Observables in the GBF

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References:

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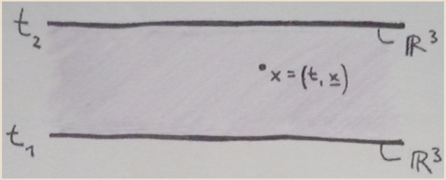
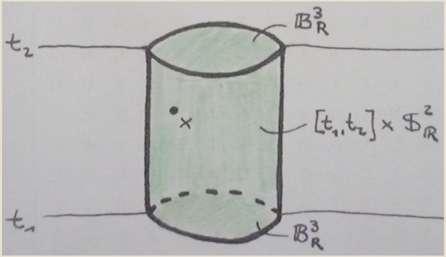
Outline

Outline



Infinite slice region

An experimenter has to prepare the initial state on the whole equal-time hypersurface, that is, on all of space, and measure the final state on all of space as well. In flat spacetime this is unnecessary due to cluster decomposition, but when the metric becomes dynamical, a priori we have no information on distances available, and thus there is a problem.



Finite hypercylinder region

Outline

The properties of observables that become visible in the Feynman path integral give rise to a concept of quantum observable, introduced in [RO2011], which fits very naturally into the GBF. Since realistic measurements are extend both in space and time, it should make sense that we locate the mathematical objects that represent observables in spacetime regions as well.

In the GBF, quantum observables are associated to a spacetime region $\mathbb{M}$ and encoded through linear observable maps $\rho_{\mathbb{M}}^{O}: \ \mathcal{H}_{\partial\mathbb{M}} \to \mathbb{C}$, just like the amplitude map of the region $\mathbb{M}$. This is expressed in the following axiom. Not any such map needs to be an observable though. Which maps qualify as observables may generally depend on the theory under consideration.

(O1) Associated to each spacetime region $\mathbb{M}$ is a real vector space $\mathcal{O}_{\mathbb{M}}$ of linear maps $\mathcal{H}_{\partial\mathbb{M}} \to \mathbb{C}$, called **observable maps**. In particular, the amplitude map is an observable map: $\rho_{\mathbb{M}} \in \mathcal{O}_{\mathbb{M}}$.

The temporal composition is generalized to **regional composition** by the **diamond product** $\diamond$ in the following axioms.

(O2a) Let $\mathbb{M}_1$ and $\mathbb{M}_2$ be regions and $\mathbb{M} = \mathbb{M}_1 \sqcup \mathbb{M}_2$ their disjoint union. Then, there is an injective bilinear map $\diamond : \ \mathcal{O}_{\mathbb{M}_1} \times \mathcal{O}_{\mathbb{M}_2} \ \hookrightarrow \ \mathcal{O}_{\mathbb{M}}$ such that for all $\rho_{\mathbb{M}_1}^{O_1} \in \mathcal{O}_{\mathbb{M}_1}$ and $\rho_{\mathbb{M}_2}^{O_2} \in \mathcal{O}_{\mathbb{M}_2}$ and for all $\psi_1 \in \mathcal{H}_{\partial\mathbb{M}_1}$ and all $\psi_2 \in \mathcal{H}_{\partial\mathbb{M}_2}$ we have

$$\left(\rho_{\mathbb{M}_1}^{O_1} \diamond \rho_{\mathbb{M}_2}^{O_2}\right)(\psi_{\partial\mathbb{M}_1} \otimes \psi_{\partial\mathbb{M}_2}) \ = \ \rho_{\mathbb{M}_1}^{O_1}(\psi_{\partial\mathbb{M}_1})\, \rho_{\mathbb{M}_2}^{O_2}(\psi_{\partial\mathbb{M}_2}). \tag{1}$$

The diamond product is required to be associative in the obvious way.

