General Boundary Quantum Field Theory in Minkowski spacetime

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DC, R. Oeckl, Phys. Lett. B665 (2008), arXiv:0710.5203.

Contents

Schrödinger-Feynman quantization

- Schrödinger representation + Feynman path integral quantization
  The state space $\mathcal{H}_\Sigma$ for a hypersurface Σ is the space of functions on field configurations Q_Σ on Σ.
- We write the inner product there as

$$\langle\psi_2|\psi_1\rangle = \int_{Q_\Sigma} \mathcal{D}\varphi\, \psi_1(\varphi)\overline{\psi_2(\varphi)}.$$

- The amplitude for a region M and a state ψ in the state space $\mathcal{H}_{\partial M}$ associated to the boundary ∂M of M is

$$\rho_M(\psi) = \int_{Q_\Sigma} \mathcal{D}\varphi\, \psi(\varphi) Z_M(\varphi),$$

  where the hypersurface Σ represents the boundary of M: $\Sigma = \partial M$.
- Z_M is the propagator given by the Feynman path integral,

$$Z_M(\varphi) = \int_{K_M, \phi|_\Sigma = \varphi} \mathcal{D}\phi\, e^{\mathrm{i} S_M(\phi)}, \quad \forall \varphi \in Q_\Sigma.$$

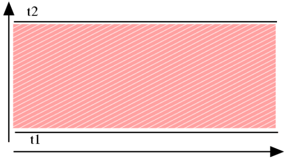
  The integral is over the space K_M of field configurations ϕ in the interior of M s.t. ϕ agrees with φ on the boundary Σ.

S-matrix in the standard setting

- We consider a massive Klein-Gordon field in Minkowski spacetime.

- The usual situation
  Spacetime region: $M = [t_1, t_2] \times \mathbb{R}^3$
  Boundary: $\partial M = \Sigma_1 \cup \overline{\Sigma}_2$
  State space: $\mathcal{H}_{\partial M} = \mathcal{H}_{\Sigma_1} \otimes \mathcal{H}^*_{\Sigma_2}$

  $\psi_{1,2}$ wave functions of field config.
  $\varphi_{1,2}$ at times $t_{1,2}$



- Standard transition amplitudes take the form

$$\langle\psi_2|U_{[t_1,t_2]}|\psi_1\rangle = \rho_{[t_1,t_2]}(\overline{\psi_2}\otimes\psi_1) = \int \mathcal{D}\varphi_1\,\mathcal{D}\varphi_2\;\psi_1(\varphi_1)\,\overline{\psi_2(\varphi_2)}\,Z_{[t_1,t_2]}(\varphi_1,\varphi_2),$$

  $U_{[t_1,t_2]}$: time-evolution operator from time t_1 to time t_2.
  $Z_{[t_1,t_2]}$: field propagator s.t. vacuum-to-vacuum amplitude equals one.

Free classical theory

- Free action of the real massive Klein-Gordon field

$$S_{[t_1,t_2],0}(\phi) = \frac{1}{2}\int \mathrm{d}^4x \left((\partial_0\phi)(\partial_0\phi) - \sum_{i\geq 1}(\partial_i\phi)(\partial_i\phi) - m^2\phi^2 \right).$$

  the integral is extended over the region under consideration.
- Bounded solutions in the region M are parametrized by plane waves,

$$\phi(x,t) = \int \frac{\mathrm{d}^3k}{\sqrt{(2\pi)^3 2\omega}} \left(a(k)e^{-\mathrm{i}\omega t}e^{\mathrm{i}kx} + c.c. \right).$$

  where $\omega = \sqrt{-\sum_i \partial_i^2 + m^2}$.
- A classical solution that reduces to the field config. φ_1 at t_1 and φ_2 at t_2 is

$$\phi(x,t) = \frac{\sin\omega(t_2 - t)}{\sin\omega(t_2 - t_1)}\varphi_1(x) + \frac{\sin\omega(t - t_1)}{\sin\omega(t_2 - t_1)}\varphi_2(x).$$

- The symplectic structure on L_t reads

$$\omega_t(\phi_1,\phi_2) = \frac{1}{2}\int \mathrm{d}^3x \left(\phi_1 \overset{\leftrightarrow}{\partial_t} \phi_2 \right)$$

