Probabilities in the GBF

Max Dohse

Centro de Ciencias Matematicas (CCM)
UNAM Campus Morelia

References:

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Outline

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Before reviewing the probability interpretation of standard QFT, we recall the main ingredients of the GBF version of **Schrödinger-Feynman quantization (SFQ).** To each hypersurface Σ on spacetime is associated its **space of field configurations** $\mathcal{C}_\Sigma$ on it, and also each region $\mathbb{M}$ has its space of field configurations $\mathcal{C}_\mathbb{M}$. This space is the same for the hypersurface with opposite orientation: $\mathcal{C}_{\Sigma_\tau} = \mathcal{C}_{\overline{\Sigma_\tau}}$. If we foliate spacetime using a foliation parameter τ, we thus can write $\mathcal{C}_\tau$ instead of $\mathcal{C}_{\Sigma_\tau}$ when we refer to a constant-τ hypersurface.

The associated quantum **state space** $\mathcal{H}^{\mathrm{S}}_{\Sigma}$ then consists of the **Schrödinger wave function(al)s** $\psi^{\mathrm{S}}_{\Sigma} : \ \mathcal{C}_{\Sigma} \to \mathbb{C}$. We think of the state space of the oppositely oriented hypersurface as being the same: $\mathcal{H}^{\mathrm{S}}_{\Sigma_\tau} = \mathcal{H}^{\mathrm{S}}_{\overline{\Sigma_\tau}}$, and therefore again write $\mathcal{H}^{\mathrm{S}}_{\tau}$ instead of $\mathcal{H}^{\mathrm{S}}_{\Sigma_\tau}$ when we refer to a constant-τ hypersurface. The involution $\iota^{\mathrm{S}}_{\Sigma_\tau}$ relating states on Σ_τ to states on $\overline{\Sigma_\tau}$ is given by complex conjugation. We shall denote a state on Σ_τ and its $\iota^{\mathrm{S}}_{\Sigma_\tau}$-image on $\overline{\Sigma_\tau}$ by the same letter, and indicate by its subscript the orientation. That is, for $\varphi_\tau \in \mathcal{C}_\tau$ we write

$$(\iota^{\mathrm{S}}_{\Sigma_\tau} \psi^{\mathrm{S}}_{\Sigma_\tau})(\varphi_\tau) = \psi^{\mathrm{S}}_{\overline{\Sigma_\tau}}(\varphi_\tau) = \overline{\psi^{\mathrm{S}}_{\Sigma_\tau}(\varphi_\tau)} \, .$$

This just says that we can move the bar from over a hypersurface towards over the state and vice versa.
The state space $\mathcal{H}^{\mathrm{S}}_{\Sigma_\tau}$ is turned into a Hilbert space by the **inner product**

$$\left\langle \eta^{\mathrm{S}}_{\Sigma_\tau}, \, \zeta^{\mathrm{S}}_{\Sigma_\tau} \right\rangle_{\Sigma_\tau} = \int_{\mathcal{C}_\tau} \mathcal{D}\varphi \; \overline{\eta^{\mathrm{S}}_{\Sigma_\tau}(\varphi)} \, \zeta^{\mathrm{S}}_{\Sigma_\tau}(\varphi) \, .$$

To each region $\mathbb{M}$ in spacetime with boundary $\partial\mathbb{M}$ can thus be associated its boundary state space $\mathcal{H}^{\mathrm{S}}_{\partial\mathbb{M}}$. As a generalization of the usual quantum-mechanical transition amplitude, each boundary state $\psi^{\mathrm{S}}_{\partial\mathbb{M}}$ is given a complex amplitude via the linear **amplitude map** $\rho^{\mathrm{S}}_{\mathbb{M}} : \ \mathcal{H}^{\mathrm{S}}_{\partial\mathbb{M}} \ \to \ \mathbb{C}$ of the region:

$$\rho^{\mathrm{S}}_{\mathbb{M}}(\psi^{\mathrm{S}}_{\partial\mathbb{M}}) \ = \int\limits_{\mathcal{C}_{\partial\mathbb{M}}} \mathcal{D}\varphi \ \ \psi^{\mathrm{S}}_{\partial\mathbb{M}}(\varphi) \ Z_{\mathbb{M}}(\varphi) \ .$$

The quantity $Z_{\mathbb{M}}$ is called the **field propagator** of the region $\mathbb{M}$. It is computed as a **Feynman path integral** of the action:

$$Z_{\mathbb{M}}(\varphi) \ = \int\limits_{\phi|_{\partial\mathbb{M}}=\varphi} \mathcal{D}\phi \ \ \exp\left(\mathrm{i} S_{\mathbb{M}}(\phi)\right) \ .$$

The amplitude map $\rho_{\mathbb{M}}$ induces a map $\widetilde{\rho}_{\mathbb{M}} : \ \mathcal{H}_{\Sigma_1} \ \to \ \mathcal{H}_{\Sigma_2}$ by

$$\left\langle \zeta^{\mathrm{S}}_{\overline{\Sigma}_2} , \ \iota^{\mathrm{S}}_{\Sigma_2} \, \widetilde{\rho}_{\mathbb{M}} \, \eta^{\mathrm{S}}_{\Sigma_1} \right\rangle_{\overline{\Sigma}_2} \stackrel{!}{=} \ \rho_{\mathbb{M}}(\eta^{\mathrm{S}}_{\Sigma_1} \otimes \zeta^{\mathrm{S}}_{\overline{\Sigma}_2}) \qquad \forall \ \eta^{\mathrm{S}}_{\Sigma_1} \in \mathcal{H}_{\Sigma_1} \zeta^{\mathrm{S}}_{\overline{\Sigma}_2} \in \mathcal{H}_{\overline{\Sigma}_2} .$$

In the standard situation $\widetilde{\rho}_{\mathbb{M}}$ is just the time evolution operator $\hat{\mathcal{U}}$.

The rest of this section is basically Section 4.1 of [RO2005]. To discuss the probability interpretation, we start with a review of it in the standard formulation. (In this elementary discussion of probabilities we assume for simplicity that state spaces are finite dimensional. This avoids difficulties of the infinite dimensional case which might require the introduction of probability densities etc.)

We consider a slice $\mathbb{M}_{[t_1,t_2]}$ of Minkowski spacetime bounded by two spacelike equal-time hyperplanes Σ_1 and $\overline{\Sigma}_2$ at times $t_{1,2}$. We canonically orient all equal-time hypersurfaces Σ_t backwards in time, and by a bar as in $\overline{\Sigma}_2$ we thus indicate forward orientation. We also recall that canonically the boundary of a region is outwards oriented, therefore $\partial\mathbb{M}_{[t_1,t_2]} = \Sigma_1 \sqcup \overline{\Sigma}_2$.

Let $\psi_{\Sigma_1} = |\,\psi\rangle_{\Sigma_1} \in \mathcal{H}_{\Sigma_1}$ be the *normalized* ket-state of a quantum system at time t_1 and ${}^*\eta_{\overline{\Sigma}_2} = {}_{\overline{\Sigma}_2}\langle\,\eta\,| \in \mathcal{H}^*_{\overline{\Sigma}_2}$ a normalized bra-state at time t_2. Usually of course one considers only one state space, i.e., $\mathcal{H}_{\Sigma_1}$ and $\mathcal{H}_{\overline{\Sigma}_2}$ are canonically identified via time-translation symmetry. However here we will distinguish them formally in order to aid the later comparison with the General Boundary Formulation. The associated **transition amplitude** A is given by

$$A = {}_{\overline{\Sigma}_2}\langle\,\eta\,|\,\hat{\mathcal{U}}_{[t_1,t_2]}\,|\,\psi\rangle_{\Sigma_1} = {}^*\eta_{\overline{\Sigma}_2}(\widetilde{\rho}_{\mathbb{M}}\psi_{\Sigma_1})$$

where $\hat{\mathcal{U}}_{[t_1,t_2]} : \quad \mathcal{H}_{\Sigma_1} \rightarrow \mathcal{H}_{\overline{\Sigma}_2}$ is the **time-evolution operator**. The associated probability P is the modulus square of the transition amplitude: $P = |A|^2$.

What is the physical meaning of P? The simplest interpretation of this quantity is as expressing the probability of finding the *normalized* state $\eta_{\overline{\Sigma}_2}$ at time t_2 given that the normalized state ψ_{Σ_1} was prepared at time t_1. Thus, we are dealing with a **conditional probability**. More specifically, such a probability usually depends on two types of data: data describing knowledge or **preparation** and data describing observation, i.e., a **measurement** which fixes the answer to a question.

