# Algebraic Quantum Field Theory

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# Presentation's Structure

# Motivation

## Objective

The main idea is to formulate the quantum theory in such a way that the observables become the relevant objects and the quantum states are "secondary." Now, the states are taken to "act" on operators to produce numbers.

## Usefulness

- Mathematically precise description of the structure of quantum field theories.
- Tool for studying the foundations of QFT.
- Work with issues related to nonlocality, the particle concept, the field concept, and inequivalent representations.
- Fundamental part in the formulation of QFT in curved spacetime.

# Wightman Axioms

1. Hilbert Space
   - There is a separable Hilbert space $\mathcal{H}$. The states of the theory are described by unit rays in $\mathcal{H}$.
   - There is a unitary, positive-energy representation $U$ of the Poincaré group on $\mathcal{H}$.
   - There exists an invariant, vacuum-vector $|\Psi_0\rangle = U|\Psi_0\rangle \in \mathcal{H}$.
   - The space of invariant vectors is one-dimensional (uniqueness of the vacuum).
2. The quantum field $\phi$ is an operator-valued distribution.
3. Vectors of the form $\phi(f_1)\cdots\phi(f_n)|\Psi_0\rangle$ , for $f \in \mathcal{S}$ and arbitrary $n$ span $\mathcal{H}$.
4. The field $\phi$ transforms covariantly under $U$:

$$U(\Lambda, a)\phi(f)U(\Lambda, a)^* = \phi(\{\Lambda, a\}f)$$

   where $(\{\Lambda, a\}f)(x) = f(\Lambda^{-1}(x - a))$.
5. The field $\phi$ is local $\rightarrow$ relativistic causality $\rightarrow$ CCR: $[\phi(f), \phi(g)]_\pm = 0$ if the supports of $f$ and $g$ are spacelike separated.

We use the basic fields to associate to each open region $\mathcal{O}$ in spacetime an algebra $\mathfrak{A}(\mathcal{O})$ of operators on Hilbert space.

$$\mathcal{O} \rightarrow \mathfrak{A}(\mathcal{O})$$

- Where $\mathcal{O}$ denotes an open, finitely extended region of Minkowski space.
- The theory is characterized by a net of algebras $\mathfrak{A}$, where any $\mathfrak{A}(\mathcal{O})$ algebra is generated by all $\Phi(f)$ "operator valued distributions" that are "smeared out" with test functions $f$ having their support in the region $\mathcal{O}$.

**Poincaré invariance** means now that to a transformation $g$ there corresponds an automorphism $\alpha_g$ of the abstract net with the property

$$\alpha_g \mathfrak{A}(\mathcal{O}) = \mathfrak{A}(g\mathcal{O})$$

$\alpha_g$ maps the elements of $\mathfrak{A}(\mathcal{O})$ onto the elements of the algebra of the transformed region $g\mathcal{O}$ in such a way that all algebraic relations are conserved.

A **representation** $\pi$ of $\mathfrak{A}$ is a homomorphism from the net $\mathfrak{A}$ to a net of operator algebras $\pi(\mathfrak{A})$ i.e. $\pi$ assigns to each algebraic element $A$ its "representor" $\pi(A)$, an operator acting in a Hilbert space.
Given a representation the automorphism $\alpha_g$ is called **implementable** in the representation $\pi$ if exist a unitary operator $U(g)$ acting in the representation space such that

$$U(g)\pi(A)U^{-1}(g) = \pi(\alpha_g A)$$

# Haag-Kastler Axioms

- The abstract net should posses an irreducible representation $\pi_0$ in which $\alpha_g$ is implementable and $U(g)$ satisfies the two Wightman axioms:
  - There is a unitary, positive-energy representation $U$ of the Poincaré group on $\mathcal{H}$.
  - There exists an invariant, vacuum-vector $|\Psi_0\rangle = U|\Psi_0\rangle \in \mathcal{H}$.
- **Isotony:** if $O_1 \subset O_2$ then $\mathfrak{A}(O_1) \subset \mathfrak{A}(O_2)$.
- **Additivity property**

