Constructive quantum field theory

Daniele Colosi

Mathematical Physics Seminar

Contents

Introduction

Question

- "Does there exist a mathematically-complete, non-linear relativistic quantum field theory in Minkowski spacetime of dimensions 4?"

Question

- "Does there exist a mathematically-complete, non-linear relativistic quantum field theory in Minkowski spacetime of dimensions 4?"
- Remains one of the most important unresolved questions in physics.

First attempts - I

Mathematicians and physicists attempted in the 50's to formulate a mathematical framework for QFT $\rightarrow$ Axiomatic QFT.(Wightman, Jost, Haag)

First attempts - I

Mathematicians and physicists attempted in the 50's to formulate a mathematical framework for QFT $\rightarrow$ Axiomatic QFT.(Wightman, Jost, Haag)
Axiomatic QFT is an attempt to construct relativistic QFT on firm mathematical foundation by assuming several general properties, the axioms, that the relativistic QFT should have.

First attempts - I

Mathematicians and physicists attempted in the 50's to formulate a mathematical framework for QFT $\rightarrow$ Axiomatic QFT.(Wightman, Jost, Haag)
Axiomatic QFT is an attempt to construct relativistic QFT on firm mathematical foundation by assuming several general properties, the axioms, that the relativistic QFT should have.

Formulation of reasonable principles for any quantum theory (Wightman axioms):

1. There is a separable Hilbert space $\mathcal{H}$. The states of the theory are described by unit rays in $\mathcal{H}$.
2. There is a unitary, positive-energy representation U of the Poincaré group on $\mathcal{H}$.
3. There exists an invariant, vacuum-vector $\Omega = U\Omega \in \mathcal{H}$.
4. The quantum field ϕ is an operator-valued distribution.
5. Vectors of the form $\phi(f_1)\cdots\phi(f_n)\Omega$, for $f \in \mathcal{S}$ and arbitrary n span $\mathcal{H}$.
6. The field ϕ transforms covariantly under U: $U(\Lambda, a)\phi(f)U(\Lambda, a)^* = \phi(\{\Lambda, a\}f)$ where $(\{\Lambda, a\}f)(x) = f(\Lambda^{-1}(x - a))$.
7. The field ϕ is local $\rightarrow$ relativistic causality $\rightarrow$ CCR: $\phi(f)\phi(g) = \phi(g)\phi(f)$ if the supports of f and g are spacelike separated.
8. The space of invariant vectors Ω is one-dimensional (uniqueness of the vacuum).

First attempts - II

- Important results: spin-statistics, CPT theorem.
- Lehmann, Symanzik, Zimmermann, and Haag and Ruelle: incorporation of scattering theory.
- Haag later introduced a more general axiomatic approach (emphasis on the algebraic properties of ϕ).
- Limitations: axioms formulated only for free fields and some interacting models in d=2 are successfully treated.

First attempts - II

- Important results: spin-statistics, CPT theorem.
- Lehmann, Symanzik, Zimmermann, and Haag and Ruelle: incorporation of scattering theory.
- Haag later introduced a more general axiomatic approach (emphasis on the algebraic properties of ϕ).
- Limitations: axioms formulated only for free fields and some interacting models in d=2 are successfully treated.

$$\Downarrow$$

- A new approach: Constructive quantum field theory. (mid-1960's)
- Glimm and Jaffe: study of non-trivial field theories satisfying the axioms: $P(\phi)_2, Y_2, (\phi^4)_3$.
- Hamiltonian approach: operator theoretic method in Minkowski space.
- Euclidean approach: study of the functional integration representation of the matrix element of the kernel e^{-tH}.

Euclidean invariant functional integration

- Euclidean symmetry
  - Poincaré symmetry: preservation of the Minkowski quadratic form $t^2 - \underline{x}^2$.
  - Analytic continuation to imaginary time: $t \to \mathrm{i}t$.
  - Euclidean symmetry: preservation of the quadratic form $-x^2 = \sum_{i=1}^{d} x_i^2$.
  - Schwinger: Euclidean field theory = analytic continuation of Minkowski field theory.
  - The converse?