Mathematically, for two *independent* events A and B, the conditional probability $P(A|B)$ of finding A given B is defined as

$$P(A|B) := \frac{P(A \text{ AND } B)}{P(B)} \, .$$

Example 1.1: In order to make this more explicit, for the simplest case let us write this probability as $P(\eta_{\overline{\Sigma}_2}|\psi_{\Sigma_1})$ (read: the probability of [observing] $\eta_{\overline{\Sigma}_2}$ conditional on [the preparation of] ψ_{Σ_1}):

$$P\left(\eta_{\overline{\Sigma}_2}\middle|\psi_{\Sigma_1}\right) = \left|\, {}_{\overline{\Sigma}_2}\langle\,\eta\,|\,\hat{\mathcal{U}}_{[t_1,t_2]}\,|\,\psi\,\rangle_{\Sigma_1}\right|^2.$$

An important ingredient of this interpretation is that the cumulative probability of all exclusive alternatives is 1. The meaning of the latter is specified with the help of the inner product which defines orthonormality. Let $\{\xi_{\overline{\Sigma}_2,h}\}_{h\,\in\,H_2}$ be an orthonormal basis of $\mathcal{H}_{\overline{\Sigma}_2}$, representing a complete set of mutually exclusive measurement outcomes, then this implies

$$1 \;=\; \sum_{h\,\in\,H_2} P\Big(\xi_{\overline{\Sigma}_2,h}\Big|\psi_{\Sigma_1}\Big) \;=\; \sum_{h\,\in\,H_2} \Big|\;{}_{\overline{\Sigma}_2}\langle\,\xi_h\,|\,\hat{\mathcal{U}}_{[t_1\,,\,t_2]}\,|\,\psi\rangle_{\Sigma_1}\Big|^2 \,.$$

Example 1.2: As an obvious extension of the Example 1.1 suppose now that we know a priori that only certain measurement outcomes might occur. (We might select a suitable subset of performed measurements in order to exclude other outcomes.) A way to formalize this is to say that the possible measurement outcomes lie in a closed subspace $\mathcal{P}_{\overline{\Sigma}_2}$ of $\mathcal{H}_{\overline{\Sigma}_2}$. Suppose that the orthonormal base of $\mathcal{H}_{\overline{\Sigma}_2}$ restricts to an ONB $\{\xi_{\overline{\Sigma}_2,p}\}_{p \in P_2 \subseteq H_2}$ of $\mathcal{P}_{\overline{\Sigma}_2}$.

We are now interested in the probability of measuring an outcome specified by a single state $\xi_{\overline{\Sigma}_2,m} \in \mathcal{P}_{\overline{\Sigma}_2}$ with $m \in P_2$ conditional both on the prepared state being ψ_{Σ_1} and knowing that the outcome must lie in $\mathcal{P}_{\overline{\Sigma}_2}$. We denote this conditional probability by $P(\xi_{\overline{\Sigma}_2,m}|\psi_{\Sigma_1},\mathcal{P}_{\overline{\Sigma}_2})$. According to the definition of a conditional probability, in order to obtain it we must divide the conditional probability $P(\xi_{\overline{\Sigma}_2,m}|\psi_{\Sigma_1})$ by the probability $P(\mathcal{P}_{\overline{\Sigma}_2}|\psi_{\Sigma_1})$ that the outcome of the measurement lies in $\mathcal{P}_{\overline{\Sigma}_2}$ given the prepared state is ψ_{Σ_1}. The latter is simply

$$\begin{aligned} P\left(\mathcal{P}_{\overline{\Sigma}_2}\middle|\psi_{\Sigma_1}\right) &= \sum_{p\,\in\,P_2} P\left(\xi_{\overline{\Sigma}_2,p}\middle|\psi_{\Sigma_1}\right) \\ &= \sum_{p\,\in\,P_2} \left|\ {}_{\overline{\Sigma}_2}\langle\,\xi_p\,|\hat{\mathcal{U}}_{[t_1,t_2]}\,|\,\psi\rangle_{\Sigma_1}\right|^2 \end{aligned}$$

$$0 < P\left(\mathcal{P}_{\overline{\Sigma}_2}\middle|\psi_{\Sigma_1}\right) \leq 1 \quad \Leftrightarrow \quad \varnothing \neq P_2 \subseteq H_2$$

Supposing the result is not zero, which would imply the impossibility of obtaining any measurement outcome in $\mathcal{P}_{\overline{\Sigma}_2}$ and thus the meaninglessness of the quantity $P(\xi_m|\psi_{\Sigma_1}, \mathcal{P}_{\overline{\Sigma}_2})$, this implies

$$
\begin{aligned}
P\Big(\xi_{\overline{\Sigma}_2,m}\Big|\psi_{\Sigma_1}, \mathcal{P}_{\overline{\Sigma}_2}\Big) &= \frac{P\Big(\xi_{\overline{\Sigma}_2,m}\Big|\psi_{\Sigma_1}\Big)}{P\Big(\mathcal{P}_{\overline{\Sigma}_2}\Big|\psi_{\Sigma_1}\Big)} \\
&= \frac{\Big|\, {}_{\overline{\Sigma}_2}\langle\, \xi_m \,|\, \hat{\mathcal{U}}_{[t_1,t_2]} \,|\, \psi \rangle_{\Sigma_1} \Big|^2}{\sum\limits_{p \,\in\, P_2} \Big|\, {}_{\overline{\Sigma}_2}\langle\, \xi_p \,|\, \hat{\mathcal{U}}_{[t_1,t_2]} \,|\, \psi \rangle_{\Sigma_1} \Big|^2} \,.
\end{aligned}
$$

Since the denominator is positive and smaller than 1, we have $P\Big(\xi_{\overline{\Sigma}_2,m}\Big|\psi_{\Sigma_1}, \mathcal{P}_{\overline{\Sigma}_2}\Big) > P\Big(\xi_{\overline{\Sigma}_2,m}\Big|\psi_{\Sigma_1}\Big)$, as expected.

Example 1.3: We can further modify the Example 1.2 by testing not against a single state, but a closed subspace $\mathcal{M}_{\overline{\Sigma}_2} \subseteq \mathcal{P}_{\overline{\Sigma}_2}$. We denote the associated conditional probability by $P(\mathcal{M}_{\overline{\Sigma}_2} | \psi_{\Sigma_1}, \mathcal{P}_{\overline{\Sigma}_2})$. This is obviously the sum of conditional probabilities $P(\xi_{\overline{\Sigma}_2,m} | \psi_{\Sigma_1}, \mathcal{P}_{\overline{\Sigma}_2})$ for an orthonormal basis $\{\xi_{\overline{\Sigma}_2,m}\}_{m \in M_2 \subseteq P_2}$ of $\mathcal{M}_{\overline{\Sigma}_2}$ (to which again we suppose the ONB of $\mathcal{P}_{\overline{\Sigma}_2}$ to be restricting):

$$P\Big(\mathcal{M}_{\overline{\Sigma}_2} \Big| \psi_{\Sigma_1}, \mathcal{P}_{\overline{\Sigma}_2}\Big) = \frac{\sum\limits_{m \in M_2} \Big| {}_{\overline{\Sigma}_2}\langle \xi_m \,|\, \hat{\mathcal{U}}_{[t_1,t_2]} \,|\, \psi \rangle_{\Sigma_1} \Big|^2}{\sum\limits_{p \in P_2} \Big| {}_{\overline{\Sigma}_2}\langle \xi_p \,|\, \hat{\mathcal{U}}_{[t_1,t_2]} \,|\, \psi \rangle_{\Sigma_1} \Big|^2}$$

$$0 \le P\Big(\mathcal{M}_{\overline{\Sigma}_2} \Big| \psi_{\Sigma_1}, \mathcal{P}_{\overline{\Sigma}_2}\Big) \le 1 \quad \Leftrightarrow \quad M_2 \subseteq P_2 \subseteq H_2.$$

Example 1.4: A conceptually different extension is the following: suppose now that $\{\xi_{\Sigma_1,h}\}_{h\in H_1}$ is an orthonormal basis of $\mathcal{H}_{\Sigma_1}$, then the quantity

$$P\left(\xi_{\Sigma_1,h}\middle|\eta_{\overline{\Sigma}_2}\right) = \left|\,{}_{\overline{\Sigma}_2}\langle\,\eta\,|\,\hat{\mathcal{U}}_{[t_1,t_2]}\,|\,\xi_h\rangle_{\Sigma_1}\right|^2$$

describes the conditional probability of the prepared state having been $\xi_{\Sigma_1,h}$ given that $\eta_{\overline{\Sigma}_2}$ was measured. This may be understood in the following sense. Suppose somebody prepared a large sample of measurements with random choices of initial states $\xi_{\Sigma_1,h}$. We then perform measurements as to whether the final state is $\eta_{\overline{\Sigma}_2}$ or not (the latter meaning that it is orthogonal to $\eta_{\overline{\Sigma}_2}$). The probability distribution of the initial states $\xi_{\Sigma_1,h}$ in the sample of measurements resulting in $\eta_{\overline{\Sigma}_2}$ is then given by $P(\xi_{\Sigma_1,h}|\eta_{\overline{\Sigma}_2})$.

These four examples are supposed to illustrate two points. Firstly, the modulus square of a transition amplitude can be interpreted as a conditional probability in various different ways. Secondly, the roles of different parts of a measurement process, in respect to which is considered the conditional one and which the depending one, are not fixed. In particular, the interpretation is not restricted to "final state conditional on initial state".

Outline

This section is basically Section 4.2 of [RO2005]. In the General Boundary Formulation the dependence of probabilities on preparation data and observation data is preserved. The considerations of the previous section together with the GBF context lead to the following formulation of the probability interpretation.
Consider a process taking place in a spacetime region $\mathbb{M}$ with a boundary $\partial\mathbb{M}$. Let $\mathcal{H}_{\partial\mathbb{M}}$ be the generalized state space describing the given physical system or measurement setup, i.e., the state space associated with the boundary $\partial\mathbb{M}$. Then, both types of data are encoded through closed subspaces of $\mathcal{H}_{\partial\mathbb{M}}$: we suppose that a certain "prepared" knowledge about the process amounts to the specification of the closed **preparation subspace** $\mathcal{P}_{\partial\mathbb{M}} \subset \mathcal{H}_{\partial\mathbb{M}}$. That is, we assume that we know the state describing the measurement process to be part of that subspace.