Field propagator for the free theory

- Field propagator

$$Z_{[t_1,t_2],0}(\varphi_1,\varphi_2) = \int_{\phi|_{t_i}=\varphi_i} \mathcal{D}\phi\, e^{\mathrm{i}S_{[t_1,t_2],0}(\phi)},$$

the integral can be evaluated by shifting the integration variable by a classical solution matching the boundary configurations:

$$Z_{[t_1,t_2],0}(\varphi_1,\varphi_2) = N_{[t_1,t_2],0} \exp\left(-\frac{1}{2}\int \mathrm{d}^3x \, \begin{pmatrix}\varphi_1 & \varphi_2\end{pmatrix} W_{[t_1,t_2]} \begin{pmatrix}\varphi_1 \\ \varphi_2\end{pmatrix}\right),$$

where $W_{[t_1,t_2]}$ is the operator valued 2×2 matrix

$$W_{[t_1,t_2]} = \frac{-\mathrm{i}\omega}{\sin\omega(t_2-t_1)} \begin{pmatrix}\cos\omega(t_2-t_1) & -1 \\ -1 & \cos\omega(t_2-t_1)\end{pmatrix}.$$

- The evolution described by this field propagator is unitary.

Quantum theory

- Vacuum state $\psi_{t,0} \in \mathcal{H}_t$

$$\psi_{t,0}(\varphi) = C_t \exp\left(-\frac{1}{2}\int \mathrm{d}^3x\, \varphi(x)(\omega\varphi)(x)\right),$$

where C_t is a normalization factor. This state corresponds to the standard Minkowski vacuum state in the Schrödinger representation.

- Coherent states in the interaction picture (time-independent description of free states)

$$\psi_{t,\eta}(\varphi) = K_{t,\eta} \exp\left(\int \frac{\mathrm{d}^3x\, \mathrm{d}^3k}{(2\pi)^3}\, \eta(k)\, e^{-\mathrm{i}(Et-kx)}\, \varphi(x)\right) \psi_{t,0}(\varphi),$$

η complex function; $K_{t,\eta}$ normalization factor.

S-matrix in the standard setting: Free theory

- S-matrix for the free theory

$$\langle \psi_{\eta_2} | \mathcal{S}_0 | \psi_{\eta_1} \rangle = \lim_{\substack{t_1 \to -\infty \\ t_2 \to +\infty}} \langle \psi_{t_2,\eta_2} | U_{[t_2,t_1],0} | \psi_{t_1,\eta_1} \rangle$$

$$= \lim_{\substack{t_1 \to -\infty \\ t_2 \to +\infty}} \rho_{[t_1,t_2],0}(\overline{\psi_2} \otimes \psi_1)$$

$$= \exp\left(\int \frac{\mathrm{d}^3 k}{(2\pi)^3 2E} \left(\eta_1(k)\,\overline{\eta_2(k)} - \frac{1}{2}|\eta_1(k)|^2 - \frac{1}{2}|\eta_2(k)|^2 \right) \right),$$

where $E = \sqrt{k^2 + m^2}$.

S-matrix in the standard setting: Source interaction (I)

- Klein-Gordon field interacting with a source field μ; the action takes the form
$$S_{[t_1,t_2],\mu}(\phi) = S_{[t_1,t_2],0}(\phi) + \int \mathrm{d}^4x\, \mu(x)\phi(x),$$
the support of μ vanishes outside the interval $[t_1, t_2]$.
- The field propagator for the theory with the source interaction takes the form
$$Z_{[t_1,t_2],\mu}(\varphi_1, \varphi_2) = \int_{\substack{\phi|_{t_1}=\varphi_1 \\ \phi|_{t_2}=\varphi_2}} \mathcal{D}\phi\, e^{\mathrm{i}S_{[t_1,t_2],\mu}(\phi)}.$$
Shifting the integration variable by a classical solution ϕ_{cl} of the homogeneous Klein-Gordon equation interpolating between φ_1 at t_1 and φ_2 at t_2,
$$Z_{[t_1,t_2],\mu}(\varphi_1, \varphi_2) = Z_{[t_1,t_2],0}(\varphi_1, \varphi_2)\, e^{\mathrm{i}\int \mathrm{d}^4x\, \mu(x)\phi_{\mathrm{cl}}(x)}\, e^{\frac{\mathrm{i}}{2}\int \mathrm{d}^4x\, \mu(x)\alpha(x)},$$
where α is a solution of the inhomogeneous Klein-Gordon equation
$$(\Box + m^2)\alpha(t, x) = \mu(t, x),$$
with boundary conditions $\alpha(t_1, x) = 0$ and $\alpha(t_2, x) = 0$ for all $x \in \mathbb{R}^3$.