Outline

Since the formula for GBF expectation values is based on the GBF formula for probabilities, let us begin by recalling the essentials of GBF probabilities. Given a spacetime region $\mathbb{M}$, any **preparation** or knowledge about the measurement is encoded through a closed **preparation subspace** $\mathcal{P}_{\partial\mathbb{M}} \subseteq \mathcal{H}_{\partial\mathbb{M}}$. Similarly, the **observation** or question is encoded in the closed **measurement subspace** $\mathcal{M}_{\partial\mathbb{M}} \subseteq \mathcal{H}_{\partial\mathbb{M}}$. The conditional probability of observing $\mathcal{M}_{\partial\mathbb{M}}$ given that $\mathcal{P}_{\partial\mathbb{M}}$ is prepared is given by

$$P(\mathcal{M}_{\partial\mathbb{M}}|\mathcal{P}_{\partial\mathbb{M}}) = \frac{\left\| \rho_{\mathbb{M}} \circ \hat{\mathrm{P}}_{\mathcal{P}_{\partial\mathbb{M}}} \circ \hat{\mathrm{P}}_{\mathcal{M}_{\partial\mathbb{M}}} \right\|^2}{\left\| \rho_{\mathbb{M}} \circ \hat{\mathrm{P}}_{\mathcal{P}_{\partial\mathbb{M}}} \right\|^2} . \tag{2}$$

Therein, $\hat{\mathrm{P}}_{\mathcal{P}_{\partial\mathbb{M}}}$ and $\hat{\mathrm{P}}_{\mathcal{M}_{\partial\mathbb{M}}}$ are the orthogonal projectors onto the subspaces $\mathcal{P}_{\partial\mathbb{M}}$ and $\mathcal{M}_{\partial\mathbb{M}}$ respectively. The norm in (2) is the one of $\mathcal{H}^*_{\partial\mathbb{M}}$. the mathematical meaning of (2) is thus clear.

Also, the additional assumption $\mathcal{M}_{\partial\mathbb{M}} \subseteq \mathcal{P}_{\partial\mathbb{M}}$ can be made. It means only asking questions in a way that takes into account fully what we already know. Formula (2) can be rewritten for $\mathcal{M}_{\partial\mathbb{M}} \subseteq \mathcal{P}_{\partial\mathbb{M}}$ as follows:

$$P(\mathcal{M}_{\partial\mathbb{M}}|\mathcal{P}_{\partial\mathbb{M}}) = \frac{\left\langle \rho_{\mathbb{M}} \circ \hat{\mathrm{P}}_{\mathcal{P}_{\partial\mathbb{M}}},\ \rho_{\mathbb{M}} \circ \hat{\mathrm{P}}_{\mathcal{M}_{\partial\mathbb{M}}} \right\rangle}{\left\| \rho_{\mathbb{M}} \circ \hat{\mathrm{P}}_{\mathcal{P}_{\partial\mathbb{M}}} \right\|^2} . \tag{3}$$

As the norm, the inner product therein is the one of the dual boundary state space $\mathcal{H}^*_{\partial\mathbb{M}}$. Whenever the projectors $P_{\mathcal{P}_{\partial\mathbb{M}}}$ and $P_{\mathcal{M}_{\partial\mathbb{M}}}$ commute, (3) coincides with (2).

As do probabilities, the expectation values of observables also depend on the preparation of the physical system. Given an observable $\rho_{\mathbb{M}}^{O} \in \mathcal{O}_{\mathbb{M}}$ and a preparation subspace $\mathcal{P}_{\partial\mathbb{M}} \subseteq \mathcal{H}_{\partial\mathbb{M}}$, the **expectation value** of $\rho_{\mathbb{M}}^{O}$ on condition that $\mathcal{P}_{\partial\mathbb{M}}$ was prepared is defined as

$$\left\langle \rho_{\mathbb{M}}^{O} \right\rangle_{\mathcal{P}_{\partial\mathbb{M}}} := \frac{\left\langle \rho_{\mathbb{M}} \circ \hat{\mathrm{P}}_{\mathcal{P}_{\partial\mathbb{M}}},\ \rho_{\mathbb{M}}^{O} \right\rangle}{\left\| \rho_{\mathbb{M}} \circ \hat{\mathrm{P}}_{\mathcal{P}_{\partial\mathbb{M}}} \right\|^{2}} . \tag{4}$$

The expectation value is linear in the observable. Further, we want the GBF probabilities to arise as a special case of expectation values. Given a closed measurement subspace $\mathcal{M}_{\partial\mathbb{M}} \subseteq \mathcal{H}_{\partial\mathbb{M}}$ and setting $\rho_{\mathbb{M}}^{O} = \rho_{\mathbb{M}} \circ \hat{\mathrm{P}}_{\mathcal{M}_{\partial\mathbb{M}}}$, we see that the expectation value (4) reproduces exactly the probability (3), and represents the conditional probability to observe $\mathcal{M}_{\partial\mathbb{M}}$ given that $\mathcal{P}_{\partial\mathbb{M}}$ is prepared. As for probabilities, it can be shown that the GBF expectation values recover those of the standard formulation in the standard setting (Minkowski slice region).