Say we are now interested in answering the question whether the measurement outcome corresponds to a closed **measurement subspace** $\mathcal{M}_{\partial\mathbb{M}} \subseteq \mathcal{P}_{\partial\mathbb{M}} \subset \mathcal{H}_{\partial\mathbb{M}}$. That is, we are interested in the conditional probability $P(\mathcal{M}_{\partial\mathbb{M}}|\mathcal{P}_{\partial\mathbb{M}})$ of the measurement process being described by the measurement subspace $\mathcal{M}_{\partial\mathbb{M}}$, given that its preparation is described by the preparation subspace $\mathcal{P}_{\partial\mathbb{M}}$.
Let again be $\{\xi_h\}_{h\,\in\,H}$ an orthonormal basis of $\mathcal{H}_{\partial\mathbb{M}}$ which reduces to an ONB $\{\xi_p\}_{p\,\in\,P\,\subseteq\,H}$ of $\mathcal{P}_{\partial\mathbb{M}}$ and further to an ONB $\{\xi_m\}_{m\,\in\,M\,\subseteq\,P\,\subset\,H}$ of $\mathcal{M}_{\partial\mathbb{M}}$. Then a first way of expressing $P(\mathcal{M}_{\partial\mathbb{M}}|\mathcal{P}_{\partial\mathbb{M}})$ is:

$$P\Big(\mathcal{M}_{\partial\mathbb{M}}\Big|\mathcal{P}_{\partial\mathbb{M}}\Big) \;=\; \frac{\displaystyle\sum_{m\,\in\,M} \big|\rho_{\mathbb{M}}(\xi_m)\big|^2}{\displaystyle\sum_{p\,\in\,P} \big|\rho_{\mathbb{M}}(\xi_p)\big|^2}$$

(One might be tempted to interpret the numerator and the denominator separately as probabilities. However, that does not appear to be meaningful in general.) As a special case, if $\mathcal{M}_{\partial\mathbb{M}}$ has dimension one, being spanned by one normalized vector ξ, we also write $P(\mathcal{M}_{\partial\mathbb{M}}|\mathcal{P}_{\partial\mathbb{M}}) \;=\; P(\xi|\mathcal{P}_{\partial\mathbb{M}})$.

Let us check that $P(\mathcal{M}_{\partial\mathbb{M}}|\mathcal{P}_{\partial\mathbb{M}})$ indeed has the properties of a **quantum mechanical probability:**
By construction we have **probabilities in the unit interval:**

$$0 \leq P(\mathcal{M}_{\partial\mathbb{M}}|\mathcal{P}_{\partial\mathbb{M}}) \leq 1\ .$$

It might now happen that the denominator is zero. This would imply that the probability of observing *anything* given the preparation $\mathcal{P}_{\partial\mathbb{M}}$ vanishes and thus the conditional probability is physically meaningless. Moreover, because of $(M \subseteq P)$ this implies that the numerator vanishes, too, and thus $P(\mathcal{M}_{\partial\mathbb{M}}|\mathcal{P}_{\partial\mathbb{M}})$ is undefined. Thus the knowledge encoded in $\mathcal{P}_{\partial\mathbb{M}}$ does not correspond to any physically allowed process.

For two mutually exclusive observations encoded by *orthogonal* subspaces $\mathcal{M}_{\partial\mathbb{M},1}$ and $\mathcal{M}_{\partial\mathbb{M},2}$, we have **additive probabilities:**

$$P\Big(\mathcal{M}_{\partial\mathbb{M},1} \oplus \mathcal{M}_{\partial\mathbb{M},2} \,\Big|\, \mathcal{P}_{\partial\mathbb{M}}\Big) \;=\; P\Big(\mathcal{M}_{\partial\mathbb{M},1} \Big| \mathcal{P}_{\partial\mathbb{M}}\Big) + P\Big(\mathcal{M}_{\partial\mathbb{M},2} \Big| \mathcal{P}_{\partial\mathbb{M}}\Big)$$

$$\Updownarrow$$

$$\hat{\mathrm{P}}_{\mathcal{M}_{\partial\mathbb{M},1}} \circ \hat{\mathrm{P}}_{\mathcal{M}_{\partial\mathbb{M},2}}\, a_{\partial\mathbb{M}} \;=\; 0 \;=\; \hat{\mathrm{P}}_{\mathcal{M}_{\partial\mathbb{M},2}} \circ \hat{\mathrm{P}}_{\mathcal{M}_{\partial\mathbb{M},1}}\, a_{\partial\mathbb{M}} \qquad \forall\, a_{\partial\mathbb{M}} \in \mathcal{H}_{\partial\mathbb{M}}\,,$$

(with $\hat{\mathrm{P}}_{\mathcal{M}_{\partial\mathbb{M},1}}$ the projection operator onto subspace $\mathcal{M}_{\partial\mathbb{M},1} \subset \mathcal{H}_{\partial\mathbb{M}}$) as can be seen at a glance by inserting the probability definition into the equation above.

Arbitrary outcome has unity probability for any (allowed) preparation and $M \subseteq P$, i.e., the probability for $M = P$ equals unity.

$$1 = P\Big(\mathcal{P}_{\partial\mathbb{M}}\Big|\mathcal{P}_{\partial\mathbb{M}}\Big) = \frac{\sum\limits_{m\,\in\,P} \big|\rho_{\mathbb{M}}(\xi_m)\big|^2}{\sum\limits_{p\,\in\,P} \big|\rho_{\mathbb{M}}(\xi_p)\big|^2} \qquad \forall\, \mathcal{P}_{\partial\mathbb{M}} \subset \mathcal{H}_{\partial\mathbb{M}}$$

If we have $\mathcal{M}_{\partial\mathbb{M},2}$ implies $\mathcal{M}_{\partial\mathbb{M},1}$ implies $\mathcal{P}_{\partial\mathbb{M}}$, then the following **probability chain rule** holds:

$$\begin{aligned} & \mathcal{M}_{\partial\mathbb{M},2} \subseteq \mathcal{M}_{\partial\mathbb{M},1} \subseteq \mathcal{P}_{\partial\mathbb{M}} \subset \mathcal{H}_{\partial\mathbb{M}} \\ \Rightarrow \qquad & P\Big(\mathcal{M}_{\partial\mathbb{M},2}\Big|\mathcal{P}_{\partial\mathbb{M}}\Big) = P\Big(\mathcal{M}_{\partial\mathbb{M},2}\Big|\mathcal{M}_{\partial\mathbb{M},1}\Big)\, P\Big(\mathcal{M}_{\partial\mathbb{M},1}\Big|\mathcal{P}_{\partial\mathbb{M}}\Big). \end{aligned}$$

This can quickly be checked by inserting the probability definition into the chain rule.

Now let's see how the GBF's probability definition reproduces the probability interpretation in the standard situation of the slice region $\mathbb{M} = \mathbb{M}_{[t_1, t_2]}$ of examples 1.1 -1.4. First of all, according to the core axioms the boundary state space factors into a tensor product of two state spaces: $\mathcal{H}_{\partial\mathbb{M}} = \mathcal{H}_{\Sigma_1} \otimes \mathcal{H}_{\overline{\Sigma}_2}$.

Example 2.1: For Example 1.1 we select a normalized state $\psi_{\Sigma_1} \in \mathcal{H}_{\Sigma_1}$ and set

$$\mathcal{P}_{\partial\mathbb{M}} = \text{"}\,\psi_{\Sigma_1} \otimes \mathcal{H}_{\overline{\Sigma}_2}\,\text{"} \quad \subset \mathcal{H}_{\partial\mathbb{M}}$$

$$\mathcal{P}_{\partial\mathbb{M}} := \left\{\alpha_\Sigma \in \mathcal{H}_{\partial\mathbb{M}} \,\middle|\, \exists\, \eta_{\overline{\Sigma}_2} \in \mathcal{H}_{\overline{\Sigma}_2} : \alpha_\Sigma = \psi_{\Sigma_1} \otimes \eta_{\overline{\Sigma}_2}\right\} \subset \mathcal{H}_{\partial\mathbb{M}}$$

Let us denote by $\{\psi_{\Sigma_1} \otimes \xi_{\overline{\Sigma}_2,p}\}_{p \,\in\, P \,=\, H_2}$ an orthonormal basis of $\mathcal{P}_{\partial\mathbb{M}}$. Then, the probability of "observing" the normalized $\eta_{\overline{\Sigma}_2} \in \mathcal{H}_{\overline{\Sigma}_2}$, which corresponds to setting

$$\mathcal{M}_{\partial\mathbb{M}} \;=\; \psi_{\Sigma_1} \otimes \eta_{\overline{\Sigma}_2}\,,$$

subject to the "preparation" of $\psi_{\Sigma_1} \in \mathcal{H}_{\Sigma_1}$ turns out as

$$P\Big(\mathcal{M}_{\partial\mathbb{M}}\Big|\mathcal{P}_{\partial\mathbb{M}}\Big) \;=\; \frac{\Big|\rho_{\mathbb{M}}(\psi_{\Sigma_1} \otimes \eta_{\overline{\Sigma}_2})\Big|^2}{\sum\limits_{p\,\in\,P}\Big|\rho_{\mathbb{M}}(\psi_{\Sigma_1} \otimes \xi_{\overline{\Sigma}_2,p})\Big|^2} \;=\; \Big|\rho_{\mathbb{M}}(\psi_{\Sigma_1} \otimes \eta_{\overline{\Sigma}_2})\Big|^2\,.$$

(We comment on unity denominator at the end of this section.)