S-matrix in the standard setting: Source interaction (II)

- The term $e^{\mathrm{i}\int \mu\phi_{\mathrm{cl}}}$ modifies the coherent states as

$$\tilde{\eta}_1(k) := \eta_1(k) + \int \mathrm{d}^3x\, e^{\mathrm{i}(Et_1 - kx)}\mu_1(x),$$
$$\tilde{\eta}_2(k) := \eta_2(k) + \int \mathrm{d}^3x\, e^{\mathrm{i}(Et_2 - kx)}\mu_2(x).$$

- The transition amplitude results to be

$$\langle\psi_{\eta_2}|\mathcal{S}_\mu|\psi_{\eta_1}\rangle = \langle\psi_{\tilde{\eta}_2}|\mathcal{S}_0|\psi_{\tilde{\eta}_1}\rangle \frac{K_{t_1,\eta_1}\overline{K_{t_2,\eta_2}}}{K_{t_1,\tilde{\eta}_1}\overline{K_{t_2,\tilde{\eta}_2}}} e^{\frac{\mathrm{i}}{2}\int \mathrm{d}^4x\,\mu(x)\alpha(x)},$$
$$= \langle\psi_{\eta_2}|\mathcal{S}_0|\psi_{\eta_1}\rangle\, e^{\mathrm{i}\int \mathrm{d}^4x\,\mu(x)\hat{\eta}(x)}\, e^{\frac{\mathrm{i}}{2}\int \mathrm{d}^4x\,\mu(x)\gamma(x)},$$

where $\hat{\eta}$ is the complex classical solution of the Klein-Gordon equation determined by η_1 and η_2 via

$$\hat{\eta}(t,x) = \int \frac{\mathrm{d}^3k}{(2\pi)^3 2E}\left(\eta_1(k)e^{-\mathrm{i}(Et-kx)} + \overline{\eta_2(k)}e^{\mathrm{i}(Et-kx)}\right).$$

S-matrix in the standard setting: Source interaction (III)

- The quantity γ is the solution of the inhomogeneous Klein-Gordon equation

$$(\Box + m^2)\gamma = \mu,$$

with boundary conditions

$$\text{for} \quad t < t_1, \quad \gamma(t,k) = e^{iEt}\int_{t_1}^{t_2} d\tau\, i\, e^{-iE\tau}\,\mu(\tau,k),$$

$$\text{for} \quad t > t_2, \quad \gamma(t,k) = e^{-iEt}\int_{t_1}^{t_2} d\tau\, i\, e^{iE\tau}\,\mu(\tau,k).$$

The function γ contains only negative energy modes at early times ($t < t_1$) and positive energy modes at late times ($t > t_2$). We recognize these as the *Feynman boundary conditions*. Thus, γ takes the form,

$$\gamma(x) = \int d^4x'\, G_F(x,x')\mu(x'),$$

G_F is the Feynman propagator normalized:$(\Box_x + m^2)G_F(x,x') = \delta^4(x - x')$.

S-matrix in the standard setting: Source interaction (IV)

- S-matrix for the theory with source interaction,

$$\langle \psi_{\eta_2} | \mathcal{S}_\mu | \psi_{\eta_1} \rangle = \langle \psi_{\eta_2} | \mathcal{S}_0 | \psi_{\eta_1} \rangle$$
$$\cdot \exp\left(\mathrm{i} \int \mathrm{d}^4 x \, \mu(x) \hat{\eta}(x) \right) \exp\left(\frac{\mathrm{i}}{2} \int \mathrm{d}^4 x \, \mathrm{d}^4 x' \mu(x) G_F(x, x') \mu(x') \right),$$

$\hat{\eta}$ is the complex solution of the Klein-Gordon given by equation

$$\hat{\eta}(t, x) = \int \frac{\mathrm{d}^3 k}{(2\pi)^3 2E} \left(\eta_1(k) e^{-\mathrm{i}(Et - kx)} + \overline{\eta_2(k)} e^{\mathrm{i}(Et - kx)} \right).$$

S-matrix in the standard setting: General interaction

- We use functional derivative techniques to work out the S-matrix in the case of a general interaction. The action of the scalar field with an arbitrary potential V can be written as

$$S(\phi) = S_0(\phi) + \int \mathrm{d}^4x\, V(x, \phi(x)) = S_0(\phi) + \int \mathrm{d}^4x\, V\left(x, \frac{\partial}{\partial\mu(x)}\right) S_\mu(\phi)\bigg|_{\mu=0},$$

