(2d-) Topological quantum field theories

A quick review

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Outline

Motivations from quantum field theory

The evolution from $\Sigma_1 = \{t_1\} \times \mathbb{R}^3$ to $\Sigma_2 = \{t_2\} \times \mathbb{R}^3$, where $\partial M = \overline{\Sigma_1} \sqcup \Sigma_2$, $M = [t_1, t_2] \times \mathbb{R}^3$. Is given by the amplitude transition

$$\rho_M : \mathcal{H}_{\Sigma_1} \otimes \mathcal{H}_{\overline{\Sigma_2}} \to \mathbb{C}$$

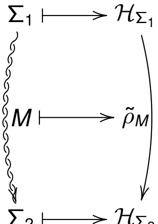
$$"\rho(\psi_1 \otimes \psi_2^*) = \int_{K_{t_1} \times K_{t_2}} \overline{\psi_2(\varphi_2)} Z_M(\varphi_1, \varphi_2) \psi_1(\varphi_1) \mathcal{D}\varphi_1 \mathcal{D}\varphi_2"$$

$$"Z_M(\varphi_1, \varphi_2) := \int_{K_{[t_1, t_2]}, \phi|_{t_i} = \varphi_i} e^{iS_M(\phi)} \mathcal{D}\phi"$$

or alternatively as an operator

$$\tilde{\rho}_M : \mathcal{H}_{\Sigma_1} \to \mathcal{H}_{\Sigma_2}, \tilde{\rho}_M \in \mathcal{H}^*_{\Sigma_1} \otimes \mathcal{H}_{\Sigma_2}$$

Thus we have a "rule"that assigns (state) vector spaces to (oriented) *hypersurfaces* Σ, and unitary (evolutionary) maps to *regions* M



Axioms from quantum theory

- (A-1) Duals:

$$\mathcal{H}_{\overline{\Sigma}} = \mathcal{H}_{\Sigma}^{*}$$

Where we take a dual vector space.
In the previous slide, this allows a coupling of ”in” state ψ_{Σ_1}, with an ”out” state $\psi_2 \in \mathcal{H}_{\Sigma_2}$.

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- (A-2) Tensor products:

$$\mathcal{H}_{\Sigma_1 \sqcup \Sigma_2} = \mathcal{H}_{\Sigma_1} \otimes \mathcal{H}_{\Sigma_2}$$

This is a tensor product and not a cartesian product i.e. quantum states demand an algebraic structure
It follows that $\mathcal{H}_{\partial M} = \mathcal{H}_{\overline{\Sigma_1} \sqcup \Sigma_2} = \mathcal{H}_{\Sigma_1}^* \otimes \mathcal{H}_{\Sigma_2}$ thus

$$\tilde{\rho}_M \in \mathcal{H}_{\partial M}$$

We consider algebraic duals and tensor products. For physically relevant theories we woukd like $\mathcal{H}_{\Sigma}$ to be *infinite dimensional* Hilbert spaces. In this case Hilbert space (continuous) duals and Hilbert space (completed) tensor products will not be considered. In the formal TQFT models we will just have *finite dimensional* Hilbert spaces, so duals and tensor products are algebraic.

Gluing axiom

(A-3a) If M is a manifold with $\partial M = \Sigma_1 \sqcup \Sigma \sqcup \overline{\Sigma}$ while $M_1 = \cup_\Sigma M$ obtained by identifying $\overline{\Sigma}$ with Σ, then

$$\tilde{\rho}_{M_1} = (\tilde{\rho}_M)_\Sigma,$$

where $(\tilde{\rho}_M)_\Sigma$ is the contraction

$$\mathcal{H}_{\Sigma_1} \otimes \mathcal{H}_\Sigma \otimes \mathcal{H}_{\overline{\Sigma}} \to \mathcal{H}_{\Sigma_1}$$