Comparing the notation to the standard formalism, i.e., recognizing

$$\rho_{\mathbb{M}}(\psi_{\Sigma_1} \otimes \eta_{\overline{\Sigma}_2}) \;=\; {}_{\overline{\Sigma}_2}\langle\, \eta \,|\, \hat{\mathcal{U}}_{[t_1,t_2]} \,|\, \psi \rangle_{\Sigma_1}$$

shows that we recover the standard result $P(\eta_{\overline{\Sigma}_2} | \psi_{\Sigma_1})$,

$$\rho_{\mathbb{M}}(\psi_{\Sigma_1} \otimes \eta_{\overline{\Sigma}_2}) \;=\; \langle \eta_{\overline{\Sigma}_2}, \iota_{\Sigma_2} \widetilde{\rho}_{\mathbb{M}} \psi_{\Sigma_1} \rangle_{\overline{\Sigma}_2} \;=\; {}^{*}\eta_{\overline{\Sigma}_2}(\iota_{\Sigma_2} \widetilde{\rho}_{\mathbb{M}} \psi_{\Sigma_1}) \;=\; {}_{\overline{\Sigma}_2}\langle\, \eta \,|\, \hat{\mathcal{U}}_{[t_1,t_2]} \,|\, \psi \rangle_{\Sigma_1}.$$

The condition that the cumulative probability of all exclusive alternatives is 1, is now reproduced by construction:

$$1 = \sum_{p \,\in\, P} P\Big(\psi_{\Sigma_1} \otimes \xi_{\overline{\Sigma}_2,p} \Big| \mathcal{P}_{\Sigma}\Big) = \frac{\sum\limits_{p \,\in\, P} \Big|\rho_{\mathbb{M}}(\psi_{\Sigma_1} \otimes \xi_{\overline{\Sigma}_2,p})\Big|^2}{\sum\limits_{p \,\in\, P} \Big|\rho_{\mathbb{M}}(\psi_{\Sigma_1} \otimes \xi_{\overline{\Sigma}_2,p})\Big|^2}.$$

Example 2.2: Similarly, Example 1.2 is recovered by setting

$$\begin{aligned} \mathcal{P}_{\partial\mathbb{M}} &= \text{"}\,\psi_{\Sigma_1} \otimes \mathcal{P}_{\overline{\Sigma}_2}\text{"} \quad \subseteq \quad \text{"}\,\psi_{\Sigma_1} \otimes \mathcal{H}_{\overline{\Sigma}_2}\text{"} \quad \subset \ \mathcal{H}_{\partial\mathbb{M}} \\ \mathcal{P}_{\partial\mathbb{M}} &:= \left\{ \alpha_{\partial\mathbb{M}} \in \mathcal{H}_{\partial\mathbb{M}} \,|\, \exists \eta_{\overline{\Sigma}_2} \in \mathcal{P}_{\overline{\Sigma}_2} : \alpha_{\partial\mathbb{M}} = \psi_{\Sigma_1} \otimes \eta_{\overline{\Sigma}_2} \right\} \subset \mathcal{H}_{\partial\mathbb{M}} \\ \mathcal{M}_{\partial\mathbb{M}} &:= \psi_{\Sigma_1} \otimes \xi_{\overline{\Sigma}_2, m} \qquad m \in P_2 \end{aligned}$$

for an orthonormal basis $\{\psi_{\Sigma_1} \otimes \xi_{\overline{\Sigma}_2, p}\}_{p \in P_2}$ of $\mathcal{P}_{\partial\mathbb{M}}$ with $(\varnothing \neq P_2 \subseteq H_2)$. Then we get agreement of $P(\xi_{\overline{\Sigma}_2, k} | \psi_{\Sigma_1}, \mathcal{P}_{\overline{\Sigma}_2})$ with

$$P\left(\mathcal{M}_{\partial\mathbb{M}} \middle| \mathcal{P}_{\partial\mathbb{M}}\right) = \frac{\left| \rho_{\mathbb{M}}(\psi_{\Sigma_1} \otimes \xi_{\overline{\Sigma}_2, m}) \right|^2}{\sum\limits_{p \in P_2} \left| \rho_{\mathbb{M}}(\psi_{\Sigma_1} \otimes \xi_{\overline{\Sigma}_2, p}) \right|^2}.$$

Example 2.3: For Example 1.3 we keep $\mathcal{P}_{\partial\mathbb{M}}$ and its orthonormal basis and assume that it restricts to an ONB $\{\psi_{\Sigma_1} \otimes \xi_{\overline{\Sigma}_2,m}\}_{m \in M_2}$ of $\mathcal{M}_{\partial\mathbb{M}}$ with $(M_2 \subseteq P_2 \subseteq H_2)$ and

$$\begin{aligned} \mathcal{M}_{\partial\mathbb{M}} &= \text{"}\,\psi_{\Sigma_1} \otimes \mathcal{M}_{\overline{\Sigma}_2}\text{"} \quad \subseteq \quad \text{"}\,\psi_{\Sigma_1} \otimes \mathcal{P}_{\overline{\Sigma}_2}\text{"} \quad \subset \; \mathcal{H}_{\partial\mathbb{M}} \\ \mathcal{M}_{\partial\mathbb{M}} &:= \left\{\alpha_{\partial\mathbb{M}} \in \mathcal{H}_{\partial\mathbb{M}} \,\middle|\, \exists \eta_{\overline{\Sigma}_2} \in \mathcal{M}_{\overline{\Sigma}_2} : \alpha_{\partial\mathbb{M}} = \psi_{\Sigma_1} \otimes \eta_{\overline{\Sigma}_2}\right\} \subset \mathcal{H}_{\partial\mathbb{M}}. \end{aligned}$$

Then we recover $P(\mathcal{M}_{\overline{\Sigma}_2} | \psi_{\Sigma_1}, \mathcal{P}_{\overline{\Sigma}_2})$ via

$$P\left(\mathcal{M}_{\partial\mathbb{M}} \middle| \mathcal{P}_{\partial\mathbb{M}}\right) = \frac{\sum_{m \in M_2} \left|\rho_{\mathbb{M}}(\psi_{\Sigma_1} \otimes \xi_{\overline{\Sigma}_2,m})\right|^2}{\sum_{p \in P_2} \left|\rho_{\mathbb{M}}(\psi_{\Sigma_1} \otimes \xi_{\overline{\Sigma}_2,p})\right|^2}.$$

Example 2.4: For Example 1.4 we have

$$\mathcal{P}_{\partial\mathbb{M}} = \text{"}\,\mathcal{H}_{\Sigma_1} \otimes \eta_{\overline{\Sigma}_2}\text{"} \quad \subset \mathcal{H}_{\partial\mathbb{M}}$$

$$\mathcal{P}_{\partial\mathbb{M}} := \left\{ \alpha_{\partial\mathbb{M}} \in \mathcal{H}_{\partial\mathbb{M}} \,\middle|\, \exists \psi_{\Sigma_1} \in \mathcal{H}_{\Sigma_1} : \alpha_{\partial\mathbb{M}} = \psi_{\Sigma_1} \otimes \eta_{\overline{\Sigma}_2} \right\} \subset \mathcal{H}_{\partial\mathbb{M}}$$

and denote by $\{\xi_{\Sigma_1,p} \otimes \eta_{\overline{\Sigma}_2}\}_{p \in P_1 = H_1}$ an orthonormal basis of $\mathcal{P}_{\partial\mathbb{M}}$.

Then, the probability of "observing" (respectively "having observed" since $t_1 < t_2$) a state $\xi_{\Sigma_1,m} \in \mathcal{H}_{\Sigma_1}$, which corresponds to setting

$$\mathcal{M}_{\partial\mathbb{M}} = \xi_{\Sigma_1,m} \otimes \eta_{\overline{\Sigma}_2} ,$$

subject to the "preparation" (respectively "post-preparation") of $\eta_{\overline{\Sigma}_2} \in \mathcal{H}_{\overline{\Sigma}_2}$ turns out as

$$P\left(\mathcal{M}_{\partial\mathbb{M}} \middle| \mathcal{P}_{\partial\mathbb{M}}\right) = \frac{\left|\rho_{\mathbb{M}}(\xi_{\Sigma_1,m} \otimes \eta_{\overline{\Sigma}_2})\right|^2}{\sum\limits_{p \in P_1} \left|\rho_{\mathbb{M}}(\xi_{\Sigma_1,p} \otimes \eta_{\overline{\Sigma}_2})\right|^2} = \left|\rho_{\mathbb{M}}(\xi_{\Sigma_1,k} \otimes \eta_{\overline{\Sigma}_2})\right|^2 .$$

which recovers $P(\xi_{\Sigma_1,k}|\eta_{\overline{\Sigma}_2})$. The important point here of course consists in "post-preparing" the experimental setup at time $t_2 > t_1$ and "measuring" the initial state retroactively. This again illustrates nicely that for certain data the "preparation vs. observation" interpretation is independent of the temporal sequence of the events described in the data.