- We assume that the interaction vanishes outside the interval $[t_1, t_2]$,

$$V((t,x), \phi(t,x)) = 0, \forall x \in \mathbb{R}^3, \forall t \notin [t_1, t_2].$$

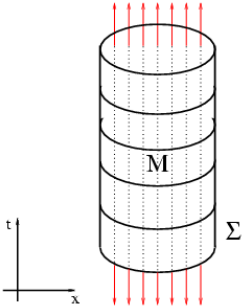
- The S-matrix

$$\langle\psi_2|\mathcal{S}_V|\psi_1\rangle = \exp\left(\mathrm{i}\int \mathrm{d}^4x\, V\left(x, -\mathrm{i}\frac{\partial}{\partial\mu(x)}\right)\right)\langle\psi_2|\mathcal{S}_\mu|\psi_1\rangle\bigg|_{\mu=0}.$$

The hypercylinder

We consider the Klein-Gordon field in a spacetime region with a connected boundary.

Spacetime region: $M = \mathbb{R} \times B^3_R$, ball of radius R in space extended over all of time, the *solid hypercylinder*.
Boundary: $\partial M = \mathbb{R} \times S^2_R$, the *hypercylinder*.
State space: $\mathcal{H}_{\partial M} = \mathcal{H}_\Sigma = \mathcal{H}_R$.



Solutions of the Klein-Gordon equation

$$\phi(t, r, \Omega) = \int_{-\infty}^{\infty} \mathrm{d}E \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \alpha_{l,m}(E) e^{-iEt} f_l(pr) Y_l^m(\Omega),$$

where f_l denotes a certain kind of spherical Bessel function. Ω is a collective notation for the angle coordinates (θ, ϕ). Y_l^m denotes the spherical harmonic.

Classical solutions

Different types of spherical Bessel functions are used depending of the value of the energy

- If $E^2 > m^2 \rightarrow$ real momentum, (ordinary) spherical Bessel functions of the first and second kind: $j_l(pr)$ and $n_l(pr)$
- If $E^2 < m^2 \rightarrow$ real imaginary, modified spherical Bessel functions of the first and second kind: $i_l^+(\mathrm{i}pr)$ and $i_l^-(\mathrm{i}pr)$

We introduce a unified notation

$$a_l(E,r) := \begin{cases} j_l(r\sqrt{E^2 - m^2}) & \text{if } E^2 > m^2, \text{regular at the origin} \\ i_l^+(r\sqrt{m^2 - E^2}) & \text{if } E^2 < m^2, \text{regular at the origin} \end{cases}$$

and

$$b_l(E,r) := \begin{cases} n_l(r\sqrt{E^2 - m^2}) & \text{if } E^2 > m^2, \text{singular at the origin} \\ i_l^-(r\sqrt{m^2 - E^2}) & \text{if } E^2 < m^2, \text{singular at the origin} \end{cases}$$

as well as $\quad c_l(E,r) := a_l(E,r) + \mathrm{i}\, b_l(E,r),\;$ and $\; p := \begin{cases} \sqrt{E^2 - m^2} & \text{if } E^2 > m^2, \\ \mathrm{i}\sqrt{m^2 - E^2} & \text{if } E^2 < m^2. \end{cases}$

Classical theory

- Free action

$$S_{R,0}(\phi) = -\frac{1}{2}\int \mathrm{d}t\,\mathrm{d}\Omega\, R^2 \phi(t,R,\Omega)(\partial_r\phi)(t,R,\Omega).$$

- Classical solution well defined inside the solid hypercylinder in terms of boundary configurations

$$\phi(t,\Omega,r) = \frac{a_l(E,r)}{a_l(E,R)}\varphi(t,\Omega).$$

Quantum theory

- Free field propagator

$$Z_{R,0}(\varphi) = N_{R,0} \exp\left(-\frac{1}{2}\int \mathrm{d}t\,\mathrm{d}\Omega\,\varphi(t,\Omega)\,\mathrm{i}R^2\frac{a_l'(E,R)}{a_l(E,R)}\,\varphi(t,\Omega)\right),$$

where a_l' is the derivative of a_l w.r.t. r and $R^2\frac{a_l'(E,R)}{a_l(E,R)}$ is to be understood as an operator via the mode decomposition of the field.

