Quantum Field Theory in the General Boundary Formulation

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Contents

Outline

Quantum Field Theory

- QFT (in Minkowski space) has been constructed according to the principles of quantum mechanics and special relativity.
- QFT is the mathematical framework to describe fundamental interactions (except gravity): According to contemporary physics, the universe is made up of matter fields (fermions) and interaction fields (bosons).
- Very successful theories have been formulated in this framework: QED (the theoretical and experimental values of the magnetic moment of the electron agree to within one part in 10^{10}), the Standard Model, etc.

GBF

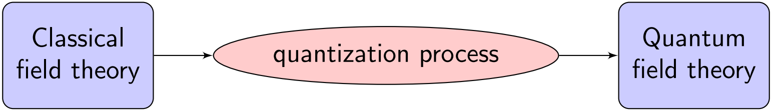
- The GBF is an axiomatic formulation of quantum theory which combines the mathematical framework of Topological Quantum Field Theory (see Homero's seminar) with a generalization of the Born's rule to extract probabilities.
- The spacetime background metric does not play any fundamental role in the GBF.
- However, a general boundary quantum theory can be implemented for studying the dynamics of fields defined on a spacetime with a definite metric background.

General Boundary QFT

The general boundary formulation of QFT appears to be interesting for several reasons,

1. the ability to reproduce known results obtained in QFT represents a fundamental test for the GBF;
2. the versatility of the GBF, where general spacetime regions are considered, offers a new perspective on QFT and a better understanding of its geometrical aspects (clarification of the holographic principle, boundaries, horizons);
3. it can treat situations where standard QFT fails:
    - QFT in presence of a static black hole: rigorous treatment implementable with the hypercylinder geometry,
    - S-matrix in Anti-de Sitter spacetime;
4. it may solve some of the interpretation problems of background independent QFT (problem of time, local description of dynamics).

How to get a QFT?



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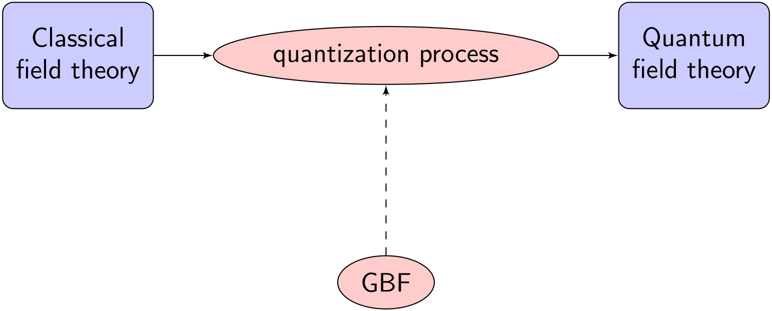

How to proceed?

1. The starting point is a classical field teory, namely:
   - Spacetime regions with a fixed metric
   - One or more fields satisfying some e.o.m.
2. The GBF provides two quantization prescriptions:
   - The Schrödinger-Feynman quantization
   - The holomorphic quantization
3. The resulting quantum field theory satisfies the axioms of the GBF.

Outline

Basic structures

In the GBF algebraic structures are associated to geometric ones.

Geometric structures (representing pieces of spacetime):

- **hypersurfaces**: oriented manifolds of dimension $d-1$
- **regions**: oriented manifolds of dimension d with boundary

Algebraic structures:

- To each hypersurface Σ associate a Hilbert space $\mathcal{H}_\Sigma$ of states.
- To each region M with boundary ∂M associate a linear amplitude map $\rho_M : \mathcal{H}_{\partial M} \to \mathbb{C}$
- As in AQFT, observables are associated to spacetime regions: An observable O in a region M is a linear map $\rho_M^O : \mathcal{H}_{\partial M} \to \mathbb{C}$, called observable map.

Axioms and recovering of standard results

These algebraic structures are subject to a number of axioms, in the spirit of **TQFT**.