Thus **preparation** in a generalized sense can be expressed as ”fixing some (input and output) parts” of an experiment while leaving some parts unfixed. In other words, preparation is fixing the question asked to the physical system *and* fixing the possible answers. **Measurement** can then be seen as ”fixing the parts left undetermined” in the preparation, or determining which of the possible answers occured.

Here input (output) denotes anything flowing into (out of) the spacetime region within which the experiment is conducted.

We recall that in Example 1.1 we had chosen ψ_{Σ_1} to be normalized while $\{\xi_{\overline{\Sigma}_2,p}\}_{p\,\in\,P_2\,=\,H_2}$ was an orthonormal basis of $\mathcal{H}_{\overline{\Sigma}_2}$. Moreover, the induced map $\widetilde{\rho}_{\mathbb{M}} : \mathcal{H}_{\Sigma_1} \to \mathcal{H}_{\Sigma_2}$ should be an isomorphism which conserves the inner product. This then implies for these examples that

$$1 = \sum_{p\,\in\,P_2} \left|\rho_{\mathbb{M}}(\psi_{\Sigma_1} \otimes \xi_{\overline{\Sigma}_2,p})\right|^2 = \sum_{p\,\in\,P_2} \left|\langle \xi_{\overline{\Sigma}_2,p},\, \iota_{\Sigma_2}\widetilde{\rho}_{\mathbb{M}}\psi_{\Sigma_1}\rangle_{\overline{\Sigma}_2}\right|^2$$

Either $\iota_{\Sigma_2}\widetilde{\rho}_{\mathbb{M}}$ maps ψ_{Σ_1} directly to one vector of the orthonormal base $\{\xi_{\overline{\Sigma}_2,p}\}$ (which can of course be arranged by choosing the ONB adequately) and hence the sum is over Kronecker deltas all vanishing but one, or the sum is achieved by summing over all inner products (now each < 1) of $\iota_{\Sigma_2}\widetilde{\rho}_{\mathbb{M}}\psi_{\Sigma_1}$ with the vectors $\{\xi_{\overline{\Sigma}_2,p}\}$ of the ONB. By similar reasoning, the normalization factor in Example 1.4 equals unity, too (but not the ones in the second and third example, since therein not the whole Hilbert space $\mathcal{H}_{\overline{\Sigma}_2}$ is covered by $\mathcal{P}_{\Sigma}$).

Outline

This section is basically Section 3 of [RO2007]. We consider the same setting as in the previous subsection: with preparation encoded by a closed preparation subspace $\mathcal{P}_{\partial\mathbb{M}} \subset \mathcal{H}_{\partial\mathbb{M}}$ and measurement outcomes by a closed measurement subspace $\mathcal{M}_{\partial\mathbb{M}} \subset \mathcal{H}_{\partial\mathbb{M}}$. And again we are interested in the conditional probability $P(\mathcal{M}_{\partial\mathbb{M}}|\mathcal{P}_{\partial\mathbb{M}})$ of the measurement process being described by $\mathcal{M}_{\partial\mathbb{M}}$ given that its preparation is described by $\mathcal{P}_{\partial\mathbb{M}}$. The second (equivalent) way for expressing this probability is

$$P\Big(\mathcal{M}_{\partial\mathbb{M}}\Big|\mathcal{P}_{\partial\mathbb{M}}\Big) \;=\; \frac{\Big\|\, \rho_{\mathbb{M}} \circ \hat{\mathrm{P}}_{\mathcal{P}_{\partial\mathbb{M}}} \circ \hat{\mathrm{P}}_{\mathcal{M}_{\partial\mathbb{M}}} \,\Big\|^2}{\Big\|\, \rho_{\mathbb{M}} \circ \hat{\mathrm{P}}_{\mathcal{P}_{\partial\mathbb{M}}} \,\Big\|^2}$$

with $\hat{\mathrm{P}}_{\mathcal{P}_{\partial\mathbb{M}}}$ and $\hat{\mathrm{P}}_{\mathcal{M}_{\partial\mathbb{M}}}$ being the orthogonal projectors onto the subspaces $\mathcal{P}_{\partial\mathbb{M}}$ respectively $\mathcal{M}_{\partial\mathbb{M}}$, while $\circ$ denotes the composition of maps.

There is a slight difference between the presentation of the probability interpretation in this section and the one in Section 2. There, the subspace $\mathcal{M}_{\partial\mathbb{M}}$ was restricted to be a subspace of $\mathcal{P}_{\partial\mathbb{M}}$. This restriction is lifted here, which represents more a formal than a physical difference. Conceptually, making $\mathcal{M}_{\partial\mathbb{M}}$ a subspace of $\mathcal{P}_{\partial\mathbb{M}}$ just means taking into account the knowledge about the preparation when the question is asked. In particular, if $\hat{\mathrm{P}}_{\mathcal{P}_{\partial\mathbb{M}}}$ and $\hat{\mathrm{P}}_{\mathcal{M}_{\partial\mathbb{M}}}$ commute, then we can replace $\mathcal{M}_{\partial\mathbb{M}}$ by $\mathcal{M}_{\partial\mathbb{M}} \cap \mathcal{P}_{\partial\mathbb{M}}$ without any change to $P(\mathcal{M}_{\partial\mathbb{M}}|\mathcal{P}_{\partial\mathbb{M}})$. Thus the projector prescription for the probabilities is somewhat more flexible with respect to which measurement subspaces $\mathcal{M}_{\partial\mathbb{M}}$ it allows.

The expressions in denumerator and denominator which the norm is taken of are linear maps $\mathcal{H}_{\partial\mathbb{M}} \to \mathbb{C}$ and thus elements of $\mathcal{H}^*_{\partial\mathbb{M}}$. The norm of such maps is defined here as follows: let

$$\beta_{\partial\mathbb{M}} \in \mathcal{H}^*_{\partial\mathbb{M}} : \quad \mathcal{H}_{\partial\mathbb{M}} \to \mathbb{C}$$

be a bounded linear map. Then there exists its dual ${}^*\beta_{\partial\mathbb{M}} \in \mathcal{H}_{\partial\mathbb{M}}$ such that

$$\beta_{\partial\mathbb{M}}(\psi_{\partial\mathbb{M}}) = \langle {}^*\beta_{\partial\mathbb{M}}, \psi_{\partial\mathbb{M}} \rangle_{\partial\mathbb{M}} \qquad \forall\, \psi_{\partial\mathbb{M}} \in \mathcal{H}_{\partial\mathbb{M}}$$

and we define

$$\| \beta_{\partial\mathbb{M}} \|_{\mathcal{H}^*_{\partial\mathbb{M}}} := \| {}^*\beta_{\partial\mathbb{M}} \|_{\mathcal{H}_{\partial\mathbb{M}}} .$$

However, the amplitude map $\rho_{\mathbb{M}}$ is generically not bounded. Thus $\mathcal{P}_{\partial\mathbb{M}}$ must be "small enough" such that it makes $\rho_{\mathbb{M}} \circ \hat{\mathrm{P}}_{\mathcal{P}_{\partial\mathbb{M}}}$ into a bounded map. This condition is satisfied in standard situations.

We recall that projectors are orthogonal if and only if they are hermitian. Thus, for any map $\beta_{\partial\mathbb{M}}$ as above, any orthogonal projector $\hat{\mathrm{P}}_{\mathcal{S}_{\partial\mathbb{M}}}$ onto a subspace $\mathcal{S}_{\partial\mathbb{M}} \subset \mathcal{H}_{\partial\mathbb{M}}$ and for all $\psi_{\partial\mathbb{M}} \in \mathcal{H}_{\partial\mathbb{M}}$ we have

$$(\beta_{\partial\mathbb{M}} \circ \hat{\mathrm{P}}_{\mathcal{S}_{\partial\mathbb{M}}})\psi_{\partial\mathbb{M}} = \left\langle {}^*(\beta_{\partial\mathbb{M}} \circ \hat{\mathrm{P}}_{\mathcal{S}_{\partial\mathbb{M}}}),\ \psi_{\partial\mathbb{M}} \right\rangle_{\partial\mathbb{M}}$$
$$= \beta_{\partial\mathbb{M}}\,(\hat{\mathrm{P}}_{\mathcal{S}_{\partial\mathbb{M}}}\psi_{\partial\mathbb{M}}) = \left\langle {}^*\beta_{\partial\mathbb{M}}\,,\ \hat{\mathrm{P}}_{\mathcal{S}_{\partial\mathbb{M}}}\psi_{\partial\mathbb{M}} \right\rangle_{\partial\mathbb{M}} = \left\langle \hat{\mathrm{P}}_{\mathcal{S}_{\partial\mathbb{M}}}{}^*\beta_{\partial\mathbb{M}},\ \psi_{\partial\mathbb{M}} \right\rangle_{\partial\mathbb{M}}$$

$$\Longrightarrow \qquad {}^*(\beta_{\partial\mathbb{M}} \circ \hat{\mathrm{P}}_{\mathcal{S}_{\partial\mathbb{M}}}) = \hat{\mathrm{P}}_{\mathcal{S}_{\partial\mathbb{M}}}{}^*\beta_{\Sigma} \qquad \in \mathcal{H}_{\partial\mathbb{M}}$$