Euclidean invariant functional integration

- Euclidean symmetry
  - Poincaré symmetry: preservation of the Minkowski quadratic form $t^2 - \underline{x}^2$.
  - Analytic continuation to imaginary time: $t \to \mathrm{i}t$.
  - Euclidean symmetry: preservation of the quadratic form $-x^2 = \sum_{i=1}^{d} x_i^2$.
  - Schwinger: Euclidean field theory = analytic continuation of Minkowski field theory.
  - The converse?
- General framework of Euclidean field theory
  - Path integral = functional integration over the space of histories of the field
  - heuristic measure:
$$\text{“}\mathrm{d}\mu(\Phi) = \frac{1}{Z} e^{-\mathrm{S}(\Phi)} \prod_x \mathrm{d}\Phi(x)\text{”},$$
    $\mathrm{S}(\Phi)$: Euclidean-invariant action functional, Z: partition function, $\prod_x \mathrm{d}\Phi(x)$: average over field configurations.
  - None of these three factors has a a mathematical meaning, but the product does.
  - One can directly study $\mathrm{d}\mu(\Phi)$ or its Fourier transform
$$S(f) = \int e^{\mathrm{i}\Phi(f)} \mathrm{d}\mu(\Phi),$$
    or its moments
$$S_n(f_1, \ldots, f_n) = \int \Phi(f_1) \cdots \Phi(f_n) \mathrm{d}\mu(\Phi).$$
    These moments are called Euclidean Green's functions or Schwinger functions.

Problem and solution

Important question

- Having a classical Euclidean invariant field Φ and a distribution $\mathrm{d}\mu(\Phi)$, does the corresponding Minkowski QFT exists? (field acting on a Hilbert space with an appropriate representation of the Poincaré group, invariant vacuum)

Problem and solution

Important question

- Having a classical Euclidean invariant field Φ and a distribution $\mathrm{d}\mu(\Phi)$, does the corresponding Minkowski QFT exists? (field acting on a Hilbert space with an appropriate representation of the Poincaré group, invariant vacuum)
- Osterwalder and Schrader: yes!

- Osterwalder and Schrader provide a set of axioms that characterize appropriate measures ($\rightarrow$ well-defined Minkowski field theories).
- The framework works well for interacting QFT in d=2 and d=3:
  - rigorous construction of the $\lambda\Phi^4$ and the Yukawa models;
  - rigorous construction of the Gross-Neveu model in d=3.
- In d=4 still an open problem.
- The approach is inadequate for theories such as gauge field theories (Yang-Mills) with a non-linear space of histories of the field. Indeed all the OS axioms use the linearity of the space.
- Proposals exist to extend the framework to theories with a non-linear space of histories.

Introduction to OS axioms

- Fields are appropriately described in terms of distribution:

$$\phi(f) = \int_{\mathbb{R}^d} \phi(x) f(x) \, \mathrm{d}^d x,$$

- $\phi \in \mathcal{S}'(\mathbb{R}^d)$, $\mathcal{S}'$: space of tempered distribution, continuous linear functional on $\mathcal{S}$.
- $f \in \mathcal{S}(\mathbb{R}^d)$, $\mathcal{S}$: space of test function, Schwartz space of rapidly decreasing functions on $\mathbb{R}^d$.
- Generating functional $S : \mathcal{S} \to \mathbb{C}$,

$$S(f) = \int e^{\mathrm{i}\phi(f)} \mathrm{d}\mu,$$

  is the inverse Fourier transform of a Borel probability measure $\mathrm{d}\mu$ on $\mathcal{S}'$.
- Bochner-Minlos theorem: establishes a one to one correspondence between measures on $\mathcal{S}'$ and generating functionals on $\mathcal{S}$.
- The Osterwalder-Schrader axioms restrict the class of possible measures $\mathrm{d}\mu$, in order to guarantee that from the measure it is possible to construct the Hilbert space (unique vacuum, well-defined Hamiltonian, appropriate Green functions).

Osterwalder-Schrader axioms (1973 - 1975) - I

- **OS0** Analyticity

  For every finite set of test function $f_j \in \mathcal{S}$, $j = 1, \ldots, N$ and complex numbers $z = z_1, \ldots, z_N \in \mathbb{C}^N$, the function

  $$S\left(\sum_{j=1}^{N} z_j f_j\right)$$

  is entire on $\mathbb{C}^N$. S is infinitely differentiable, hence the correlators exist. In other words, $\mathrm{d}\mu$ decays faster than any exponential.