(A-3b) When two regions M_1, M_2 with $\partial M_1 = \overline{\Sigma_1} \sqcup \Sigma$, with $\partial M_1 = \overline{\Sigma_1} \sqcup \Sigma, \partial M_2 = \overline{\Sigma'} \sqcup \Sigma_2$ are glued along a boundary diffeomorphism $f : \Sigma \to \overline{\Sigma'}$, we obtain $\tilde{\rho}_{M_1 \cup_f M_2} \in \mathcal{H}^*_{\Sigma_1} \otimes \mathcal{H}_{\Sigma_2}$ as the contraction $(\tilde{\rho}_{M_1}, \tilde{\rho}_{M_2})$ where

$$(\cdot,\cdot) : \mathcal{H}^*_{\Sigma_1} \otimes (\mathcal{H}_\Sigma \otimes \mathcal{H}^*_\Sigma) \otimes \mathcal{H}_{\Sigma_2} \to \mathcal{H}^*_{\Sigma_1} \otimes \mathcal{H}_{\Sigma_2}$$

This axiom encodes the "time evolution" composition

$$''Z_{M_1 \cup_f M_2} = \int_{K_\Sigma} Z_{M_1}(\varphi_1, \varphi_\Sigma) Z_{M_2}(\varphi_\Sigma, \varphi_2) \mathcal{D}\varphi''_\Sigma$$

with $e^{iS_{M_1 \cup M_2}} = e^{iS_{M_1}} e^{iS_{M_2}}$.

More abstract axioms

- (A-4a) For the empty hypersurface $\emptyset = \Sigma$,

$$\mathcal{H}_{\emptyset} = \mathbb{C}$$

It follows that for $\partial\left(M_1 \cup_f M_2\right) = \emptyset$, $\tilde{\rho}_{M_1 \cup_f M_2} = \tilde{\rho}_{M_1} \otimes \tilde{\rho}_{M_2}$

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- (A-4b) For the empty region with empty boundary $\tilde{\rho}_\emptyset = 1 : \mathbb{C} \to \mathbb{C}$
  Also for the empty region with *non* empty boundary $\overline{\Sigma} \sqcup \Sigma$,
  $\tilde{\rho}_\emptyset = \text{id} \in \mathcal{H}^*_\Sigma \otimes \mathcal{H}_\Sigma$.

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- (A-5a) There are hermitian structures in $\mathcal{H}_\Sigma$ and $\mathcal{H}^*_\Sigma$ compatible with duals i.e. there are conjugate-linear maps $\iota_\Sigma : \mathcal{H}_\Sigma \to \mathcal{H}^*_\Sigma$, $\iota_{\overline{\Sigma}} : \mathcal{H}^*_\Sigma \to \mathcal{H}_\Sigma$, such that $\iota_{\overline{\Sigma}} \circ \iota_\Sigma = id$ and the following diagrams commutes

$$\begin{array}{ccccc}
\mathcal{H}^*_\Sigma & \otimes & \mathcal{H}_\Sigma & \longrightarrow & \mathbb{C} \\
\downarrow \iota_{\overline{\Sigma}} & & \downarrow id & & \downarrow z \mapsto \overline{z} \\
\mathcal{H}_\Sigma & \otimes & \mathcal{H}_\Sigma & \xrightarrow{\langle | \rangle} & \mathbb{C}
\end{array}$$

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  \mathcal{H}_\Sigma & \otimes & \mathcal{H}_\Sigma & \xrightarrow{\langle | \rangle} & \mathbb{C}
  \end{array}$$

- (A-5b) $\tilde{\rho}_M$ is unitary

"Topological field theories"

- (A-6) For any cylinder region $\Sigma \times [0,1]$,

$$\tilde{\rho}_{\Sigma\times[0,1]} = id \in \mathcal{H}_\Sigma^* \otimes \mathcal{H}_\Sigma,$$

where $\partial(\Sigma \times [0,1]) = \overline{\Sigma} \sqcup \Sigma$.
(Physics commment: This axiom encodes "kinematical theories", i.e. theories with null Hamiltonian where the action $S_M(\varphi), \varphi \in K_M$ is *diffeomorphisms invariant*

$$S_M(f^*\varphi) = S_M(\varphi), \qquad \text{for all } f : M \circlearrowleft,\ f \in Diff_0(M), \varphi \in K_M$$