In the same way we can show that for several projectors $\{\hat{\mathrm{P}}_{\mathcal{S}_{\partial\mathbb{M},j}}\}_{j=1,\ldots,k}$ we have

$${}^*(\beta_{\partial\mathbb{M}} \circ \hat{\mathrm{P}}_{\mathcal{S}_{\partial\mathbb{M},1}} \ldots \hat{\mathrm{P}}_{\mathcal{S}_{\partial\mathbb{M},k}}) = \hat{\mathrm{P}}_{\mathcal{S}_{\partial\mathbb{M},k}} \ldots \hat{\mathrm{P}}_{\mathcal{S}_{\partial\mathbb{M},1}}\,({}^*\beta_{\partial\mathbb{M}}).$$

$P(\mathcal{M}_{\partial\mathbb{M}}|\mathcal{P}_{\partial\mathbb{M}})$ again has the properties of a quantum mechanical probability:
By construction we have **probabilities in the unit interval:**

$$0 \leq P(\mathcal{M}_{\partial\mathbb{M}}|\mathcal{P}_{\partial\mathbb{M}}) \leq 1 \; .$$

It might again happen that the denominator is zero. This would imply that the probability of observing *anything* given the preparation subspace $\mathcal{P}_{\partial\mathbb{M}}$ vanishes and thus the conditional probability is physically meaningless. Moreover this implies that the numerator vanishes, too, and thus $P(\mathcal{M}_{\partial\mathbb{M}}|\mathcal{P}_{\partial\mathbb{M}})$ is undefined. Thus the knowledge encoded in $\mathcal{P}_{\partial\mathbb{M}}$ does not correspond to any physically allowed process.

For two mutually exclusive observations encoded by *orthogonal* subspaces $\mathcal{M}_{\partial\mathbb{M},1}$ and $\mathcal{M}_{\partial\mathbb{M},2}$ we have **additive probabilities**:

$$P\Big(\mathcal{M}_{\partial\mathbb{M},1} \oplus \mathcal{M}_{\partial\mathbb{M},2} \,\Big|\, \mathcal{P}_{\partial\mathbb{M}}\Big) \;=\; P\Big(\mathcal{M}_{\partial\mathbb{M},1}\Big|\mathcal{P}_{\partial\mathbb{M}}\Big) + P\Big(\mathcal{M}_{\partial\mathbb{M},2}\Big|\mathcal{P}_{\partial\mathbb{M}}\Big)$$

$$\Updownarrow$$

$$\hat{\mathrm{P}}_{\mathcal{M}_{\partial\mathbb{M},1}} \circ \hat{\mathrm{P}}_{\mathcal{M}_{\partial\mathbb{M},2}}\, a_{\partial\mathbb{M}} \;=\; 0 \;=\; \hat{\mathrm{P}}_{\mathcal{M}_{\partial\mathbb{M},2}} \circ \hat{\mathrm{P}}_{\mathcal{M}_{\partial\mathbb{M},1}}\, a_{\partial\mathbb{M}} \qquad\qquad \forall\; a_{\partial\mathbb{M}} \in \mathcal{H}_{\partial\mathbb{M}}$$

because of

$$P\Big(\mathcal{M}_{\partial\mathbb{M},1} \oplus \mathcal{M}_{\partial\mathbb{M},2} \,\Big|\, \mathcal{P}_{\partial\mathbb{M}}\Big) \;=\; \frac{\Big\|\, \rho_{\mathbb{M}} \circ \hat{\mathrm{P}}_{\mathcal{P}_{\partial\mathbb{M}}} \circ \hat{\mathrm{P}}_{\mathcal{M}_{\partial\mathbb{M},1} \oplus \mathcal{M}_{\partial\mathbb{M},2}} \,\Big\|^2}{\Big\|\, \rho_{\mathbb{M}} \circ \hat{\mathrm{P}}_{\mathcal{P}_{\partial\mathbb{M}}} \,\Big\|^2}$$

$$= \frac{\Big\|\, \rho_{\mathbb{M}} \circ \hat{\mathrm{P}}_{\mathcal{P}_{\partial\mathbb{M}}} \circ (\hat{\mathrm{P}}_{\mathcal{M}_{\partial\mathbb{M},1}} + \hat{\mathrm{P}}_{\mathcal{M}_{\partial\mathbb{M},2}}) \,\Big\|^2}{\Big\|\, \rho_{\mathbb{M}} \circ \hat{\mathrm{P}}_{\mathcal{P}_{\partial\mathbb{M}}} \,\Big\|^2}$$

$$= \frac{\Big\langle {}^*\Big(\rho_{\mathbb{M}} \circ \hat{\mathrm{P}}_{\mathcal{P}_{\partial\mathbb{M}}} \circ (\hat{\mathrm{P}}_{\mathcal{M}_{\partial\mathbb{M},1}} + \hat{\mathrm{P}}_{\mathcal{M}_{\partial\mathbb{M},2}})\Big),\; {}^*\Big(\rho_{\mathbb{M}} \circ \hat{\mathrm{P}}_{\mathcal{P}_{\partial\mathbb{M}}} \circ (\hat{\mathrm{P}}_{\mathcal{M}_{\partial\mathbb{M},1}} + \hat{\mathrm{P}}_{\mathcal{M}_{\partial\mathbb{M},2}})\Big) \Big\rangle_{\partial\mathbb{M}}}{\Big\|\, \rho_{\mathbb{M}} \circ \hat{\mathrm{P}}_{\mathcal{P}_{\partial\mathbb{M}}} \,\Big\|^2}$$

$$= P\Big(\mathcal{M}_{\partial\mathbb{M},1}\Big|\mathcal{P}_{\partial\mathbb{M}}\Big) + P\Big(\mathcal{M}_{\partial\mathbb{M},2}\Big|\mathcal{P}_{\partial\mathbb{M}}\Big) + (2\,\mathbb{R}\mathrm{e}\; N) \,/\, \Big\|\, \rho_{\mathbb{M}} \circ \hat{\mathrm{P}}_{\mathcal{P}_{\partial\mathbb{M}}} \,\Big\|^2$$

wherein N vanishes:

$$
\begin{aligned}
N &= \left\langle {}^*\!\left(\rho_{\mathbb{M}} \circ \hat{\mathrm{P}}_{\mathcal{P}_{\partial\mathbb{M}}} \circ \hat{\mathrm{P}}_{\mathcal{M}_{\partial\mathbb{M},1}}\right),\ {}^*\!\left(\rho_{\mathbb{M}} \circ \hat{\mathrm{P}}_{\mathcal{P}_{\partial\mathbb{M}}} \circ \hat{\mathrm{P}}_{\mathcal{M}_{\partial\mathbb{M},2}}\right) \right\rangle_{\partial\mathbb{M}} \\
&= \left\langle \hat{\mathrm{P}}_{\mathcal{M}_{\partial\mathbb{M},1}} \circ \hat{\mathrm{P}}_{\mathcal{P}_{\partial\mathbb{M}}}\, {}^*\rho_{\mathbb{M}},\ \hat{\mathrm{P}}_{\mathcal{M}_{\partial\mathbb{M},2}} \circ \hat{\mathrm{P}}_{\mathcal{P}_{\partial\mathbb{M}}}\, {}^*\rho_{\mathbb{M}} \right\rangle_{\partial\mathbb{M}} \\
&= \left\langle \hat{\mathrm{P}}_{\mathcal{P}_{\partial\mathbb{M}}} \circ \underbrace{\hat{\mathrm{P}}_{\mathcal{M}_{\partial\mathbb{M},2}} \circ \hat{\mathrm{P}}_{\mathcal{M}_{\partial\mathbb{M},1}}}_{=0} \circ \hat{\mathrm{P}}_{\mathcal{P}_{\partial\mathbb{M}}} \circ {}^*\rho_{\mathbb{M}},\ {}^*\rho_{\mathbb{M}} \right\rangle_{\partial\mathbb{M}} = 0.
\end{aligned}
$$

Arbitrary outcome has unity probability for any (allowed) preparation subspace:

$$1 \;=\; P\Big(\mathcal{H}_{\partial\mathbb{M}}\Big|\mathcal{P}_{\partial\mathbb{M}}\Big) \;=\; \frac{\left\|\, \rho_{\mathbb{M}} \circ \hat{\mathrm{P}}_{\mathcal{P}_\Sigma} \circ \overbrace{\hat{\mathrm{P}}_{\mathcal{H}_{\partial\mathbb{M}}}}^{\mathrm{Id}_{\mathcal{H}_{\partial\mathbb{M}}}} \,\right\|^2}{\left\|\, \rho_{\mathbb{M}} \circ \hat{\mathrm{P}}_{\mathcal{P}_\Sigma} \,\right\|^2} \qquad \forall\, \mathcal{P}_{\partial\mathbb{M}} \,\subset\, \mathcal{H}_{\partial\mathbb{M}}$$

If we have $\mathcal{M}_{\partial\mathbb{M},2}$ implies $\mathcal{M}_{\partial\mathbb{M},1}$ implies $\mathcal{P}_{\partial\mathbb{M}}$, then the following **probability chain rule** holds:

$$\begin{aligned} & \mathcal{M}_{\partial\mathbb{M},2} \subseteq \mathcal{M}_{\partial\mathbb{M},1} \subseteq \mathcal{P}_{\partial\mathbb{M}} \subset \mathcal{H}_{\partial\mathbb{M}} \\ \Rightarrow \quad & P\left(\mathcal{M}_{\partial\mathbb{M},2}\middle|\mathcal{P}_{\partial\mathbb{M}}\right) = P\left(\mathcal{M}_{\partial\mathbb{M},2}\middle|\mathcal{M}_{\partial\mathbb{M},1}\right) P\left(\mathcal{M}_{\partial\mathbb{M},1}\middle|\mathcal{P}_{\partial\mathbb{M}}\right). \end{aligned}$$

This can quickly be checked by inserting the probability definition the chain rule and applying the premise to the projection operators.

