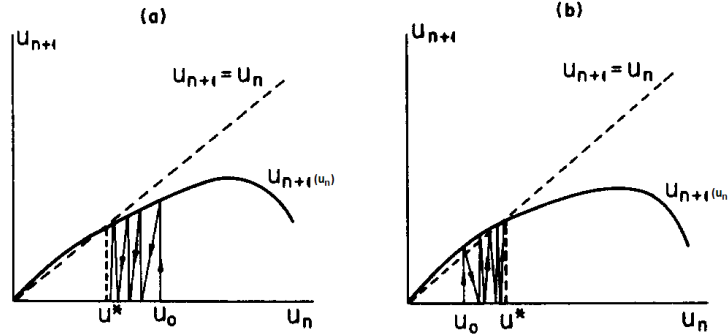


Fig. 4.4. The iteration formula for  $u_n$ .

- (a) If  $u_0 > u^*$ , the iteration scheme leads  $u_n$  to  $u^*$  from above.  
 (b) If  $u_0 < u^*$ ,  $u_n$  approaches  $u^*$  from below.

- as the figure illustrates, the value of  $u_k$  after many iterations is near the fixpoint  $u^*$  independent of the original coupling  $u_0$  (as long as  $u_0$  lies within the basin of attraction of the non-Gaussian fixpoint), this means that the effective coupling strength (within the basin of attraction) is determined by the fixpoint, and not by the original coupling  $u_0$

## 4.2 Linearized recursion relations

- recall definitions (after rescaling  $j$  to 1) of temperature-dependent coupling constants

$$r \equiv r_0 := \frac{b}{j} - 2d = \frac{B}{k_b T} - 2d$$

$$u \equiv u_0 := l/j^2 = L/(k_b T)$$

- in non-Gaussian case  $u \neq 0$  we can be at a critical temperature without having  $r_0$  and  $u_0$  at fixed point values, because for being at  $T_{\text{crit}}$  one only needs to adjust one parameter, but for being at fixed point one has to adjust two parameters
- the role of the fixpoint for  $T = T_{\text{crit}}$  consists in that with increasing number  $k$  of iterations  $r_k$  and  $u_k$  approach the fixpoint values, while for  $T \neq T_{\text{crit}}$  they show another limiting behavior
- now consider specific interaction before iteration starts, with  $T$  being only variable (i.e.:  $d, B, L$  fixed), then  $r_0$  and  $u_0$  depend on  $T$  analytically as above
- now we want to study the theory for temperatures near  $T_{\text{crit}}$  in order to calculate the critical exponent  $\nu$
- first step is to study sequence  $\{(r_k(T), u_k(T))\}$  generated by repeatedly iterating the recursion relations and starting from  $(r_0(T), u_0(T))$

- we expect  $r_k(T_{\text{crit}}) \xrightarrow{k \rightarrow \infty} r^*$  and  $u_k(T_{\text{crit}}) \xrightarrow{k \rightarrow \infty} u^*$  as on line A in figure 4.5

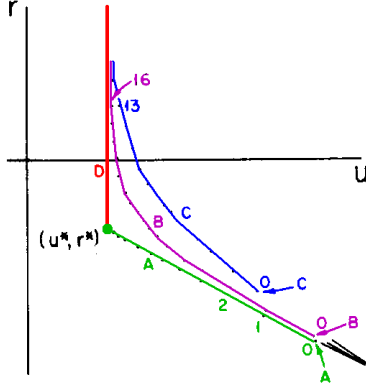


Fig. 4.5.  
 Plot of the iteration scheme for three different initial choices of parameters. **Sequence A** ( $T=T_c$ ) goes into the fixpoint  $(u^*, r^*)$ . **Sequences B and C** begin for choices of  $u$  and  $r$  slightly removed from criticality. These sequences eventually deviate far from the fixed point but approach the **unique curve D**.

- because the recursion relations are analytic, we expect linear behavior for fixed  $k$  and  $T$  close enough to  $T_{\text{crit}}$ :

$$r_k(T) = r_k(T_{\text{crit}}) + \rho_k \overbrace{(T - T_{\text{crit}})}^{\tau}$$

$$u_k(T) = u_k(T_{\text{crit}}) + \mu_k \overbrace{(T - T_{\text{crit}})}^{\tau}$$

- now if  $k$  is sufficiently large, then

$$r_k(T_{\text{crit}}) \approx r^* \quad \Rightarrow \quad r_k(T) \approx r^*$$

$$u_k(T_{\text{crit}}) \approx u^* \quad \Rightarrow \quad u_k(T) \approx u^*$$

- this suggests we should study the recursion formulae close to  $T_{\text{crit}}$  for large  $k$ , because then we can linearize the recursion formulae
- in matrix form this reads

$$\begin{pmatrix} r_{k+1} - r^* \\ u_{k+1} - u^* \end{pmatrix} \approx M \begin{pmatrix} r_k - r^* \\ u_k - u^* \end{pmatrix}$$

$$M = \begin{pmatrix} 4 - \frac{12cu^*}{(1+r^*)^2} & \frac{12c}{(1+r^*)} \\ \frac{2\epsilon 18cu^{*2}}{(1+r^*)^3} & 2\epsilon - \frac{2\epsilon 18cu^*}{(1+r^*)^2} \end{pmatrix}$$