$$\mathfrak{A}(\mathcal{O}_1 \cup \mathcal{O}_2) = \mathfrak{A}(\mathcal{O}_1) \vee \mathfrak{A}(\mathcal{O}_2)$$

  where the symbol $\vee$ on the right hand side denote the operator algebra generate by two algebras $\mathfrak{A}(\mathcal{O}_i), i = 1, 2$.
- **Hermiticity** means that $\mathfrak{A}(\mathcal{O})$ is an involutive algebra (a $*$-algebra). One has within each $\mathfrak{A}(\mathcal{O})$ the involution $A \to A^*$.

# Haag-Kastler Axioms

- **Transformation properties of fields**, becomes

$$U(a,\alpha)\mathfrak{A}(\mathcal{O})U^{-1}(a,\alpha) = \mathfrak{A}(\Lambda(\alpha)\mathcal{O} + a)$$

  The geometry symmetry operations map the algebra of one region onto the algebra of the transforme region.
- **Causality:** $\mathfrak{A}(\mathcal{O}_1)$ is compatible with $\mathfrak{A}(\mathcal{O}_2)$ when the two regions lie space-like. In others words, two observables associated with space-like separated regions are compatible.
- **Completeness** is translated to the algebra as: Any operator should be approximated as linear combination of products of the algebra elements.
- **Primitive Causality:** Let $\hat{\mathcal{O}}$ denote the causal completion of $\mathcal{O}$ then

$$\mathfrak{A}(\hat{\mathcal{O}}) = \mathfrak{A}(\mathcal{O})$$

# Fundamental Elements

## Algebra

We may define de algebra $\mathfrak{A}_{loc} = \cup \mathfrak{A}(\mathcal{O})$ of "all local observables" and the $C^*$-algebra

$$\mathfrak{A} = \bar{\mathfrak{A}}_{loc}$$

the completion of $\mathfrak{A}_{loc}$ in the norm topology.

## States

States $\omega : \mathfrak{A} \to \mathbb{C}$ are positive linear funtionals ($\omega(A^*A) \geq 0 \ \forall A \in \mathfrak{A}$) such that $\omega(1) = 1$. The value of the state $\omega$ acting on the observable $A$ can be interpreted as the expectation value of the operator $A$ on the state $\omega$,i.e., $\langle A \rangle = \omega(A)$.

# GNS-Construction

*Let $\mathfrak{A}$ be a $C^*$-algebra with unit and let $\omega : \mathfrak{A} \to \mathbb{C}$ be a state. Then there exist a Hilbert space $\mathcal{H}$, a representation $\pi : \mathfrak{A} \to L(\mathcal{H})$ and a vector $|\Psi_0\rangle \in \mathcal{H}$ such that,*

$$\omega(A) = \langle \Psi_0, \pi(A)\Psi_0 \rangle_{\mathcal{H}}$$

*Furthermore, the vector $|\Psi_0\rangle$ is cyclic. The triplet $(\mathcal{H}, \pi, |\Psi_0\rangle)$ with this properties is unique (up to unitary equivalence).*

First, we use $\omega$ to define a (pre-)inner-product on $\mathfrak{A}$ by $(A_1, A_2) = \omega(A_1^* A_2)$. Then we take the left ideal $I_\omega = \{A \in \mathfrak{A} : \omega(A^*A) = 0\} \subseteq \mathfrak{A}$ and define a pre-Hilbert space as the quotient vector space $\mathfrak{A}/I_\omega$, now this space has a induced inner-product and can be use to complete $\mathfrak{A}/I_\omega$ and define a Hilbert space $\mathcal{H}_\omega$.

In this Hilbert space $\mathcal{H}_\omega$ there is a natural representation $\pi_\omega : \mathfrak{A} \to BL(\mathcal{H}_\omega, \mathcal{H}_\omega)$ is define as an extended of the continuos homomorphism $\tilde{\pi}_\omega(a) : \mathfrak{A}/I_\omega \to \mathfrak{A}/I_\omega$ whit action $\tilde{\pi}_\omega(a) : [b] \mapsto [ab]$.