- If $\overline{\Sigma}$ denote Σ with opposite orientation, then $\mathcal{H}_{\overline{\Sigma}} = \mathcal{H}^*_{\Sigma}$.
- **(Decomposition rule)** If $\Sigma = \Sigma_1 \cup \Sigma_2$, then $\mathcal{H}_\Sigma = \mathcal{H}_{\Sigma_1} \otimes \mathcal{H}_{\Sigma_2}$.
- **(Gluing rule)** If M and N are adjacent regions, then $\rho_{M\cup N} = \rho_M \circ \rho_N$. The composition $\circ$ involves a sum over a complete basis on the boundary hypersurface Σ shared by M and N.

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- Standard transition amplitudes of QFT can be recover from the GBF: $\rho_{[t_1, t_2]}(\psi_{t_1} \otimes \eta_{t_2}) = \langle \eta | U(t_1, t_2) | \psi \rangle$.
- A consistent probability interpretation can be implemented, standard probabilities recovered.
- Conventional expectation values of observables can be recovered.

Classical field theory

We consider a **linear real scalar field theory** in a spacetime region M of an N-dimensional Lorentzian manifold $(\mathcal{M}, g)$.

- Action: $S[\phi] = \int_M \mathrm{d}^N x \mathcal{L}(\phi, \partial\phi, x)$.
- With an hypersurface Σ we associate L_Σ, the space of solutions of the e.o.m. defined in a neighborhood of Σ. L_Σ is a vector space.
- The symplectic potential is the one-form on L_Σ,

$$(\theta_\Sigma)_\phi(X) := \int_\Sigma \mathrm{d}^{N-1}\sigma \, X(x(\sigma)) \left(n^\mu \frac{\delta \mathcal{L}}{\delta \partial_\mu \phi} \right) (x(\sigma)),$$

  where n^μ is the unit normal vector to Σ.
- For every two elements of L_Σ there is the bilinear map $[\cdot,\cdot]_\Sigma : L_\Sigma \times L_\Sigma \to \mathbb{R}$ defined such that $[\xi, \eta]_\Sigma := (\theta_\Sigma)_\xi(\eta)$
- L_Σ is equipped with the symplectic structure $\omega_\Sigma : L_\Sigma \times L_\Sigma \to \mathbb{R}$ given by $\omega_\Sigma(\xi, \eta) := \frac{1}{2}[\xi, \eta]_\Sigma - \frac{1}{2}[\eta, \xi]_\Sigma$.

Complex structure

The passage from the classical to quantum theory needs the specification of a compatible complex structure J_Σ represented by the linear map $J_\Sigma : L_\Sigma \to L_\Sigma$ such that

$$J_\Sigma^2 = -\mathrm{id}, \qquad \omega_\Sigma(J_\Sigma \cdot, J_\Sigma \cdot) = \omega_\Sigma(\cdot, \cdot)$$

and $\omega_\Sigma(\cdot, J_\Sigma \cdot)$ is a positive definite bi-linear map.

Schrödinger-Feynman quantization

- Quantum states are represented by wave functionals of field configurations.
- We define the «space of momentum», $M_\Sigma \subset L_\Sigma$, as

$$M_\Sigma := \{\eta \in L_\Sigma : [\xi, \eta] = 0 \, \forall \xi \in L_\Sigma\}.$$

- We consider the quotient space $Q_\Sigma := L_\Sigma / M_\Sigma$ which corresponds the space of all field configurations on Σ.
- Next, we define the bilinear map

$$\begin{aligned} \Omega_\Sigma : & Q_\Sigma \times Q_\Sigma \to \mathbb{C}, \\ & (\varphi, \varphi') \mapsto 2\omega_\Sigma(j_\Sigma(\varphi), J_\Sigma j_\Sigma(\varphi')) - i[j_\Sigma(\varphi), \varphi']_\Sigma, \end{aligned}$$

where j_Σ is the unique linear map $Q_\Sigma \to L_\Sigma$ such that $q_\Sigma \circ j_\Sigma = \mathrm{id}_{Q_\Sigma}$ and q_Σ is the quotient map $L_\Sigma \to Q_\Sigma$.