Null hamiltonian evolution in Minkowski space-time yields the identification $\mathcal{H}_{t_1} \simeq \mathcal{H}_{t_2} \simeq \mathcal{H}$. Thus solutions $\tilde{\rho}_m$ should be interpreted as vacuum states)

- Example: BF theories modulo gauge. For simplicity take $\dim M = 2$,

$$S_{BF}(B,A) = \int_M B \wedge F^A,\ B \in \Omega^0(M,\mathbb{R}), A \in \Omega^1(M,\mathfrak{g})$$

where $\mathfrak{g}$ is the Lie algebra of G, and A is a connection for a $G-$principal bundle (for physicists $\varphi = B \times A$ is a field configuration and F^A is the field strength.) Then for every $f \in Diff_0(M)$

$$S_{BF}(f^*B, f^*A) = \int_M f^*B \wedge F^{f^*A} = \int_M f^*(B \wedge F^A) = \int_{f(M)} B \wedge F^A = \int_M B \wedge F^A$$

"Area almost topological field theories"

Example: Yang-Mills theories. For simplicity take M compact, $\dim M = 2$ with area form $\omega \in \Omega^2(M, \mathbb{R})$

$$S_{YM}(A) = \int_M tr(F^A \wedge *F^A) = \int_M tr(F^2)\omega, \; A \in \Omega^1(M, \mathfrak{g})$$

Then for every

$$f \in Diff_{0,\omega} M := \{f \in Diff_0 M \mid f^*\omega = \omega\}$$

we have the invariance

$$S_{YM}(f^*A) = \int_M tr(F \circ f)^2 f^*\omega = \int_M tr(F^2)\omega = S_{YM}(A)$$

Theorem (Moser)

Let ω, ω' two area forms in M such that they give the same total area $t = \int_M \omega = \int_M \omega'$, then there exists a diffeomorphism isotopy $f_s \in Diff_0 M, 0 \leq s \leq 1$ such that $f_1^*\omega' = \omega$.

Corollary

The Yang-Mills action S_{YM} of the cylinder $\tilde{\rho}_{\Sigma\times[0,1]}$, depends only on the total area t.
The transition amplitude of the cylinder $\tilde{\rho}_{t,\Sigma\times[0,1]} : \mathcal{H}_\Sigma \to \mathcal{H}_\Sigma$, depends just on the total area t of the cylinder $\Sigma \times [0,1]$

Objects

Simplifying (mathematical) hypothesis: A) regions M and hypersurfaces Σ correspond to a compact manifolds.
B) There is no "time" foliation in M, we rather regard space-time M as an "evolving movie" of space Σ where space topology may change.
Recall the assignment $M \mapsto \tilde{\rho}_M$, $\Sigma \mapsto \mathcal{H}_\Sigma$, the "source" for this rule can be described as the *n−dimensional cobordisms* **Cob**(n). This category consists of
Objects *Obj*(**Cob**(n)) : hypersurfaces Σ are identified with diffeomorphism classes of *oriented closed* ($\partial\Sigma = \emptyset$) $(n-1)-$dimensional manifolds
There exists a operation of objects: $\Sigma_1, \Sigma_2 \mapsto \Sigma_1 \sqcup \Sigma_2$. The empty object is a unity of the operation:

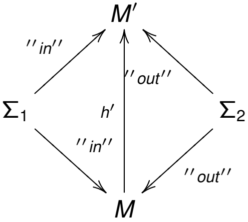
$$\emptyset \sqcup \Sigma = \Sigma = \Sigma \sqcup \emptyset$$