Outline

In Classical Mechanics, observables are functions on the phase space of a physical system, taken at some instant of time. In standard nonrelativistic Quantum Mechanics (QM) there is only **one** state space, which is the Hilbert space that consists in the states describing the physical system at an instant of time in "all of space". (We refer to the Schrödinger picture, where the states are time-dependent and the operators are time-independent). Observables are then certain operators on this Hilbert space, and are usually constructed from classical observables through some method of quantization. That is, we encode a measurement process on the system by an associated operator on the state space.

Saying that an operator is applied to a state at some time t then means that the corresponding measurement is applied to the system at this time. This implies, that measurements composed of two measurements $\hat{A}$ and $\hat{B}$ are represented on the state space by the operator product of $\hat{A}$ and $\hat{B}$. If we begin with measuring $\hat{A}$ and then measure $\hat{B}$, then the operator product $\hat{B}\hat{A}$ represents the composed measurement. If we reverse the order of measurements, then the operator product $\hat{A}\hat{B}$ represents the composed measurement. The operator product thus describes the **temporal** composition of measurements on a system. Generically the result depends on the sequence or order of the measurements. That is, generically $\hat{B}\hat{A} \neq \hat{A}\hat{B}$. The information on the difference of these results is encoded in the commutation relations of the operators.

Considering classical field theory in a relativistic setting, observables are now not functions on phase space at a certain moment in **time**, but represent field configurations at some point in **spacetime**. In QFT observables are now operators on the one and only Hilbert state space. They are labeled by a point in spacetime, for example $\hat{\phi}(x)$. There is now only one meaningful way to associate an operator product to composed measurements: the **time-ordered** product of the constituent measurements operators. The time-ordered product is commutative:

$$\mathbf{T}\hat{\phi}(x)\hat{\phi}(y) = \mathbf{T}\hat{\phi}(y)\hat{\phi}(x) := \begin{cases} \hat{\phi}(x)\hat{\phi}(y) & t_x > t_y \\ \frac{1}{2}\hat{\phi}(x)\hat{\phi}(y) + \frac{1}{2}\hat{\phi}(y)\hat{\phi}(x) & t_x = t_y \\ \hat{\phi}(y)\hat{\phi}(x) & t_x < t_y \end{cases} .$$

According to [RO2012], this strongly suggests that the operator point of view is not the most natural one here. In standard QFT treatments, first there is constructed a noncommutative algebra of field operators, starting from equal-time commutation relations. As the condition of equal times is not Poincaré invariant, the equal-time commutation relations then are generalized for operators at different times. For consistence it is necessary that all operators commute whose spacetime labels are spacelike separated:

$$[\hat{A}(x),\, \hat{B}(y)] \overset{!}{=} 0 \qquad \forall\ x, y \text{ spacelike separated.} \tag{5}$$

This condition is Poincaré invariant.

Usually the time-ordered product is considered as derived from the noncommutative operator product. However, only the time-ordered product has a direct operational meaning and appears in amplitudes and S-matrix elements of QFT. Further, the noncommutative product can be derived from the time-ordered product. The equal-time commutators can be obtained as the limit

$$[\hat{A}(t,\underline{x}),\, \hat{B}(t,\underline{y})] \;=\; \lim_{\epsilon\to+0}\Big(\mathbf{T}\hat{A}(t{+}\epsilon,\underline{x})\hat{B}(t{-}\epsilon,\underline{y}) - \mathbf{T}\hat{A}(t{-}\epsilon,\underline{x})\hat{B}(t{+}\epsilon,\underline{y})\Big).$$

Therefore in a special relativistic setting, there are good reasons to regard the time-ordered operator product of observables as more fundamental than the noncommutative one. This suggests to try to formulate the theory of observables in terms of the time-ordered product rather than the noncommutative one. In a (quantum) general relativistic setting with no predefined background metric a condition such as (5) makes no longer sense, making the postulation of a noncommutative algebra structure for observables even more questionable.