Schrödinger-Feynman quantization II

- The Hilbert space $\mathcal{H}_\Sigma$ is defined as the closure of the set of all coherent states

$$K_\xi(\varphi) = \exp\left(\Omega_\Sigma(q_\Sigma(\xi), \varphi) + \mathrm{i}[\xi, \varphi]_\Sigma - \frac{1}{2}\Omega_\Sigma(q_\Sigma(\xi), q_\Sigma(\xi)) - \frac{\mathrm{i}}{2}[\xi, \xi]_\Sigma - \frac{1}{2}\Omega_\Sigma(\varphi, \varphi)\right),$$

with respect to the inner product

$$\langle K_\xi, K_{\xi'}\rangle := \int_{Q_\Sigma} \mathcal{D}\varphi\, \overline{K_\xi(\varphi)}\, K_{\xi'}(\varphi),$$

where the bar denotes complex conjugation.

- The vacuum state K_0 is defined as the coherent state with $\xi = 0$,

$$K_0(\varphi) = \exp\left(-\frac{1}{2}\Omega_\Sigma(\varphi, \varphi)\right).$$

Schrödinger-Feynman quantization III

- Dynamics is encoded in an amplitude map for a region M, $\rho_M : \mathcal{H}_\Sigma \to \mathbb{C}$, for a state $\psi \in \mathcal{H}_\Sigma$, (where now Σ denotes the boundary of M)

$$\rho_M(\psi) = \int_{Q_\Sigma} \mathcal{D}\varphi\, \psi(\varphi) \int_{K_M, \phi|_\Sigma = \varphi} \mathcal{D}\phi\, e^{\mathrm{i} S_M(\phi)}.$$

  The inner integral is over the space K_M of space-time field configurations ϕ in the interior of M which agree with φ on the boundary Σ.
- A classical observable F in M is modeled as a function on K_M. The quantization of F is the linear map $\rho_M^F : \mathcal{H}_\Sigma \to \mathbb{C}$ defined as

$$\rho_M^F(\psi) = \int_{Q_\Sigma} \mathcal{D}\varphi\, \psi(\varphi) \int_{K_M, \phi|_\Sigma = \varphi} \mathcal{D}\phi\, F(\phi) e^{\mathrm{i} S_M(\phi)}.$$

Holomorphic quantization

- Linear field theory: L_Σ is the vector space of solutions near the hypersurface Σ.
- The complex structure J_Σ and the symplectic structure ω_Σ are combined to a real inner product

$$g_\Sigma(\cdot,\cdot) = 2\omega_\Sigma(\cdot, J_\Sigma \cdot),$$

  and assume that this form is positive definite.
- Next, we define the sesquilinear form

$$\{\cdot,\cdot\}_\Sigma = g_\Sigma(\cdot,\cdot) + 2\mathrm{i}\omega_\Sigma(\cdot,\cdot).$$

- The completion of L_Σ with the inner product $\{\cdot,\cdot\}_\Sigma$ turns it into a complex Hilbert space.

Holomorphic quantization II

- The Hilbert space $\mathcal{H}_\Sigma$ is the set of square integrable holomorphic functions on L_Σ, and is given by the closure of the set of all coherent states
$$K_\xi(\phi) := e^{\frac{1}{2}\{\xi,\phi\}_\Sigma},$$
where $\xi \in L_\Sigma$ and the closure is taken with respect to the inner product
$$\langle K_\xi, K_{\xi'}\rangle := \int_{L_\Sigma} \mathrm{d}\nu_\Sigma(\phi)\, \overline{K_\xi(\phi)} K_{\xi'}(\phi),$$
where $\mathrm{d}\nu_\Sigma$ can be represented formally as $\mathrm{d}\nu_\Sigma(\phi) = \mathrm{d}\mu_\Sigma(\phi) e^{\frac{1}{4}g_\Sigma(\phi,\phi)}$ with a certain translation invariant measure $\mathrm{d}\mu_\Sigma$.

