Hamiltonian Systems with Dressing Symmetries

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The Cotangent Bundle $M = T^*Q$ is possibly the most important type of symplectic manifold for physical applications.
If $Q = G$ Lie group then the cotangent bundle $T^*G = G \times \mathfrak{g}^*$ is trivial.
We can define over the cotangent bundle:

- *The Lioville Form* $\theta = \sum p_i dq_i$.
- The canonical 2-form $\omega_0 = -d\theta = \sum dq_i \wedge dp_i$.

This is the symplectic structure over T^*Q.

Theorem

Let Q be a manifold and $f : Q \to Q$ a diffeomorphism; we can define the **lift** *$F : T^*Q \to T^*Q$ such that this diagram commute*

$$\begin{tikzcd} T^*Q \arrow[r, "F"] \arrow[d, "\pi"'] & T^*Q \arrow[d, "\pi"] \\ Q \arrow[r, "f"] & Q \end{tikzcd}$$

and F^* is symplectic. In this case satisfies $F^*\theta = \theta$

Let (M, ω) be a symplectic manifold and $\rho : G \times M \to M$ a symplectic action of the Lie group G on M; that is, $(\rho_g)^* \omega = \omega$. we say that a map

$$\Phi : \quad M \to \mathfrak{g}^*$$

is a **Momentum Map** for the group action provided that for every $X \in \mathfrak{g}$

$$i_{X^M} \omega = d\phi_X$$

where ϕ_X is construct as

$$\begin{aligned} \Phi : \quad & M \to \mathfrak{g}^* \\ & m \longmapsto \Phi(m) : \quad \begin{aligned}[t] & \mathfrak{g} \to \mathbb{R} \\ & X \longmapsto \langle \Phi(m), X \rangle_{\mathfrak{g}} = \phi_X(m) \end{aligned} \end{aligned}$$

and

$$(X^M)_{m_0} = \frac{d}{dt} \rho(e^{tX}, m_0) \Big|_{t=0} \qquad , \text{with } X \in \mathfrak{g}$$

is the tangent vector at $m_0 \in M$ over the curve $m(t) = \rho(e^{tX}, m_0)$.

Definition

A momentum mapping Φ *is called* **Ad*-equivariant** *provided*

$$\Phi(\rho(g,m)) = Ad^*_{g^{-1}}(\Phi(m))$$

for every $g \in G$*; that is, the following diagram commutes:*

$$\begin{array}{ccc} \mathfrak{g}^* & \xrightarrow{Ad^*_{g^{-1}}} & \mathfrak{g}^* \\ \Phi \big\uparrow & & \big\uparrow \Phi \\ M & \xrightarrow{\rho_g} & M \end{array}$$

This will permit maps the Hamiltonian vector fields from M to $\mathfrak{g}^*$.

- Concrete momentum mappings one normally constructs are Ad*-equivariant . However, there are "exotic" case where Φ is not Ad*-equivariant , [1]

Definition (Adjoint Representation)

$$\begin{array}{ccc} G & \xrightarrow{Ad} & Aut(\mathfrak{g}) \\ {\scriptstyle exp}\uparrow & & \uparrow{\scriptstyle exp} \\ \mathfrak{g} & \xrightarrow{ad} & End(\mathfrak{g}) \end{array}$$

$$Ad_g X = \frac{d}{dt}(g\,e^{tX} g^{-1})\Big|_{t=0} = gXg^{-1}$$
$$ad_X Y = \frac{d}{dt}(Ad_{e^{tX}} Y)\Big|_{t=0} = [X, Y]$$

with $X, Y \in \mathfrak{g}, g \in G$.

Definition (Coadjoint Representation)

$$\begin{array}{ccc} G & \xrightarrow{Ad^*} & Aut(\mathfrak{g}^*) \\ {\scriptstyle exp}\uparrow & & \uparrow{\scriptstyle exp} \\ \mathfrak{g} & \xrightarrow{ad^*} & End(\mathfrak{g}^*) \end{array}$$

$$\langle Ad_g^* \xi, Y\rangle = \langle \xi, Ad_{g^{-1}} Y\rangle$$
$$\langle ad_X^* \xi, Y\rangle = -\langle \xi, ad_X Y\rangle$$

with $X, Y \in \mathfrak{g}, \xi \in \mathfrak{g}^*, g \in G$.