- **OS1** Regularity

  For some p, $1 \leq p \leq 2$, for some constant c, and for all $f \in \mathcal{S}$,

  $$|S(f)| \leq \exp c(\| f \|_{L^1} + \| f \|_{L_p}^p).$$

  This axiom introduces a bound on the growth of the correlation functions, restricts their singularities.

- **OS2** Euclidean invariance

  The Euclidean group E of $\mathbb{R}^d$ acts on $\mathcal{S}(\mathbb{R}^d)$ and $\mathcal{S}'(\mathbb{R}^d)$ by

$$\begin{aligned}(\tilde{\varphi}_g f)(x) &= f(g^{-1}x),\\ (\varphi_g \phi)(f) &= \phi(\tilde{\varphi}_{g^{-1}} f),\end{aligned}$$

  where g is an element of the Euclidean group E, and gx denotes the standard action of E on $\mathbb{R}^d$ by translations, rotations and reflections, $f \in \mathcal{S}(\mathbb{R}^d)$ and $\phi \in \mathcal{S}'(\mathbb{R}^d)$.
  $S(f)$ is invariant under Euclidean symmetry of $\mathbb{R}^d$:

$$S(f) = S(\tilde{\varphi}_g f), \qquad \forall g \in E.$$

  Equivalently the measure is Euclidean invariant,

$$\varphi_g * \mu = \mu.$$

We introduce exponential functionals

$$\mathcal{A} = \left\{ A(\phi) = \sum_{j=1}^{N} c_j \, \exp(\phi(f_j)), c_j \in \mathbb{C}, f_j \in \mathcal{S} \right\}. \tag{1}$$

An element $A \in \mathcal{A}$ maps $\phi \in \mathcal{S}'$ into $\mathbb{C}$. $\mathcal{A}$ is an algebra.
By **OS0** the functions A are all integrable and are in $L_p(\mathcal{S}', \mathrm{d}\mu), \forall p < \infty$. It is not difficult to show that by the Euclidean invariance of $\mathrm{d}\mu$, Euclidean transformations define a continuous unitary group on $L_2(\mathcal{S}', \mathrm{d}\mu)$, and

$$(U_g A)(\phi) = A(\varphi_g \phi),$$

where $g \in E$.

Euclidean invariance yields Poincaré invariance when the fields are analytically continued.

- **OS3** Reflection positivity

Consider $\mathcal{A}_+ \subset \mathcal{A}$:

$$\mathcal{A}_+ = \left\{ A(\phi) \text{of the form (1)} : f_j \in \mathcal{S}(\mathbb{R}^d_+), \mathbb{R}^d_+ = \{\underline{x}, t : t > 0\} \right\}.$$

We assume that the time reflection $\theta : \{\underline{x}, t\} \to \{\underline{x}, -t\}$ satisfies

$$0 \leq \langle U_\theta A, A \rangle_{L_2} = \int \overline{(U_\theta A)} A \, \mathrm{d}\mu. \tag{2}$$

This axiom is equivalent to the property that $S(f)$ satisfies

$$0 \leq \sum_{i,j=1}^{n} \overline{c_j} c_i S(f_j - \theta f_i),$$

for every choice of n real functions $f_j \in \mathcal{S}(\mathbb{R}^d_+)$ and complex constants c_j.

Reflection positivity (probably the most important axiom) yields the positivity of the inner product in the Hilbert space, with a non-negative self-adjoint Hamiltonian acting on it, for the corresponding Minkowski field theory.

- Construction of the Hilbert space

Consider the subspace $\mathcal{E}_+$ of $L_2(\mathcal{S}'(\mathbb{R}^d), \mathrm{d}\mu)$ of vectors $A \in \mathcal{A}_+$.
There is a degenerate inner product on $\mathcal{E}_+$, given by

$$b(A, B) = \langle U_\theta A, B \rangle_{L_2} = \int \overline{(U_\theta A)} B \mathrm{d}\mu. \tag{3}$$

It is positive by (2). Let $\mathcal{N}$ be the subspace of vectors in $\mathcal{E}_+$ which are null in the inner product (3).
The Hilbert space $\mathcal{H}$ is obtained by taking the quotient $\mathcal{E}_+/\mathcal{N}$ and completing it with respect to the inner product (3). The inner product in $\mathcal{H}$ is positive.