Now let us verify again how the GBF probabilities via projectors reproduce the examples of the standard formulation. Again, following the core axioms we have to suppose for the standard case that the state space factors into a tensor product of two state spaces: $\mathcal{H}_{\partial\mathbb{M}} = \mathcal{H}_{\Sigma_1} \otimes \mathcal{H}_{\overline{\Sigma}_2}$.

Example 3.1: For Example 1.1 we select two normalized states $\psi_{\Sigma_1} \in \mathcal{H}_{\Sigma_1}$ and $\eta_{\overline{\Sigma}_2} \in \mathcal{H}_{\overline{\Sigma}_2}$ and set

$$\begin{aligned}
\mathcal{P}_{\partial\mathbb{M}} &= \text{''}\, \psi_{\Sigma_1} \otimes \mathcal{H}_{\overline{\Sigma}_2} \text{''} \quad \subset \mathcal{H}_{\partial\mathbb{M}} \\
\mathcal{M}_{\partial\mathbb{M}} &= \text{''}\, \mathcal{H}_{\Sigma_1} \otimes \eta_{\overline{\Sigma}_2} \text{''} \quad \subset \mathcal{H}_{\partial\mathbb{M}} \\
\mathcal{P}_{\partial\mathbb{M}} &:= \left\{ \alpha_{\partial\mathbb{M}} \in \mathcal{H}_{\partial\mathbb{M}} \,\middle|\, \exists\, \beta_{\overline{\Sigma}_2} \in \mathcal{H}_{\overline{\Sigma}_2} : \alpha_{\partial\mathbb{M}} = \psi_{\Sigma_1} \otimes \beta_{\overline{\Sigma}_2} \right\} \subset \mathcal{H}_{\partial\mathbb{M}} \\
\mathcal{M}_{\partial\mathbb{M}} &:= \left\{ \alpha_{\partial\mathbb{M}} \in \mathcal{H}_{\partial\mathbb{M}} \,\middle|\, \exists\, \beta_{\Sigma_1} \in \mathcal{H}_{\Sigma_1} : \alpha_{\partial\mathbb{M}} = \beta_{\Sigma_1} \otimes \eta_{\overline{\Sigma}_2} \right\} \subset \mathcal{H}_{\partial\mathbb{M}}
\end{aligned}$$

Let us now denote by $\{\xi_{\Sigma_1,j} \otimes \xi_{\overline{\Sigma}_2,k}\}_{j,k=1,\ldots,(\dim \mathcal{H}_{\Sigma_{1,2}})}$ an orthonormal basis of $\mathcal{H}_{\partial\mathbb{M}}$ with $\xi_{\overline{\Sigma}_2,k} = \iota_{\overline{\Sigma}_2} \widetilde{\rho}_{\mathbb{M}} \xi_{\Sigma_1,k}$, and further such that ψ_{Σ_1} is one of the $\xi_{\Sigma_1,j}$. Then, the probability of "observing" $\eta_{\overline{\Sigma}_2} \in \mathcal{H}_{\overline{\Sigma}_2}$ subject to the "preparation" of $\psi_{\Sigma_1} \in \mathcal{H}_{\Sigma_1}$ after the calculation below turns out as:

$$P\Big(\mathcal{M}_{\partial\mathbb{M}}\Big|\mathcal{P}_{\partial\mathbb{M}}\Big) = \Big|\Big\langle \eta_{\overline{\Sigma}_2},\ \psi_{\overline{\Sigma}_2} \Big\rangle_{\overline{\Sigma}_2}\Big|^2 = \Big|\,{}_{\overline{\Sigma}_2}\Big\langle \eta \,\Big|\, \hat{\mathcal{U}}_{[t_1,t_2]} \,\Big|\, \psi \Big\rangle_{\Sigma_1}\Big|^2$$

which again recovers the standard result $P(\eta_{\overline{\Sigma}_2}|\psi_{\Sigma_1})$.

The calculation goes as follows: the two relevant projectors are

$$\begin{aligned}
\hat{\mathrm{P}}_{\mathcal{P}_{\partial\mathbb{M}}} &= \psi_{\Sigma_1}\langle\psi_{\Sigma_1},\ \cdot\,\rangle_{\Sigma_1}\otimes\mathrm{Id}_{\mathcal{H}_{\overline{\Sigma}_2}} \\
&= \psi_{\Sigma_1}\langle\psi_{\Sigma_1},\ \cdot\,\rangle_{\Sigma_1}\otimes\sum_{k=1}^{\dim\,\mathcal{H}_{\overline{\Sigma}_2}}\xi_{\overline{\Sigma}_2,k}\langle\xi_{\overline{\Sigma}_2,k},\ \cdot\,\rangle_{\overline{\Sigma}_2} \\
\hat{\mathrm{P}}_{\mathcal{M}_{\partial\mathbb{M}}} &= \mathrm{Id}_{\mathcal{H}_{\Sigma_1}}\otimes\eta_{\overline{\Sigma}_2}\langle\eta_{\overline{\Sigma}_2},\ \cdot\,\rangle_{\overline{\Sigma}_2} \\
&= \sum_{k=1}^{\dim\,\mathcal{H}_{\Sigma_1}}\xi_{\Sigma_1,k}\langle\xi_{\Sigma_1,k},\ \cdot\,\rangle_{\Sigma_1}\otimes\eta_{\overline{\Sigma}_2}\langle\eta_{\overline{\Sigma}_2},\ \cdot\,\rangle_{\overline{\Sigma}_2}
\end{aligned}$$

The probability for Example 3.1 via projectors then is:

$$
\begin{aligned}
P(\mathcal{M}_{\partial\mathbb{M}}|\mathcal{P}_{\partial\mathbb{M}}) &= \frac{\left\| \rho_{\mathbb{M}} \circ \hat{\mathrm{P}}_{\mathcal{P}_{\partial\mathbb{M}}} \circ \hat{\mathrm{P}}_{\mathcal{M}_{\partial\mathbb{M}}} \right\|^2}{\left\| \rho_{\mathbb{M}} \circ \hat{\mathrm{P}}_{\mathcal{P}_{\partial\mathbb{M}}} \right\|^2} \\
&= \frac{\left\langle {}^*\left(\rho_{\mathbb{M}} \hat{\mathrm{P}}_{\mathcal{P}_{\partial\mathbb{M}}} \hat{\mathrm{P}}_{\mathcal{M}_{\partial\mathbb{M}}}\right), \ {}^*\left(\rho_{\mathbb{M}} \hat{\mathrm{P}}_{\mathcal{P}_{\partial\mathbb{M}}} \hat{\mathrm{P}}_{\mathcal{M}_{\partial\mathbb{M}}}\right) \right\rangle_{\partial\mathbb{M}}}{\left\langle {}^*\left(\rho_{\mathbb{M}} \hat{\mathrm{P}}_{\mathcal{P}_{\partial\mathbb{M}}}\right), \ {}^*\left(\rho_{\mathbb{M}} \hat{\mathrm{P}}_{\mathcal{P}_{\partial\mathbb{M}}}\right) \right\rangle_{\partial\mathbb{M}}} \\
&= \frac{\left\langle \hat{\mathrm{P}}_{\mathcal{M}_{\partial\mathbb{M}}} \hat{\mathrm{P}}_{\mathcal{P}_{\partial\mathbb{M}}} {}^*\rho_{\mathbb{M}}, \ \hat{\mathrm{P}}_{\mathcal{M}_{\partial\mathbb{M}}} \hat{\mathrm{P}}_{\mathcal{P}_{\partial\mathbb{M}}} {}^*\rho_{\mathbb{M}} \right\rangle_{\partial\mathbb{M}}}{\left\langle \hat{\mathrm{P}}_{\mathcal{P}_{\partial\mathbb{M}}} {}^*\rho_{\mathbb{M}}, \ \hat{\mathrm{P}}_{\mathcal{P}_{\partial\mathbb{M}}} {}^*\rho_{\mathbb{M}} \right\rangle_{\partial\mathbb{M}}} \\
&= \frac{\left\langle \hat{\mathrm{P}}_{\mathcal{M}_{\partial\mathbb{M}}} \hat{\mathrm{P}}_{\mathcal{P}_{\partial\mathbb{M}}} \sum_{k=1}^{\dim \mathcal{H}_{\Sigma_1}} \xi_{\Sigma_1,k} \otimes \xi_{\overline{\Sigma}_2,k}, \ \hat{\mathrm{P}}_{\mathcal{M}_{\partial\mathbb{M}}} \hat{\mathrm{P}}_{\mathcal{P}_{\partial\mathbb{M}}} \sum_{k=1}^{\dim \mathcal{H}_{\Sigma_1}} \xi_{\Sigma_1,k} \otimes \xi_{\overline{\Sigma}_2,k} \right\rangle_{\partial\mathbb{M}}}{\left\langle \hat{\mathrm{P}}_{\mathcal{P}_{\partial\mathbb{M}}} \sum_{k=1}^{\dim \mathcal{H}_{\Sigma_1}} \xi_{\Sigma_1,k} \otimes \xi_{\overline{\Sigma}_2,k}, \ \hat{\mathrm{P}}_{\mathcal{P}_{\partial\mathbb{M}}} \sum_{k=1}^{\dim \mathcal{H}_{\Sigma_1}} \xi_{\Sigma_1,k} \otimes \xi_{\overline{\Sigma}_2,k} \right\rangle_{\partial\mathbb{M}}} .
\end{aligned}
$$