Outline

In order to see the origin of the GBF's notion of observables, let us review the construction of observables in the standard formulation using Feynman's path integral. For simplicity we consider the standard situation: real scalar field theory on a slice region $\mathbb{M}_{[t_a,t_b]} \times \mathbb{R}^3$ in Minkowski spacetime. By $\mathcal{C}_{[t_a,t_b]}$ we denote the space of field $\mathcal{C}$onfigurations on $\mathbb{M}_{[t_a,t_b]}$. We consider a classical observable $\mathcal{C}_{[t_a,t_b]} \to \mathbb{R}$ that encodes an n-point function:

$$\phi \;\mapsto\; \phi(t_1, \underline{x}_1) \cdot \ldots \cdot \phi(t_n, \underline{x}_n) \qquad\qquad t_1, \ldots, t_n \in [t_a, t_b]. \tag{6}$$

Let us denote by $\tilde{\phi}(t_i, \underline{x}_i)$ the usual quantizations of the field values in the Heisenberg picture (wherein states are time-independent while operators are time-dependent). Given an initial state $\psi_a \in \mathcal{H}$ at time t_a and a final state $\eta_b \in \mathcal{H}$ at time t_b, the corresponding matrix element of the **time-ordered** product of the quantized version of (6) can be expressed by the Feynman path integral

$$\begin{aligned} &\left\langle \eta_b,\ \mathbf{T}\tilde{\phi}(t_1, \underline{x}_1) \cdot \ldots \cdot \tilde{\phi}(t_n, \underline{x}_n)\, \psi_a \right\rangle_{\mathcal{H}} \qquad (7) \\ = &\int\limits_{\mathcal{C}_{[t_a, t_b]}} \mathcal{D}\phi\ \overline{\eta_b(\phi|_{t_b})}\, \psi_a(\phi|_{t_a})\, \phi(t_1, \underline{x}_1) \cdot \ldots \cdot \phi(t_n, \underline{x}_n)\ \exp\Big(\mathrm{i} S_{[t_a, t_b]}(\phi)\Big). \end{aligned}$$

On the right hand side therein, ψ_a and η_b are the Schrödinger wave functions corresponding tho the respective states, and $\mathbf{T}$ denotes time-ordering. When initial and final state are the vacuum, (7) becomes the usual quantum n-point function of QFT.

Let us recall at this point the form of the transition amplitude using the path integral:

$$\rho_{[t_a,t_b]}(\psi_a \otimes \eta_b) = \Big\langle \eta_b,\ \psi_a \Big\rangle_{\mathcal{H}} = \int\limits_{\mathcal{C}_{[t_a,t_b]}} \mathcal{D}\phi\ \overline{\eta_b(\phi|_{t_b})}\,\psi_a(\phi|_{t_a})\ \exp\Big(\mathrm{i}S_{[t_a,t_b]}(\phi)\Big)$$

$$= \int\limits_{\mathcal{C}_{t_a}} \mathcal{D}\varphi_a \int\limits_{\mathcal{C}_{t_b}} \mathcal{D}\varphi_b\ \overline{\eta_b(\varphi_b)}\,\psi_a(\varphi_a)\ Z_{[t_a,t_b]}(\varphi_a,\varphi_b)$$

$$Z_{[t_a,t_b]}(\varphi_a,\varphi_b) = \int\limits_{\substack{\phi|_{t_a}=\varphi_a \\ \phi|_{t_b}=\varphi_b}} \mathcal{D}\phi\ \exp\Big(\mathrm{i}S_{[t_a,t_b]}(\phi)\Big). \tag{8}$$