Morphisms

Morphisms $Morph_{\mathbf{Cob}(n)}(\Sigma_1, \Sigma_2)$: regions M are identified with equivalence classes of cobordisms $M : \Sigma_1 \rightsquigarrow \Sigma_2$,

$$\Sigma_1 \xrightarrow{''in''} M \xleftarrow[''out'']{} \Sigma_2$$

The ”in” diffemorphism inverts orientation in Σ_1 with respect to the induced orientation in M, the ”out” diffeomorphism preserves orientation. The equivalence relation of cobordisms $M' \sim M$



The cobordism class of the composition of two cobordisms $M_2 \circ M_1$ is well defined trough the cobordism class of the gluing$M_1 U_f M_2$.
$\Sigma \times [0, 1] : \Sigma \rightsquigarrow \Sigma$ is the identity morphism in the object Σ.

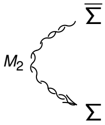
Examples of nonequivalent morphisms

1. The cylinder

$$\Sigma \overset{\Sigma\times[0,1]}{\rightsquigarrow} \Sigma$$

2. A bended cylinder $M_1 : \Sigma \sqcup \Sigma \rightsquigarrow \emptyset$, or equivalently

3. Another bended cylinder $M_2 : \emptyset \rightsquigarrow \Sigma \sqcup \Sigma$, or equivalently



Symmetric monoidal categories

The operation of objects defined by disjoint union is associative, and symmetric: there is a twist morphism

$$\tau : \Sigma_1 \sqcup \Sigma_2 \rightsquigarrow \Sigma_2 \sqcup \Sigma_1$$

There is also an operation of morphisms, $M_1 : \Sigma_1 \rightsquigarrow \Sigma_1'$ and $M_2 : \Sigma_2 \rightsquigarrow \Sigma_2'$,

$$M_1 \sqcup M_2 : \Sigma_1 \sqcup \Sigma_2 \rightsquigarrow \Sigma_1' \sqcup \Sigma_2'$$

There exists the empy morphism $\emptyset_M : \emptyset \rightsquigarrow \emptyset$ There is also a symmetry for the disjoint union of morphisms, and the unity morphism

$$\begin{array}{ccc} \Sigma_1 \sqcup \Sigma_2 & \overset{M_1 \sqcup M_2}{\rightsquigarrow} & \Sigma_1' \sqcup \Sigma_2' \\ \downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle \tau} \\ \Sigma_2 \sqcup \Sigma_1 & \overset{M_2 \sqcup M_1}{\rightsquigarrow} & \Sigma_2' \sqcup \Sigma_1' \end{array}$$

All this constitutes a *symmetric monoidal category* $(\mathbf{Cob}(n), \sqcup, \emptyset)$.

Examples from 2d TQFT

In this case $\dim M = 2$, $\dim \Sigma = 1$ and therefore $\Sigma = S^1$. Hence

$$Obj(\mathbf{Cob}(2)) = \{\emptyset, S^1, S^1 \sqcup S^1, S^1 \sqcup S^1 \sqcup S^1, \dots\} \simeq \{\mathbf{0}, \mathbf{1}, \mathbf{2}, ...\}$$

$Obj(\mathbf{Cob}(2))$ is a set!, $(Obj(\mathbf{Cob}(2)), \sqcup, \emptyset)$ is a monoid that corresponds to the additive monoid structure on the natural numbers

Proposition

Let M_1, M_2 be two compact **connected** *oriented surfaces with boundary. The surfaces M_1, M_2 are diffeomorphic iff they have the same genus g and the same number of boundary components k.*

This will allow a complete description of morphisms:

$$Morph_{\mathbf{Cob}(2)}(\Sigma_1, \Sigma_2)$$

The category of vector spaces **Vect**

On the other hand we organize the spaces $\mathcal{H}_\Sigma$ also in a category **Vect**
Objects $Obj(\mathbf{Vect})$: are vector spaces V
Morphisms are linear maps
We have as operation a tensor product $\otimes$
This is symmetric

$$V \otimes W \simeq W \otimes V$$

It has a unit

$$\mathbb{C} \otimes V \simeq V \simeq V \otimes \mathbb{C}$$

Hence $(\mathbf{Vect}, \otimes, \mathbb{C})$ is also a symmetric monoidal category