We now consider the term in the denominator:

$$\begin{aligned}\hat{\mathrm{P}}_{\mathcal{P}_{\partial\mathbb{M}}} \sum_{a=1}^{\dim \mathcal{H}_{\Sigma_1}} \xi_{\Sigma_1,a} \otimes \xi_{\overline{\Sigma}_2,a} &= \sum_{a,b=1}^{\dim \mathcal{H}_{\Sigma_1}} \psi_{\Sigma_1} \overbrace{\langle \psi_{\Sigma_1}, \xi_{\Sigma_1,a} \rangle_{\Sigma_1}}^{\text{"}\delta_{\psi,a}\text{"}} \otimes \overbrace{\xi_{\overline{\Sigma}_2,b} \langle \xi_{\overline{\Sigma}_2,b}, \xi_{\overline{\Sigma}_2,a} \rangle_{\overline{\Sigma}_2}}^{=\xi_{\overline{\Sigma}_2,a}} \\ &= \psi_{\Sigma_1} \otimes \underbrace{\iota_{\overline{\Sigma}_2} \widetilde{\rho}_{\mathbb{M}} \psi_{\Sigma_1}}_{:=\psi_{\overline{\Sigma}_2}} .\end{aligned}$$

The norm of this is unity since ψ_{Σ_1} is normalized and $\widetilde{\rho}_{\mathbb{M}}$ is an isomorphism.

Let us move on with the denumerator:

$$\hat{\mathrm{P}}_{\mathcal{M}_{\partial\mathbb{M}}}\hat{\mathrm{P}}_{\mathcal{P}_{\partial\mathbb{M}}}\sum_{a=1}^{\dim\mathcal{H}_{\Sigma_1}}\xi_{\Sigma_1,a}\otimes\xi_{\overline{\Sigma}_2,a} = \hat{\mathrm{P}}_{\mathcal{M}_{\partial\mathbb{M}}}\,\psi_{\Sigma_1}\otimes\psi_{\overline{\Sigma}_2} = \psi_{\Sigma_1}\otimes\eta_{\overline{\Sigma}_2}\langle\eta_{\overline{\Sigma}_2},\psi_{\overline{\Sigma}_2}\rangle_{\overline{\Sigma}_2}$$

The norm squared of this according to core axioms is just $\left|\langle\eta_{\overline{\Sigma}_2},\psi_{\overline{\Sigma}_2}\rangle_{\overline{\Sigma}_2}\right|^2$ and thus we have derived:

$$P(\mathcal{M}_{\partial\mathbb{M}}|\mathcal{P}_{\partial\mathbb{M}}) = \left|\left\langle\eta_{\overline{\Sigma}_2},\ \psi_{\overline{\Sigma}_2}\right\rangle_{\overline{\Sigma}_2}\right|^2 \qquad = \left|{}_{\overline{\Sigma}_2}\left\langle\eta\,\middle|\,\hat{\mathcal{U}}_{[t_1,t_2]}\,\middle|\,\psi\right\rangle_{\Sigma_1}\right|^2.$$

Example 3.2: For Example 1.2 we set

$$\begin{aligned}
\mathcal{P}_{\partial\mathbb{M}} &= \text{"}\,\psi_{\Sigma_1} \otimes \mathcal{S}_{\overline{\Sigma}_2}\,\text{"} \quad \subseteq \quad \text{"}\,\psi_{\Sigma_1} \otimes \mathcal{H}_{\overline{\Sigma}_2}\,\text{"} \quad \subset \ \mathcal{H}_{\partial\mathbb{M}} \\
\mathcal{M}_{\partial\mathbb{M}} &= \text{"}\,\mathcal{H}_{\Sigma_1} \otimes \eta_{\overline{\Sigma}_2}\,\text{"} \qquad \eta_{\overline{\Sigma}_2} \in \mathcal{S}_{\overline{\Sigma}_2} \\
\mathcal{P}_{\partial\mathbb{M}} &:= \left\{ \alpha_{\partial\mathbb{M}} \in \mathcal{H}_{\partial\mathbb{M}} \,\middle|\, \exists\, \beta_{\overline{\Sigma}_2} \in \mathcal{S}_{\overline{\Sigma}_2} : \alpha_{\partial\mathbb{M}} = \psi_{\Sigma_1} \otimes \beta_{\overline{\Sigma}_2} \right\} \subset \mathcal{H}_{\partial\mathbb{M}} \\
\mathcal{M}_{\partial\mathbb{M}} &:= \left\{ \alpha_{\partial\mathbb{M}} \in \mathcal{H}_{\partial\mathbb{M}} \,\middle|\, \exists\, \beta_{\Sigma_1} \in \mathcal{H}_{\Sigma_1} : \alpha_{\partial\mathbb{M}} = \beta_{\Sigma_1} \otimes \eta_{\overline{\Sigma}_2} \right\} \subset \mathcal{H}_{\partial\mathbb{M}}
\end{aligned}$$

We leave the calculation of the probability as a homework for the inclined readership.

Example 3.3: For Example 1.3 we keep $\mathcal{P}_{\partial\mathbb{M}}$ and set

$$\mathcal{M}_{\partial\mathbb{M}} = \text{"}\,\mathcal{H}_{\Sigma_1} \otimes \mathcal{M}_{\overline{\Sigma}_2}\,\text{"} \quad \subseteq \quad \text{"}\,\mathcal{H}_{\Sigma_1} \otimes \mathcal{P}_{\overline{\Sigma}_2}\,\text{"} \quad \subset \ \mathcal{H}_{\partial\mathbb{M}}$$

$$\mathcal{M}_{\partial\mathbb{M}} := \left\{\alpha_{\partial\mathbb{M}} \in \mathcal{H}_{\partial\mathbb{M}} \,|\, \exists\, \beta_{\Sigma_1} \in \mathcal{H}_{\Sigma_1},\, \psi_{\overline{\Sigma}_2} \in \mathcal{M}_{\overline{\Sigma}_2} : \alpha_{\partial\mathbb{M}} = \beta_{\Sigma_1} \otimes \psi_{\overline{\Sigma}_2}\right\} \subset \mathcal{H}_{\partial\mathbb{M}}\,.$$

Example 3.4: For Example 1.4 we set:

$$
\begin{aligned}
\mathcal{P}_{\partial\mathbb{M}} &= \text{"}\,\mathcal{H}_{\Sigma_1} \otimes \eta_{\overline{\Sigma}_2}\text{"} \quad \subset \mathcal{H}_{\partial\mathbb{M}} \\
\mathcal{M}_{\partial\mathbb{M}} &= \text{"}\,\psi_{\Sigma_1} \otimes \mathcal{H}_{\overline{\Sigma}_2}\text{"} \quad \subset \mathcal{H}_{\partial\mathbb{M}} \\
\mathcal{P}_{\partial\mathbb{M}} &:= \left\{ \alpha_{\partial\mathbb{M}} \in \mathcal{H}_{\partial\mathbb{M}} \,|\, \exists\, \beta_{\Sigma_1} \in \mathcal{H}_{\Sigma_1} : \alpha_{\partial\mathbb{M}} = \beta_{\Sigma_1} \otimes \eta_{\overline{\Sigma}_2} \right\} \subset \mathcal{H}_{\partial\mathbb{M}} \\
\mathcal{M}_{\partial\mathbb{M}} &:= \left\{ \alpha_{\partial\mathbb{M}} \in \mathcal{H}_{\partial\mathbb{M}} \,|\, \exists\, \beta_{\overline{\Sigma}_2} \in \mathcal{H}_{\overline{\Sigma}_2} : \alpha_{\partial\mathbb{M}} = \psi_{\Sigma_1} \otimes \beta_{\overline{\Sigma}_2} \right\} \subset \mathcal{H}_{\partial\mathbb{M}}.
\end{aligned}
$$

This reproduces the correct probabilities also for this case of retroactive measurement as the active readership can verify following the steps of the first example. In the definition of probabilities via projectors this is not unexpected, since here we can see with respect to Example 3.1 we have only interchanged $\mathcal{P}_{\partial\mathbb{M}}$ with $\mathcal{M}_{\partial\mathbb{M}}$.