The quantization occurring in (7) may thus be seen as the conversion of a classical observable $F: \ \mathcal{C}_{[t_a,t_b]} \ \to \ \mathbb{R}$ to a quantum observable represented as operator $\hat{F}_{[t_a,t_b]}$ on $\mathcal{H}$ respectively a "modified field propagator" $Z^F_{[t_a,t_b]}(\varphi_a,\varphi_b)$ giving rise to a "modified amplitude map" $\rho^F_{[t_a,t_b]}(\psi_a \otimes \eta_b)$ as in

$$\rho^F_{[t_a,t_b]}(\psi_a \otimes \eta_b) = \left\langle \eta_b, \ \hat{F}_{[t_a,t_b]}\, \psi_a \right\rangle_{\mathcal{H}} = \int\limits_{\mathcal{C}_{[t_a,t_b]}} \mathcal{D}\phi \ \overline{\eta_b(\phi|_{t_b})}\, \psi_a(\phi|_{t_a}) \ F(\phi) \ \exp\Big(\mathrm{i}S_{[t_a,t_b]}(\phi)\Big)$$

$$= \int\limits_{\mathcal{C}_{t_a}} \mathcal{D}\varphi_a \int\limits_{\mathcal{C}_{t_b}} \mathcal{D}\varphi_b \ \overline{\eta_b(\varphi_b)}\, \psi_a(\varphi_a) \ Z^F_{[t_a,t_b]}(\varphi_a,\varphi_b)$$

$$Z^F_{[t_a,t_b]}(\varphi_a,\varphi_b) = \int\limits_{\substack{\phi|_{t_a}=\varphi_a \\ \phi|_{t_b}=\varphi_b}} \mathcal{D}\phi \ F(\phi) \ \exp\Big(\mathrm{i}S_{[t_a,t_b]}(\phi)\Big). \tag{9}$$

A remarkable property of QFT in this quantization prescription is a correspondence between the composition of classical and quantum observables, termed **composition correspondence** in [RO2011]. It comes from a generic property of the path integral. Let $t_a < t_b < t_c$, further $F: \ \mathcal{C}_{[t_a, t_b]} \to \mathbb{R}$ and $G: \ \mathcal{C}_{[t_b, t_c]} \to \mathbb{R}$ be classical observables on the regions $\mathbb{M}_{[t_a, t_b]}$ respectively $\mathbb{M}_{[t_b, t_c]}$. We can extend both observables trivially to calssical observables on the union $\mathbb{M}_{[t_a, t_c]} = \mathbb{M}_{[t_a, t_b]} \cup \mathbb{M}_{[t_b, t_c]}$, and multiply then them as functions on $\mathbb{M}_{[t_a, t_c]}$. The prescription (9) then leads to the identity

$$\widehat{G \cdot F} \ = \ \hat{G} \circ \hat{F}. \tag{10}$$

That is, there is a direct correspondence between the classical composition of observables (via multiplication of functions) and the quantum composition of observables (via operator multiplication). Note that **locality properties** of the observables are crucial for the correspondence (10), because the above prescription allows only to compose in this way observables which have disjoint support.

It is important to distinguish the composition correspondence (10) from the Dirac quantization condition relating the commutator of observables quantized as operators to the quantization of the Poisson bracket of the observables. In the latter, classical observables are necessarily understood as functions on phase space. For classical observables D, E this takes the form

$$\left[\tilde{D},\ \tilde{E}\right] = -\mathrm{i}\,\widetilde{\left\{D,\ E\right\}}, \tag{11}$$

wherein the bracket indicates the Poisson bracket and quantization is denoted by a tilde. In QFT elementary observables at equal times can also be viewed as functions on phase space and as such realize the Dirac condition (11) in the form of canonical commutation relations.

Outline

An essential feature of the GBF observables is that they reproduce the observables and expectation values of the standard formulation. In nonrelativistical Quantum Mechanics, observables are associated to instants of time, that is, equal-time hypersurfaces. We model this through "infinitesimally thin" regions, which we thus call (infinitesimal) **slice regions**. Geometrically speaking, they are hypersurfaces, but treated as regions.