The functorial hypothesis

One of the first attemps to formalize TQFT is regarding the rule $\Sigma \mapsto \mathcal{H}_\Sigma$ and $M \mapsto \tilde{\rho}$ as a functor
The formal definition can be written as follows

Definition

A TQFT is a monoidal functor between monoidal symmetric categories

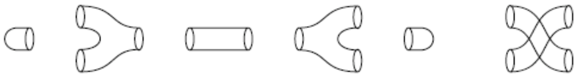
$$\mathcal{Z} : (\mathbf{Cob}(n), \sqcup, \emptyset) \Rightarrow (\mathbf{Vect}, \otimes, \mathbb{C})$$

$$\mathcal{Z}(\Sigma) := \mathcal{H}_\Sigma, \qquad \mathcal{Z}_{\Sigma_1,\Sigma_2}(M) := \tilde{\rho}_M$$

The interpretation of Atiyah's axioms as a functorial relation between categories does not cover physically relevant theories, nevertheless it clarifies conceptual insights for further research and also for calculations

Generating cobordisms in 2d

Let V be the vector space associated to $\Sigma \cong S^1$, i.e. $V = \mathcal{Z}(\mathbf{1}) \in \mathbf{Vect}$. From the topological classification of surfaces it follows that every cobordism in 2d can be obtained as a composition of one or several cobordisms in the following list. The corresponding morphisms associated by using the functor $\mathcal{Z}$ are dscribed below



$$\mathcal{Z} \Downarrow$$

$$e : \mathbb{C} \to V,\; m : V \otimes V \to V,\; id : V \to V,\; \delta : V \to V \otimes V,\; \epsilon : V \to \mathbb{C},$$

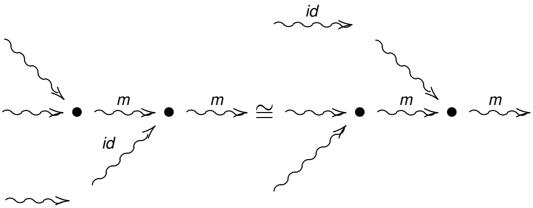
$$\tau : V \otimes V \to V \otimes V$$

They are called unit, multiplication, identity, comultiplication, trace and twisting respectively

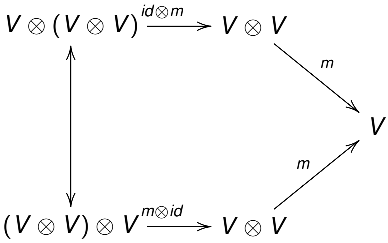
Relations among generators

The previous maps coming from generators have relations induced by the cobordism equivalence (surface classification)

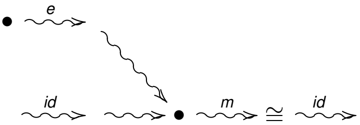
For instance the multiplication $m : V \otimes V \to V$ is associative as can be deduced from the following equivalence of cobordisms (in graph notation)



This also may be expressed as a commutative diagram

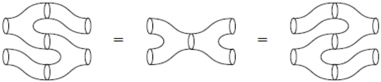


The unit *e* and multiplication *m* have the property



Frobenius relation

One important relation is Frobenius relation wich follows from the diffeomorphism



Finally we have a vector space V which is also a unital commutative algebra with additional structures such as the coproduct, the counit and the Frobenius relation. This relation encodes a compatibility between the product and the coproduct. A **(commutative) Frobenius algebra** is an algebra V provided with the operations and relations we have just mentioned. Using commutative diagrams

$$\begin{array}{ccccc} & & V \otimes (V \otimes V) & \xrightarrow{id \otimes m} & V \otimes V \\ & & \updownarrow & & \\ V \otimes V & \xrightarrow{\delta \otimes id} & (V \otimes V) \otimes V & & \end{array}$$