- after iterating the linearized recursion relations many times the relation reads

$$\begin{pmatrix} r_{k+n} - r^* \\ u_{k+n} - u^* \end{pmatrix} \approx M^n \begin{pmatrix} r_k - r^* \\ u_k - u^* \end{pmatrix}$$



- the advantage of studying  $M^n$  for large  $n$  is that  $M^n$  is completely dominated by the largest eigenvalue of  $M$
- in the limit of vanishing  $u^*$  the eigenvalues are 4 and 1, with  $4^n$  clearly dominating  $1^n$  for large  $n$
- by diagonalizing  $M$  the explicit form of  $M^n$  neglecting the eigenvalue 1 can be obtained (with  $\lambda$  being the dominant eigenvalue, and  $v, w$  some components of the eigenvectors of  $M$ ):

$$M^n = \lambda^n \begin{pmatrix} 1 & v \\ w & vw \end{pmatrix} \quad (4.1)$$

$$\begin{aligned} \lambda &= 4\left(1 - \frac{1}{3}\epsilon \ln 2\right) \\ v &= +4c\left(1 + \frac{5}{9}\epsilon \ln 2\right) \\ w &= -4c\left(1 + \frac{5}{9}\epsilon \ln 2\right) \end{aligned}$$

- because we are considering  $T \approx T_{\text{crit}}$ , for fixed  $k$  we have

$$\begin{aligned} r_k(T) - r_k(T_{\text{crit}}) &\sim (T - T_{\text{crit}}) = \tau \\ u_k(T) - u_k(T_{\text{crit}}) &\sim (T - T_{\text{crit}}) = \tau \end{aligned}$$

so that for sufficiently large  $k$

$$(r_k(T) - r^*) + v(u_k(T) - u^*) = c_k(T - T_{\text{crit}}) = c_k \tau$$

and thus from the matrix equation (4.1) we deduce

$$\begin{aligned} r_{k+n} - r^* &= \lambda^n c_k (T - T_{\text{crit}}) \\ u_{k+n} - u^* &= \lambda^n w c_k (T - T_{\text{crit}}) \end{aligned} \quad (4.2)$$

- now we can calculate the critical exponent  $\nu$ , we had seen above that the effective correlation length  $\xi(r, u)$  is defined for any choice of interaction parameters  $r$  and  $u$ , since these depend on  $T$  we have  $\xi = \xi(T)$
- we had also seen the scaling relation

$$\xi(r_{k+n}, u_{k+n}) = 2^{-(k+n)} \xi(r_0, u_0)$$

- from (4.2) we read off the following:

$$\begin{aligned} \left[ r_{k+n+1}(T) - r^* \right]_{T-T_{\text{crit}}=\tau/\lambda} &= \left[ r_{k+n}(T) - r^* \right]_{T-T_{\text{crit}}=\tau} \\ \left[ u_{k+n+1}(T) - u^* \right]_{T-T_{\text{crit}}=\tau/\lambda} &= \left[ u_{k+n}(T) - u^* \right]_{T-T_{\text{crit}}=\tau} \end{aligned}$$

or written more clearly

$$\begin{aligned} r_{k+n+1}(T_{\text{crit}}+\tau/\lambda) &= r_{k+n}(T_{\text{crit}}+\tau) \\ u_{k+n+1}(T_{\text{crit}}+\tau/\lambda) &= u_{k+n}(T_{\text{crit}}+\tau) \end{aligned}$$

- thus

$$\xi_{(r_{k+n+1}, u_{k+n+1})_{T=T_{\text{crit}}+\tau/\lambda}} = \xi_{(r_{k+n}, u_{k+n})_{T=T_{\text{crit}}+\tau}}$$

which with the scaling relation implies

$$2^{-(k+n+1)}\xi_{(T_{\text{crit}}+\tau/\lambda)} = 2^{-(k+n)}\xi_{(T_{\text{crit}}+\tau)} \quad (4.3)$$

- thus assuming power law behavior for the effective correlation range

$$\xi_{(T_{\text{crit}}+\tau)} \sim \tau^{-\nu}$$

(4.3) for arbitrarily small  $\tau$  gives us

$$(\tau/\lambda)^{-\nu} = 2\tau^{-\nu}$$

thus

$$\nu = \frac{\ln 2}{\ln \lambda} = \frac{1}{2 - \epsilon/3}$$

- up to linear order in  $\epsilon$  this writes

$$\nu \approx \frac{1}{2} + \frac{1}{12}\epsilon$$

- the artificial features of the iteration scheme (factors of  $\ln 2$ ) do not appear in  $\nu$ , which is crucial, because the critical exponent should be a property of the physical system, and not of the method of solution
- for nonzero  $\epsilon$  our  $\nu$  differs from the value  $\frac{1}{2}$  obtained in Gaussian and mean field model, experimentally ( $d=3$ )  $\nu \approx 0.6\dots 0.7$ , in the 3-dim. (2-dim.) Ising model one has  $\nu = 0.64$  (1.0), we have obtained here for ( $d=3$ ), ( $\epsilon=1$ ) that  $\nu \approx 0.58$  and for ( $d=2$ ), ( $\epsilon=2$ ) that  $\nu \approx \frac{2}{3}$ , thus it seems that for "small"  $\epsilon=1$  our approximations work well, but for "not so small"  $\epsilon=2$  they fail
- $\nu$  is independent of the initial couplings  $r_0$  and  $u_0$ , thus we have an example for universal behavior of a system