We consider the (backwards oriented) equal-time hypersurface $\Sigma_t = \{t\} \times \mathbb{R}^3$ at time t in Minkowski spacetime. The (infinitesimal) slice region consisting of Σ_t is denoted by $\hat{\Sigma}_t$. By definition, the boundary $\partial\hat{\Sigma}_t$ of the (infinitesimal) slice is equal to the disjoint union $\Sigma_t \sqcup \overline{\Sigma}_t$, implying $\mathcal{H}_{\partial\hat{\Sigma}_t} = \mathcal{H}_{\Sigma_t} \hat{\otimes} \mathcal{H}^*_{\Sigma_t}$. We denote the one and only state space of the standard formulation by $\mathcal{H}$, and identify it with $\mathcal{H}_{\Sigma_t}$.

We recall now the relation between the amplitude map $\rho_{[t_1,t_2]}$ of a slice region $\mathbb{M}_{[t_1,t_2]}$ and the time-evolution operator $\hat{\mathcal{U}}_{[t_1,t_2]}$ for any states $\psi_{\Sigma_{t_1}}$ at t_1 and $\eta_{\Sigma_{t_2}}$ at t_2:

$$\rho_{[t_1,t_2]}\Big(\psi_{\Sigma_{t_1}} \otimes \iota_{\Sigma_{t_2}} \eta_{\Sigma_{t_2}}\Big) = \Big\langle \eta_{\Sigma_{t_2}},\, \hat{\mathcal{U}}_{[t_1,t_2]}\, \psi_{\Sigma_{t_1}} \Big\rangle_{Si_{t_2}}.$$

The relation between an observable map $\rho^{O}_{\hat{\Sigma}_t}$ and the corresponding operator $\tilde{O}$ on $\mathcal{H} = \mathcal{H}_{\Sigma_t}$ is analoguous to the above relation:

$$\rho^{O}_{\hat{\Sigma}_t}\Big(\psi_{\Sigma_t} \otimes \iota_{\Sigma_t} \eta_{\Sigma_t}\Big) = \Big\langle \eta_{\Sigma_t},\; \tilde{O}\, \psi_{\Sigma_t} \Big\rangle_{\Sigma_t}.$$

We continue by recovering the expectation values of the standard setting. We thus let now $\psi_{\Sigma_t} \in \mathcal{H} = \mathcal{H}_{\Sigma_t}$ be a normalized state encoding a preparation. In the GBF language this means that the preparation subspace $\mathcal{P}_{\partial\hat{\Sigma}_t}$ becomes

$$\mathcal{P}_{\partial\hat{\Sigma}_t} = \text{"}\psi_{\Sigma_t} \otimes \mathcal{H}^*_{\Sigma_t}\text{"} = \left\{ \psi_{\Sigma_t} \otimes \xi_{\Sigma_t} \,\middle|\, \xi_{\Sigma_t} \in \mathcal{H}^*_{\Sigma_t} \right\} \subseteq \mathcal{H}_{\partial\hat{\Sigma}_t}.$$

We recall that by core axiom (T3x) the amplitude map $\rho_{\hat{\Sigma}_t}$ of the (infinitesimal) slice region $\hat{\Sigma}_t$ can be identified with the inner product on $\mathcal{H} = \mathcal{H}_{\Sigma_t}$. Thus, for normalized states $\eta_{\Sigma_t}, \zeta_{\Sigma_t} \in \mathcal{H}_{\Sigma_t}$ we get

$$\begin{aligned}\left(\rho_{\hat{\Sigma}_t} \circ \hat{\mathrm{P}}_{\mathcal{P}_{\partial\hat{\Sigma}_t}}\right)(\zeta_{\Sigma_t} \otimes \iota_{\Sigma_t}\eta_{\Sigma_t}) &= \left\langle \hat{\mathrm{P}}_{\mathcal{P}_{\partial\hat{\Sigma}_t}}\eta_{\Sigma_t},\ \hat{\mathrm{P}}_{\mathcal{P}_{\partial\hat{\Sigma}_t}}\zeta_{\Sigma_t}\right\rangle_{\mathcal{H}_{\Sigma_t}} \\ &= \left\langle \eta_{\Sigma_t},\ \hat{\mathrm{P}}_{\psi_{\Sigma_t}}\zeta_{\Sigma_t}\right\rangle_{\mathcal{H}_{\Sigma_t}},\end{aligned}$$

wherein $\hat{\mathrm{P}}_{\psi_{\Sigma_t}}$ is the orthogonal projector in $\mathcal{H}_{\Sigma_t}$ onto the subspace spanned by ψ_{Σ_t}.