b depends only on the equivalence class:
If $A, B \in \mathcal{E}_+$ and $N \in \mathcal{N}$ then $b(A + N, B) = b(A, B)$.
With the Schwarz inequality:

$$|b(a, b)| = \left| \int \overline{(U_\theta A)} B \mathrm{d}\mu \right| \leq b(A, A)^{1/2} b(B, B)^{1/2}. \tag{4}$$

- Hamiltonian

Let $^\wedge : \mathcal{E}_+ \to \mathcal{H}$ be the canonical imbedding, $A^\wedge = A + \mathcal{N}$ for $A \in \mathcal{E}_+$.
We can transfer operators S acting on $\mathcal{E}_+$ to operators $S^\wedge$ acting on $\mathcal{H}$:

$$(S(A+N))^\wedge = S^\wedge(A+N)^\wedge = S^\wedge A^\wedge = (SA)^\wedge, \qquad \forall N \in \mathcal{N}. \tag{5}$$

Alternatively with

$$\langle A^\wedge, B^\wedge \rangle_{\mathcal{H}} = b(A, B), \tag{6}$$

define

$$\langle A^\wedge, S^\wedge B^\wedge \rangle_{\mathcal{H}} = b(A, SB). \tag{7}$$

$S^\wedge$ must be defined on equivalence class,

$$S : \mathcal{D}(S) \cap \mathcal{E}_+ \to \mathcal{E}_+, \qquad \text{and} \qquad S : \mathcal{D}(S) \cap \mathcal{N} \to \mathcal{N}, \tag{8}$$

where $\mathcal{D}(S)$ is the domain of S.

Proposition:
Let $\mathrm{d}\mu$ be a probability measure on $\mathcal{S}'$ and assume reflection positivity and reflection invariance of $\mathrm{d}\mu$. Then $^\wedge$ is a contraction: $||A^\wedge||_{\mathcal{H}} \leq ||A||_{\mathcal{E}}$.

Theorem (Reconstruction of quantum mechanics):
Let $\mathrm{d}\mu$ be a probability measure on $\mathcal{S}'$ and assume reflection positivity and reflection and time translation invariance invariance of $\mathrm{d}\mu$. Then for $0 \leq t$, $T(t)$ satisfies (8) and

$$T(t)^\wedge = e^{-tH}, \tag{9}$$

where $0 \leq H = H^*$ and for $\Omega = 1^\wedge$, $H\Omega = 0$. In other words, H is a positive self-adjoint operator with ground state Ω.

- **OS4** Ergodicity

The time translation subgroup acts ergodically on the measure space $(\mathcal{S}'(\mathbb{R})^d, \mathrm{d}\mu)$, i.e., for $g = T(s) : T(s)(t', \underline{x}) = (t' + s, \underline{x})$, and for all L_1 functions $A(\phi)$,

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t A(\varphi_{T(s)} \phi_0) \mathrm{d}s = \int_{\mathcal{S}'(\mathbb{R})^d} A(\phi) \mathrm{d}\mu(\phi). \tag{10}$$

The axiom ensures that the left hand-side of (10) does not depend on the particular choice of ϕ_0.

Axioms OS2 and OS4 imply that the vacuum, Euclidean invariant function on $L_2(\mathcal{S}'(\mathbb{R}^d), \mathrm{d}\mu)$, is unique and correspond to a constant function.

Reconstruction theorem

Theorem: Let a measure $\mathrm{d}\mu$ on $\mathcal{S}'(\mathbb{R}^d)$ satisfy OS0-4. Then the real time field obtained by analytic continuation $t \to -\mathrm{i}t$, satisfies the Wightman axioms.

The Schwinger functions and Wightman functions are related by analytic continuation,

$$\int \phi_E(\underline{x}_1, t_1) \cdots \phi_E(\underline{x}_n, t_n) \mathrm{d}\mu = \langle \Omega, \phi_M(\underline{x}_1, \mathrm{i}t_1) \cdots \phi_M(\underline{x}_n, \mathrm{i}t_n) \Omega \rangle. \tag{11}$$

- The problem of defining a quantum field theory is that of finding suitable measures $\mathrm{d}\mu$ on $\mathcal{S}'$ or equivalently appropriate generating functionals S on $\mathcal{S}$.
- The set of Osterwalder-Schrader characterize to properties of S for it to be appropriate.
- It is then possible to construct a Minkowski quantum field theory satisfying the Wightman axioms (Hilbert space of states, Hamiltonian, vacuum, algebra of observables).