Finite dimension

Theorem

A Frobenius algebra V has finite dimension

Proof. Consider the Frobenius relation then it follows that the following composition should coincide with the identity

$$\begin{array}{ccccc} V & & (V \otimes V) \otimes V & \xrightarrow{\epsilon \circ m \otimes id} & \mathbb{C} \otimes V \\ \Vert & & \updownarrow & & \Vert \\ V \otimes \mathbb{C} & \xrightarrow{\delta \circ e \otimes id} & V \otimes (V \otimes V) & & V \end{array}$$

Let $(a \mid b) := \epsilon(m(a \otimes b))$ then the previous composition yields

$$a \otimes 1 \mapsto a \otimes \sum_{i=1}^{n} a_i \otimes b_i \mapsto \sum_{i=1}^{n} a_i \otimes b_i \otimes a \sum_{i1}^{n} a_i(b_i \mid a) = a$$

Therefore $a_1, \ldots, a_n$ is basis for V.

Corollary

$(a \mid \cdot) : V \to \mathbb{C}$ is non degenerate from every $a \in V$
$V \simeq V^*$
$V \simeq V**$

This last result impose an obstruction for considering V as an infinite dimensional Hilbert space. This obstruction may also be clarified by a straight calculation:

If $\delta : V \to V \otimes V$ is a coproduct defined as $\delta(|\psi_i\rangle) = |\psi_i\rangle \otimes |\psi_i\rangle$ for an orthonormal basis an linearly extended. Then:

a) This map is basis dependent.

b) Furthermore the trace $\epsilon : V \to \mathbb{C}$ can be defined (or redefined) as $\epsilon|\psi_i\rangle = 1$ and extended linearly. The Frobenius relation implies that ϵ becomes infinite. Thus a trace can exist only for finite dimensional Hilbert spaces.

In physics literature this discussion is the main limitation of the categorical interpretation of Atiyah's axioms in order to produce formalisms for physically relevant models.

Classification of 2d-TQFT

Despite the limitations we have described for the categorical point of view, one of the most appealing features of this formalism is that it provides a complete classification of 2d-TQFT. Namely we have the following theorem:

Theorem

The monoidal functors $\mathcal{Z} : \mathbf{Cob}(2) \Rightarrow \mathbf{Vect}$ *have themselves a category structure. Furthermore the following categories are equivalent*

$$\mathbf{2d} - \mathbf{TQFT} \simeq \mathbf{cFA}$$

Where the r.h.s. corresponds to the commutative Frobenius algebras category.

Natural transformations of functors $\mathcal{Z}$ correspond to morphisms in the category **2dTQFT**, equivalence of categories are defined by full, faithful and essentially surjective functors.

Explicit calculations for BF theories

Problem: Let M_g be a closed ($\partial M_g = \emptyset$) oriented surface of genus g. We are interested in calculating $\tilde{\rho}_{M_g} \in \mathbb{C}$.
I. **Bf theories:** Consider the action S_{BF} then boundary configurations correspond to the Lie group G modulo the adjoint action, thus $V = \mathcal{H}_{S^1}$ is the Hilbert spaces

$$C_{Class}(G) = \{f : G \to \mathbb{C} \mid f(g^{-1} \cdot x \cdot g) = f(x)\}$$

of class functions on G. By Peter Weil theorem there exists an ON basis $\{\chi_\alpha\}$ on $C_{Class}(G) = L^2(G)$ given by characters

$$\chi_\alpha(g) = tr(\alpha(g))$$

associated to irreducible representations α. Any state in $C_{Class}(G)$ may be written as $\psi = \sum_\alpha \psi_\alpha \chi_\alpha$.
Algebra product $m : V \otimes V \to V$ is given by convolution, characters are nilpotent mod coefficients: $m(\chi_i \otimes \chi_j) = a_{ij}\delta_{i,j}\chi_i$