In this representation the vector $|\Psi_0\rangle \in \mathcal{H}$ corresponding to $e \in \mathfrak{A}$ is a cyclic vector and satisfies $\omega(A) = \langle \Psi_0, \pi(A)\Psi_0 \rangle_{\mathcal{H}}$ for all $A \in \mathfrak{A}$.

- The theory is define in Minkowski spacetime ${}^4M$. We will perform a $3+1$ decomposition in the form $M = \Sigma \times \mathbb{R}$ whit an arbitrary embeddings $T_t : \Sigma \to {}^4M$ (the surface $\Sigma$ is topologically $\mathbb{R}^3$).
- The **phase space** can be written as $\Gamma = (\varphi, \pi)$, where $\varphi = T_t^*[\phi]$ and $\pi = T_t^*[\sqrt{h} n^a \nabla_a \phi]$.
- The **symplectic structure** takes the following form, when acting on vectors $(\varphi_1, \pi_1)$ and $(\varphi_2, \pi_2)$

$$\Omega([\varphi_1, \pi_1], [\varphi_2, \pi_2]) = \int_\Sigma (\pi_1 \varphi_2 - \pi_2 \varphi_1) d^3x$$

- We can define the **linear functions** on $\Gamma$ as follow: given a vector $Y^\alpha$ in $\Gamma$ of the form $Y^\alpha = (\varphi, \pi)^\alpha$ , and a pair $\lambda_\alpha = (-f. -g)_\alpha$, where $f$ is a scalar density and $g$ a scalar, we define the action of $\lambda$ on $Y$ as,

$$F_\lambda(Y) = -\lambda_\alpha Y^\alpha := \int_\Sigma (f\,\varphi + g\,\pi) d^3x = \Omega_{\alpha\beta} \lambda^\alpha Y^\beta = \Omega(\lambda, Y)$$

- We can compute the **Poisson brackets** as

$$\{F_\lambda(Y), G_\nu(Y)\} = \{\Omega(\lambda, Y), \Omega(\nu, Y)\} = -\Omega(\lambda, \nu)$$

- We can define a **complex structure** $J$ ($J : \Gamma \to \Gamma, J^2 = -1$) compatible with the symplectic form. The most general form is given by

$$-J(\varphi, \pi) = (A\varphi + B\pi, C\pi + D\varphi)$$

  Where $A, B, C, D$ are linear operators.

- We this structures we can defined on the phase space a positive definite **metric** $\mu(\cdot, \cdot) := \Omega(J\cdot, \cdot)$.
- The **classical observables** that are to be quantized and in terms of which the CCR are expressed

$$\varphi[f] := \int_\Sigma f\, \varphi d^3x \quad \text{and} \quad \pi[g] := \int_\Sigma g\, \pi d^3x \tag{1}$$

## $C^*$-algebra

For the case of a linear theory, the algebra ones considers is the so-called *Weyl algebra*. Each generator $W(\lambda)$ ( where $\lambda_\alpha = (-f, -g)$ ) of the Weyl algebra is the "Exponentiated" version of the linear observables (1), labeled by a phase-space vector $\lambda^\alpha$. These generator satisface the Weyl relations,

$$W(\lambda)^* = W(-\lambda), \;\; W(\lambda_1)W(\lambda_2) = e^{i\Omega(\lambda_1,\lambda_2)/2}W(\lambda_1 + \lambda_2)$$

The CCR now get replace by the quantum Weyl relations where now the operators $\hat{W}(\lambda)$ belong to the (abstract) algebra $\mathfrak{A}$.

## State

The value of the state $\omega_{Fock}$ acting on the Weyl generators $\hat{W}(\lambda)$ is given by

$$\omega_{Fock}(\hat{W}(\lambda)) = e^{-(1/4)\mu(\lambda,\lambda)} \tag{2}$$

Where $\mu(\cdot,\cdot) := \Omega(J\cdot,\cdot)$ is the positive definite metric defined on the phase space and $J$ is a *complex structure* ($J : \Gamma \to \Gamma, J^2 = -1$) compatible with the symplectic structure.