In order to construct our system we need the next elements:

- A Poisson-Lie group is a Lie group G with a Poisson structure compatible with the group operations.
- The infinitesimal version of Poisson Lie groups are the Lie bialgebras.
- With the help of Lie bialgebras we can define the dual Lie group G^* of G.
- G^* will be the symmetry group.
- With the Poisson-Lie groups we can construct double groups.
- The dressing action we will define over double groups.

Poisson Geometry

Definition

A **Poisson Structure** *over a smooth Manifold M (called* **Poisson Manifold***) is a map $\{\ ,\ \}_M : C^\infty(M) \times C^\infty(M) \longrightarrow C^\infty(M)$ that satisfies:*

- $\{f, g\} = -\{g, f\}$*, antisymmetric.*
- $\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$ *, Jacobi identity.*
- $\{fg, h\} = f\{g, h\} + \{f, h\}g$ *, Leibniz identity.*

Definition

A map $\lambda : N \to M$ that preserves the Poisson bracket is called **Poisson Map**

$$\{F_1, F_2\}_M \circ \lambda = \{F_1 \circ \lambda, F_2 \circ \lambda\}_N$$

The cartesian product $M \times N$ of Poisson manifolds is a new Poisson Manifold with bracket $\{f, g\}_{M\times N}(x, y) = \{f(\ , y), g(\ , y)\}_M(x) + \{f(x,\ ), g(x,\ )\}_N(y)$

Definition

A **Poisson-Lie group** *is Lie group G with Poisson structure such that the product $\mu : G \times G \longrightarrow G$ is a Poisson map,*

$$\{F_1 \circ \mu, F_2 \circ \mu\}_{G\times G} = \{F_1, F_2\}_G \circ \mu$$

Example

The Poisson structure on G induced a Lie algebra structure on $\mathfrak{g}^*$ given by

$$[\xi_1, \xi_2]_{\mathfrak{g}^*} = (d\{f_1, f_2\})_e \qquad , \textit{with} \quad (df_i)_e = \xi_i \in \mathfrak{g}^*.$$

This is call the **tangent Lie bialgebra***.*

Theorem

$\Rightarrow$) *If G is a Poisson-Lie group, then $\mathfrak{g}$ have a natural bialgebra structure, called tangent Lie bialgebra.*

$\Leftarrow$) *Any Lie bialgebra structure on $\mathfrak{g}$ is a tangent Lie bialgebra of an unique Poisson-Lie group G [5].*

- In general a Lie bialgebra on $\mathfrak{g}$ is defined by a Lie bracket on $\mathfrak{g}^*$ compatible with the Lie bracket on $\mathfrak{g}$. In particular the tangent Lie bialgebra satisface the compatibility condition.

Definition

Three Lie algebras $(\delta, \mathfrak{g}, \mathfrak{g}^*)$ form a **double Lie algebra** *if $\mathfrak{g}$ and $\mathfrak{g}^*$ are subalgebras of δ and $\delta = \mathfrak{g} \oplus \mathfrak{g}^*$ as vectorial space.*

Theorem

If $(\mathfrak{g}, \mathfrak{g}^*)$ is Lie bialgebra. We Denote as $\mathfrak{g} \bowtie \mathfrak{g}^*$ the vector space $\mathfrak{g} \oplus \mathfrak{g}^*$ together with a Lie bracket

$$[(X, \eta)(Z, \xi)]_{\mathcal{D}(\mathfrak{g})} = ([X, Z] - ad^*_\eta Z + ad^*_\xi X, [\eta, \xi] - ad^*_X \xi + ad^*_Z \eta)$$

Then $(\mathfrak{g} \bowtie \mathfrak{g}^*, \mathfrak{g}, \mathfrak{g}^*)$ is a double Lie algebra $\mathcal{D}(\mathfrak{g})$.