Holomorphic quantization III

- The amplitude map $\rho_M : \mathcal{H}_\Sigma \to \mathbb{C}$ associated with the spacetime region M for a state $\psi \in \mathcal{H}_\Sigma$ is given by

$$\rho_M(\psi) = \int_{L_{\tilde{M}}} \psi(\phi) \exp\left(-\frac{1}{4} g_\Sigma(\phi, \phi)\right) \mathrm{d}\mu_{\tilde{M}}(\phi).$$

  where $L_{\tilde{M}} \subseteq L_\Sigma$ is the set of all global solutions on M mapped to L_Σ by just considering the solutions in a neighborhood of Σ.
- The observable map associated to a classical observable F in a region M is

$$\rho_M^F(\psi) = \int_{L_{\tilde{M}}} \psi(\phi) F(\phi) \exp\left(-\frac{1}{4} g_\Sigma(\phi, \phi)\right) \mathrm{d}\mu_{\tilde{M}}(\phi).$$

Results

Result 1

An isomorphic can be constructed between the Hilbert spaces in the two representations.

Result 2

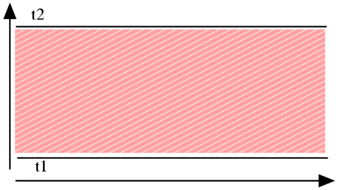
The GBF axioms are satisfied by these quantization prescriptions.

Outline

Klein-Gordon theory in Minkowski: standard setting

The S-matrix technique is used to describe interacting QFT:

Spacetime region:
$M = [t_1, t_2] \times \mathbb{R}^3$
Boundary: $\partial M = \Sigma_1 \cup \overline{\Sigma}_2$
State space: $\mathcal{H}_{\partial M} = \mathcal{H}_{\Sigma_1} \otimes \mathcal{H}^*_{\Sigma_2}$



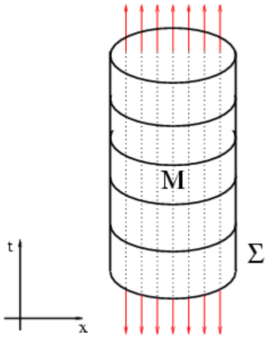
Assume interaction is relevant only between the initial time t_1 and the final time t_2. The S-matrix is the asymptotic limit of the amplitude between free states at early and at late time:

$$\boxed{\langle \psi_2 | \mathcal{S} | \psi_1 \rangle = \lim_{\substack{t_1 \to -\infty \\ t_2 \to +\infty}} \langle \psi_2 | U_{int}(t_1, t_2) | \psi_1 \rangle = \lim_{\substack{t_1 \to -\infty \\ t_2 \to +\infty}} \rho^U_{[t_1, t_2] \times \mathbb{R}^3}(\psi_2 \otimes \psi_1)}$$

Spatially asymptotic S-matrix

Similarly, we can describe interacting QFT via a spatially asymptotic amplitude. Assume interaction is relevant only within a radius R from the origin in space (but at all times). Consider then the asymptotic limit of the amplitude of a free state on the hypercylinder when the radius goes to infinity:

$$\boxed{\mathcal{S}(\psi) = \lim_{R\to\infty} \rho_R(\psi)}$$



Result

The S-matrices are equivalent when both are valid.

Some results obtained from GBQFT

- Description of quantum states on timelike hypersurfaces. This permits the quantization of evanescent waves that are "invisible" in traditional quantization prescriptions.
- Description of general interacting QFT in Minkowski spacetime.
- Description of new types of asymptotic amplitudes, generalizing the S-matrix framework.
- New representation of the Feynman propagator.
- These results have been extended to Euclidean, de Sitter, Anti-de Sitter and Rindler spaces.
- General structure of the complex structure (and vacuum state) used in the Schrödinger representation of QFT and pairing between the holomorphic and the Schrödinger representation(in col. with Max).
- Unitary evolution in curved spacetime (in col. with Robert).
- General structure of the S-matrix and the Feynman propagator in a wide class of curved spaces(in col. with Max).