Quantization in the old sense means a representation of the Weyl relations on a Hilbert space $\mathcal{H} := L^2(\bar{\mathcal{C}}, d\mu)$, where $\bar{\mathcal{C}}$ is the *quantum configuration space* and $d\mu$ is a measure on $\bar{\mathcal{C}}$. One will need to specify these objects in the construction of the theory. The form to do this is using the GNS construction.

In the general quantization procedure the abstract operators $\hat{\varphi}[f]$ and $\hat{\pi}[g]$ are represent as operators in $\mathcal{H}$.We can represent them, when acting on functionals $\Psi[\varphi] : \bar{\mathcal{C}} \to \mathbb{C}$ as

$$(\hat{\varphi}[f] \cdot \Psi)[\varphi] := \varphi[f]\Psi[\varphi]$$

and

$$(\hat{\pi}[g] \cdot \Psi)[\varphi] := -i\hbar \int g(x) \frac{\delta \Psi}{\delta \varphi(x)} d^3x + \text{multiplicative term} \tag{3}$$

The second term in the last equation, depending on configuration variable and on the details of the measure $d\mu$ on $\bar{\mathcal{C}}$.

In order to specify the measure $d\mu$ that defines de Hilbert space, it suffices to concider configuration observables. The Weyl observable $\hat{W}(\lambda)$ corresponding to $\lambda^\alpha = (0, f)^\alpha$ is represented as $R(\hat{W}(\lambda)) = e^{i\hat{\varphi}[f]}$.
The GNS construction says us that

$$\omega(\hat{W}(\lambda)) = \int_{\bar{\mathcal{C}}} d\mu \bar{\Psi}_0[R(\hat{W}(\lambda))\Psi_0] = \int_{\bar{\mathcal{C}}} d\mu e^{i\int_\Sigma f\varphi d^3x} \tag{4}$$

On the other hand the explicit form of the state (2) is

$$\omega(\hat{W}(\lambda)) = exp\left[-\frac{1}{4}\mu(\lambda,\lambda)\right] = exp\left[-\frac{1}{4}\int_\Sigma fBfd^3x\right] \tag{5}$$

Let us now compare Eqs. (4) and (5),

$$\int_{\bar{\mathcal{C}}} d\mu e^{i\int_\Sigma f\varphi d^3x} = exp\left[-\frac{1}{4}\int_\Sigma fBfd^3x\right]$$

Schematically this say us that the measure looks like

$$d\mu = e^{-\int_\Sigma \varphi B^{-1}\varphi}\mathcal{D}\varphi$$

where $\mathcal{D}\varphi$ represents a fictitious "Lebesgue-like" measure.

We still need to find the “multiplicative term" in the representation of the momentum operator (3). For that, we will need the full Weyl algebra and Eq.(2) . We have to compute $\langle R(\hat{W}(g,f))\rangle_{\text{vac}} = \langle\Psi_0, \exp(i\hat{\varphi}[f] - i\hat{\pi}[g])\Psi_0\rangle$. Using a similar analysis done for find the measure, we can show that the representation of the momentum is

$$(\hat{\pi}[g]\cdot\Psi)[\varphi] := -i\hbar\int\left(g\frac{\delta}{\delta\varphi} - \varphi(B^{-1} - iCB^{-1})g\right)\Psi[\varphi]$$

## quantum configuration space

In the case of Minkowski spacetime and flat embeddings, where $\Sigma$ is Euclidian space, the quantum configuration space is the space $\mathcal{T}^*$ of tempered distributions on $\Sigma$.

# The Future

- Implement the dynamics in the linear theory of scalar field.
- Construct the theory for interacting fields (one million of dolar in the case Yang-Mills fields).
- Generalize the axioms to curved spacetime.

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