This makes it straightforward to evaluate the denominator in the formula for GBF expectation values, which here becomes

$$\left\langle \rho^{O}_{\hat{\Sigma}_t} \right\rangle_{\mathcal{P}_{\partial\hat{\Sigma}_t}} := \frac{\left\langle \rho_{\hat{\Sigma}_t} \circ \hat{\mathrm{P}}_{\mathcal{P}_{\partial\hat{\Sigma}_t}},\ \rho^{O}_{\hat{\Sigma}_t} \right\rangle}{\left\| \rho_{\hat{\Sigma}_t} \circ \hat{\mathrm{P}}_{\mathcal{P}_{\partial\hat{\Sigma}_t}} \right\|^2} \stackrel{?}{=} \left\langle \tilde{O} \right\rangle_{\psi_{\Sigma_t}} .$$

Let $\{\xi_i\}_{i\in\mathbb{N}}$ an ONB of $\mathcal{H}_{\Sigma_t}$, chosen for convenience such that $\xi_1 = \psi_{\Sigma_t}$. Then, the denominator evaluates to unity as in the talk on probabilities in the GBF:

$$\begin{aligned}
\left\| \rho_{\hat{\Sigma}_t} \circ \hat{\mathrm{P}}_{\mathcal{P}_{\partial\hat{\Sigma}_t}} \right\|^2 &= \sum_{i,j\in\mathbb{N}} \left| \left(\rho_{\hat{\Sigma}_t} \circ \hat{\mathrm{P}}_{\mathcal{P}_{\partial\hat{\Sigma}_t}}\right)(\xi_i \otimes \iota_{\Sigma_t}\xi_j) \right|^2 \\
&= \sum_{i,j\in\mathbb{N}} \left| \left\langle \xi_j,\ \hat{\mathrm{P}}_{\psi_{\Sigma_t}} \xi_i \right\rangle_{\Sigma_t} \right|^2 \\
&= \sum_{i,j\in\mathbb{N}} \left| \left\langle \xi_j,\ \delta_{1,i}\, \xi_1 \right\rangle_{\Sigma_t} \right|^2 \\
&= 1.
\end{aligned}$$

For the numerator we observe that

$$
\begin{aligned}
\left\langle \rho_{\hat{\Sigma}_t} \circ \hat{\mathrm{P}}_{\mathcal{P}_{\partial\hat{\Sigma}_t}},\ \rho^{O}_{\hat{\Sigma}_t} \right\rangle_{\Sigma_t} &= \sum_{i,j\in\mathbb{N}} \overline{\left(\rho_{\hat{\Sigma}_t} \circ \hat{\mathrm{P}}_{\mathcal{P}_{\partial\hat{\Sigma}_t}}\right)(\xi_i \otimes \iota_{\Sigma_t}\xi_j)}\ \rho^{O}_{\hat{\Sigma}_t}(\xi_i \otimes \iota_{\Sigma_t}\xi_j) \\
&= \sum_{i,j\in\mathbb{N}} \left\langle \hat{\mathrm{P}}_{\psi_{\Sigma_t}}\xi_i,\ \xi_j \right\rangle_{\Sigma_t} \left\langle \xi_j,\ \tilde{O}\,\xi_i \right\rangle_{\Sigma_t} \\
&= \sum_{i\in\mathbb{N}} \left\langle \psi_{\Sigma_t}\delta_{1,i},\ \tilde{O}\,\xi_i \right\rangle_{\Sigma_t} \\
&= \left\langle \psi_{\Sigma_t},\ \tilde{O}\,\psi_{\Sigma_t} \right\rangle_{\Sigma_t} \\
&= \left\langle \tilde{O} \right\rangle_{\psi_{\Sigma_t}}.
\end{aligned}
$$

Hence, the GBF expectation values recover here the standard expectation value of the instantaneous operator $\tilde{O}$ with respect to the state ψ_{Σ_t}.