- From this theorem we know that the Lie bracket of cross elements is

$$[\xi, X]_{\mathcal{D}} = ad^*_X \xi - ad^*_\xi X$$

Definition

If $\mathfrak{g}^*$ is endowed with a Lie algebra structure, then the connected and simply-connected Lie group G^* with Lie algebra $\mathfrak{g}^*$ is called **Dual Group** *of G.*

Double Lie Groups

Definition

- *Given three groups* $(\mathcal{D}, G, G^*)$ *they form a* **Double Lie Group** *if* G *and* G^* *are closed subgroups of* $\mathcal{D}$ *and the map* α *is diffeomorphism*

$$\alpha : \quad \begin{aligned} G \times G^* &\rightarrow \mathcal{D} \\ (g, \widetilde{h}) &\longmapsto g\widetilde{h} \end{aligned}$$

- *For each pair* $(g, \widetilde{h})$ *with* $g\widetilde{h} \in \mathcal{D}$ *there is* $\widetilde{h}g \in \mathcal{D}$ *s.t.*

$$\widetilde{h}g = g^{\widetilde{h}}\widetilde{h}^{g} \,, \qquad \text{with} \quad g, g^{\widetilde{h}} \in G \text{ and } \widetilde{h}, \widetilde{h}^{g} \in G^*$$

  With this we can define the **Dressing Action** *as:*

$$dr : \quad \begin{aligned} G^* \times G &\rightarrow G \\ (\widetilde{h}, g) &\longmapsto Dr(\widetilde{h}, g) = g^{\widetilde{h}} \end{aligned}$$

- *Given* G *and* G^* *Poisson-Lie Groups with tangent Lie bialgebra* $(\mathfrak{g}, \mathfrak{g}^*)$ *and* $(\mathfrak{g}^*, \mathfrak{g})$ *respectively. We denote by* $G \bowtie G^*$ *the Lie Group with tangent Lie algebra* $\mathfrak{g} \bowtie \mathfrak{g}^*$.

Dressing Action and Momentum Map on T^*G

Theorem

The map

$$\begin{aligned} dr: \quad & G^* \times G \to G \\ & (\widetilde{h}, g) \longmapsto dr(\widetilde{h}, g) = g^{\widetilde{h}} \end{aligned}$$

Is a left action of G^* on G and lifts to the trivialization $T^*G \equiv G \times \mathfrak{g}^*$ as

$$\begin{aligned} Dr: \quad & G^* \times T^*G \to T^*G \\ & (\widetilde{h}, (g, \xi)) \longmapsto Dr^{G \times \mathfrak{g}^*}(\widetilde{h}, (g, \xi)) = (g^{\widetilde{h}}, Ad_{\widetilde{h}^g}\xi) \end{aligned}$$

With Ad^*-equivariant momentum map given by

$$\Phi_B(g, \eta) = \Pi_{\mathfrak{g}} Ad^{*\mathcal{D}}_{g^{-1}} \eta$$

All of this writen in body coordinates i.e. left trivialization.

Collective Motion

Now concider a **Hamiltonian System** (T^*G, ω, H) with momentum map $\Phi : T^*G \to \mathfrak{g}$ associated to the dressing action of G^* on G. Suposse that H is a **Collective Hamiltonian** i.e.

$$H = \mathcal{F} \circ \Phi \qquad , \text{with} \quad \mathcal{F} : \mathfrak{g} \to \mathbb{R}$$

we can associate to $\mathcal{F}$ a map $\mathcal{L}_{\mathcal{F}} : \mathfrak{g} \to \mathfrak{g}^*$, such that for any $X \in \mathfrak{g}$

$$\langle X, \mathcal{L}_{\mathcal{F}}(Y)\rangle_{\mathfrak{g}^*} = \langle (d\mathcal{F})_Y, X\rangle_{\mathfrak{g}} = \left.\frac{d\mathcal{F}(X + tY)}{dt}\right|_{t=0}$$

With this we can construct the function

$$\mathcal{L}_{\mathcal{F}} \circ \Phi : T^*G \to \mathfrak{g}^*.$$

Proposition

The Hamiltonian vector field defined by $H = \mathcal{F} \circ \Phi$ on T^*G is

$$V_H|_{(g,\eta)} = (Dr_{(g,\eta)})_*[\mathcal{L}_{\mathcal{F}} \circ \Phi](g, \eta)$$

with $(g, \eta) \in T^*G \equiv G \times \mathfrak{g}^*$.