- Vacuum state

$$\psi_{R,0}(\varphi) = C_R \exp\left(-\frac{1}{2}\int \mathrm{d}t\,\mathrm{d}\Omega\,\varphi(t,\Omega)(B_R\varphi)(t,\Omega)\right),$$

C_R: normalization factor
B_R: family of operators indexed by R given by

$$B_R = -\mathrm{i}R^2\frac{c_l'(E,R)}{c_l(E,R)}.$$

- Coherent states in the interaction picture (radius-independent description of free states)

$$\psi_{R,\xi}(\varphi) = K_{R,\xi} \exp\left(\int_{|E|\geq m}\mathrm{d}E\sum_{l,m}\frac{\xi_{l,m}(E)}{c_l(pR)}\,\varphi_{l,m}(E)\right)\psi_{R,0}(\varphi),$$

$K_{R,\xi}$: normalization factor
$\xi_{l,m}(E)$: complex function s.t. $\xi_{l,m}(E) = 0$ if $|E| < m$.

S-matrix on the hypercylinder: Free theory

- Amplitude of the coherent state for the solid hypercylinder

$$\rho_{R,0}(\psi_{R,\xi}) = \exp\left(\int_{|E|\geq m} \mathrm{d}E \sum_{l,m} \frac{p}{8\pi} \left(\xi_{l,m}(E)\xi_{l,-m}(-E) - |\xi_{l,m}(E)|^2\right)\right).$$

By construction this expression is independent of the radius R. The limit $R \to \infty$ gives the asymptotic amplitude for the free theory,

$$\mathcal{S}_0(\psi_\xi) = \lim_{R\to\infty} \rho_{R,0}(\psi_{R,\xi}) = \exp\left(\int \mathrm{d}E \sum_{l,m} \frac{p}{8\pi} \left(\xi_{l,m}(E)\xi_{l,-m}(-E) - |\xi_{l,m}(E)|^2\right)\right).$$

S-matrix on the hypercylinder: Source interaction (I)

- We consider an interaction with a source field μ that vanishes outside the solid hypercylinder ($r \geq R$).
- The amplitude associated with the solid hypercylinder $\mathbb{R} \times B_R^3$ is

$$\rho_{R,\mu}(\psi_{R,\xi}) = \int \mathcal{D}\varphi \, \psi_{R,\xi}(\varphi) \, Z_{R,\mu}(\varphi).$$

- The field propagator is evaluated by shifting the integration variable by a solution ϕ_{cl} matching the boundary data, $\phi_{\text{cl}}|_R = \varphi$

$$Z_{R,\mu}(\varphi) = Z_{R,0}(\varphi) \, e^{\mathrm{i} \int \mathrm{d}^4x \, \mu(x) \, \phi_{\text{cl}}(x)} \, e^{\frac{\mathrm{i}}{2} \int \mathrm{d}^4x \, \mu(x) \alpha(x)},$$

  α satisfies the inhomogeneous Klein-Gordon equation with vanishing boundary conditions at radius R,

$$(\Box + m^2)\alpha = \mu, \quad \text{and} \quad \alpha|_R = 0.$$

- The the amplitude has the form

$$\rho_{R,\mu}(\psi_{R,\xi}) = \rho_{R,0}(\psi_{R,\xi}) \, e^{\mathrm{i} \int \mathrm{d}^4x \, \mu(x) \hat{\xi}(x)} \, e^{\frac{\mathrm{i}}{2} \int \mathrm{d}^4x \, \mu(x) \gamma(x)}$$

S-matrix on the hypercylinder: Source interaction (II)

- The quantity γ solves the inhomogeneous Klein-Gordon equation

$$(\Box + m^2)\gamma = \mu,$$

with boundary conditions (expressed in momentum space)

$$\gamma_{l,m}(E,r)\Big|_{r>R} = \mathrm{i}\, p \int_0^\infty \mathrm{d}r\, r^2\, a_l(E,r)\, c_l(E,r)\, \mu_{l,m}(E,r). \qquad (1)$$

- The solution is

$$\gamma(x) = \int \mathrm{d}^4x'\, G_F(x,x')\mu(x'),$$

G_F is the Feynman propagator

$\Longrightarrow$ the spatially asymptotic boundary conditions (1) are *equivalent* to the usual temporally asymptotic Feynman boundary conditions.