We consider as "building blocks" of the surface M as the unit, outwards bended tube, the triple product each one is represented in the vector space morphisms:

a) Cap:

$$e = \sum_\alpha f_\alpha \chi_\alpha = \sum_\alpha w_\alpha \in V, \qquad w_\alpha = f_\alpha \chi_\alpha$$

b) Three holed sphere:

$$t = \sum_\alpha c_\alpha \chi_\alpha \otimes \chi_\alpha \otimes \chi_\alpha \in V \otimes V \otimes V$$

c) Two holed sphere:

$$\mu = \sum_\alpha \chi_\alpha^* \otimes \chi_\alpha^* \in V^* \otimes V^*$$

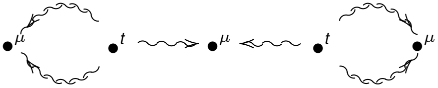
The identity relation yields the relation among coefficients $c_\alpha f_\alpha = 1$
A direct calculation yields

$$\mu(w_\alpha, w_\alpha) = f_\alpha^2 = c_\alpha^{-2}$$

Also

$$t = \sum_\alpha c_\alpha^4 w_\alpha \otimes w_\alpha \otimes w_\alpha$$

For instance take $g = 2$, a contraction of suitable oriented graphs yields the contraction of the tensor $(\mu \otimes \mu \otimes \mu, t \otimes t) = \sum_\alpha c_\alpha^2$



For general genus we have the contraction of $3(g-1)$ times μ with $2(g-1)$ times t hence

$$\mathcal{Z}(M_g) = \tilde{\rho}_M = \sum_\alpha c_\alpha^{2(g-1)}$$

After some calculations from group representation theory $c_\alpha = \frac{vol(G)}{(2\pi)^{\dim G} \dim \alpha}$

Calculations in other models

II. **Dijkgraaf-Witten model**: it uses finite gauge group G, then $V = C_{class}(G)$ is a Frobenius algebra. Here the categorical interpretation of 2d TQFT is fully incorporated. Nice calculations of representation theory yield

$$\mathcal{Z}(M_g) = |G|^{2g-2} \sum \frac{1}{(\dim \alpha)^{2g-2}}$$

III. **Electromagnetism:** recall that $T > 0$ is the total area of the region

a) Cap:

$$e = \sum_{\alpha \in \mathbb{Z}} e^{-T\alpha^2/2} \chi_\alpha$$

where $\chi_\alpha(\theta) = e^{i\theta\alpha}$, $\alpha \in \mathbb{Z}, \theta \in [0, 2\pi)$.

b) Three holed sphere:

$$t = \sum_{\alpha \in \mathbb{Z}} e^{-T\alpha^2/2} \chi_\alpha \otimes \chi_\alpha \otimes \chi_\alpha$$

c) Two holed sphere:

$$\mu = \sum_{\alpha \in \mathbb{Z}} e^{-T\alpha^2/2} \chi_\alpha^* \otimes \chi_\alpha^*$$

Invariant

$$\mathcal{Z}(M_g) = \sum_{\alpha \in \mathbb{Z}} e^{-T\alpha^2}$$

IV. **2d Yang Mills:**

a) Cap:

$$e = \sum_\alpha \dim \alpha e^{-T/2C_2(\alpha)} \chi_\alpha$$

b) Three holed sphere:

$$t = \sum_\alpha e^{-T/2C_2(\alpha)} / \dim \alpha \chi_\alpha \otimes \chi_\alpha \otimes \chi_\alpha$$

c) Two holed sphere:

$$\mu = \sum_\alpha e^{-T/2C_2(\alpha)} \chi_\alpha^* \otimes \chi_\alpha^*$$

Invariant

$$\mathcal{Z}(M_g) = \sum_\alpha \frac{e^{-TC_2(\alpha)}}{(\dim \alpha)^{2g-2}}$$

More general building blocks for example balls, may be considered in any dimension, but we have to describe TQFT with corners

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