Collective Motion

The image of the Hamiltonian vector field V_H through the momentum map Φ is

$$(d\Phi)_m V_H|_m = -ad^*_{\mathcal{L}_{\mathcal{F}}(\Phi(m))}\Phi(m)$$

This means that the momentum map maps the Hamiltonian vector field to the coadjoint orbits on $\mathfrak{g}$.

In other words, if $m(t)$ is a trajectory ($\dot{m}(t) = V_H|_{m(t)}$) on T^*G then the momentum map maps $m(t)$ to $\gamma(t) = \Phi(m(t))$ and satisfies

$$\dot{\gamma}(t) = -ad^*_{\mathcal{L}_{\mathcal{F}}(\gamma(t))}\gamma(t)$$

This is the Hamiltonian system on the coadjoint orbits of $\mathfrak{g}$ with Hamiltonian $H_{\mathfrak{g}} = \mathcal{F}$.

Conclusion

The momentum map maps the Collective Hamiltonian system $H = \mathcal{F} \circ \Phi$ to a Hamiltonia system on coadjoint orbits with Hamiltonian fuction $H_{\mathfrak{g}} = \mathcal{F}$.

Collective Motion

Now we can construct a collective Hamiltonian invariant under the actions of G^*.

- The momentum map in body coordinates is

$$\Phi_B(g,\eta) = \Pi_{\mathfrak{g}} Ad^{*\mathcal{D}}_{g^{-1}}\eta \qquad ,\text{with} \quad (g,\eta) \in G \times \mathfrak{g}^*$$

- $\mathcal{F} : \mathfrak{g} \to \mathbb{R}$, is $\mathcal{F}(X) = (X, \mathcal{B}X)_{\mathfrak{g}}$, where the map $\mathcal{B} : \mathfrak{g} \to \mathfrak{g}$ is:
  - Symmetric, $(X, \mathcal{B}Y) = (\mathcal{B}X, Y)$.
  - Ad^*-equivariant , $Ad^{*\mathcal{D}}_{g^{-1}}\mathcal{B}X = \mathcal{B}Ad^{*\mathcal{D}}_{g^{-1}}X$

The collective Hamiltonian is given by

$$H(g,\eta) = \mathcal{F}(\Phi(g,\eta)) = (\Pi_{\mathfrak{g}} Ad^{*\mathcal{D}}_{g^{-1}}\eta, \mathcal{B}\Pi_{\mathfrak{g}} Ad^{*\mathcal{D}}_{g^{-1}}\eta)_{\mathfrak{g}}$$

The equations of motion are given by the Hamiltonian vector field $V_H|_{(g,\eta)} = (\dot{g}, \dot{\eta})$

$$g^{-1}\dot{g} = -\delta H$$
$$\dot{\eta} = ad^*_{\delta H}\eta - g\Delta H$$

with $dH = (\Delta H, \delta H) \in T^*_g(G \times \mathfrak{g})$

Summary

- We studied the geometrical formulation of classical mechanics with emphasis in configuration spaces that are Lie groups.
- We applied this formalism to Poisson-Lie groups and Double groups.
- We study the dressing actions of G^* on G, implementing it on the phase space T^*G and we calculated the momentum map associated to the dressing symmetry.
- We proposed a collective system with dressing symmetry, and we found its Hamilton equations.

This formalism can be applied to:

- *Loop groups* (Field Theory on $1 + 1$ dimensions).
- T-Duality.

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