S-matrix on the hypercylinder: Source interaction (III)

- Amplitude on the hypercylinder

$$\rho_{R,\mu}(\psi_{R,\xi}) = \rho_{R,0}(\psi_{R,\xi}) \cdot \exp\left(\mathrm{i}\int \mathrm{d}^4x\, \mu(x)\hat{\xi}(x)\right) \exp\left(\frac{\mathrm{i}}{2}\int \mathrm{d}^4x\, \mathrm{d}^4x'\, \mu(x) G_F(x,x')\mu(x')\right), \quad (2)$$

where $\hat{\xi}$ is a complex solution of the Klein-Gordon equation parametrized by functions ξ (correspondence between complex solutions and coherent states),

$$\hat{\xi}(t,r,\Omega) := \int_{|E|\geq m} \mathrm{d}E \sum_{l,m} \frac{p}{2\pi} \xi_{l,m}(E) j_l(pr) e^{\mathrm{i}Et} Y_l^{-m}(\Omega).$$

- No explicit dependence on the radius R is present in (2). The limit $R \to \infty$ gives the asymptotic amplitude in the case of a source interaction,

$$\mathcal{S}_\mu(\psi_\xi) = \mathcal{S}_0(\psi_\xi) \exp\left(\mathrm{i}\int \mathrm{d}^4x\, \mu(x)\hat{\xi}(x)\right) \exp\left(\frac{\mathrm{i}}{2}\int \mathrm{d}^4x\, \mathrm{d}^4x'\, \mu(x) G_F(x,x')\mu(x')\right).$$

S-matrix on the hypercylinder: General interaction

- We consider a general interaction vanishing outside a finite spatial region

$$V((t,x),\phi(t,x)) = 0, \quad \text{if } |x| \geq R.$$

- We use functional derivative techniques to work out the amplitude

$$\rho_{R,V}(\psi_R) = \exp\left(\mathrm{i}\int \mathrm{d}^4x\, V\left(x, -\mathrm{i}\frac{\partial}{\partial\mu(x)}\right)\right)\rho_{R,\mu}(\psi_R)\Big|_{\mu=0}.$$

- No dependence on $R \Rightarrow$ the aymptotic amplitude for a general interaction

$$\mathcal{S}_V(\psi) = \exp\left(\mathrm{i}\int \mathrm{d}^4x\, V\left(x, -\mathrm{i}\frac{\partial}{\partial\mu(x)}\right)\right)\mathcal{S}_\mu(\psi)\Big|_{\mu=0}.$$

Equality of asymptotic amplitudes

The asymptotic amplitudes with source interaction in the two settings are very similar:

- standard setting
  $\langle\psi_{\eta_2}|\mathcal{S}_\mu|\psi_{\eta_1}\rangle = \langle\psi_{\eta_2}|\mathcal{S}_0|\psi_{\eta_1}\rangle\, e^{\mathrm{i}\int \mathrm{d}^4x\,\mu(x)\hat{\eta}(x)}\, e^{\frac{\mathrm{i}}{2}\int \mathrm{d}^4x\,\mathrm{d}^4x'\,\mu(x)G_F(x,x')\mu(x')}$,
- hypercylinder
  $\rho_{R,\mu}(\psi_{R,\xi}) = \rho_{R,0}(\psi_{R,\xi})\, e^{\mathrm{i}\int \mathrm{d}^4x\,\mu(x)\hat{\xi}(x)}\, e^{\frac{\mathrm{i}}{2}\int \mathrm{d}^4x\,\mathrm{d}^4x'\,\mu(x)G_F(x,x')\mu(x')}$,
- The same Feynman propagator appears in both expressions
- We can identify asymptotic states at temporal and at spatial infinity,
  $\hat{\xi} = \hat{\eta}$
  $\Rightarrow$ isomorphism of Hilbert spaces $\mathcal{H}_1 \otimes \mathcal{H}_2^* \to \mathcal{H}_R$
- Indeed this isomorphism makes the free amplitudes equals

$$\langle\psi_{\eta_2,t_2}|U_{0,[t_1,t_2]}|\psi_{\eta_1,t_1}\rangle\Big|_{\hat{\eta}=\hat{\xi}} = \rho_{R,0}(\Psi_{R,\xi}). \qquad (3)$$

- Consequently an n-particles states in $\mathcal{H}_1 \otimes \mathcal{H}_2^*$ can be expressed as a linear combination of n-particles states in $\mathcal{H}_R$, and viceversa!

Summary

- General interacting QFTs in Minkowski fit into the GBF
- New representation of the Feynman propagator and the S-matrix using the hypercylinder geometry
- Existence of an isomorphism between the state space associated with two connected spacelike hypersurfaces and the state space associated one connected timelike hypersurface.
- New perspective on QFT: crossing symmetry is implicit in the hypercylinder case (no distinction between in- and out-states)
- The "hypercylinder" quantization scheme has been applied to dS and AdS QFTs.