Discrete models and General Boundary Field Theory

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$2d$-Ising model as a sum over paths

$$
\begin{aligned}
< s(p1)\cdot \ldots \cdot s(pn) > &= \frac{1}{Z}\sum_{s} s_{\nu(p1)}\cdot \ldots \cdot s_{\nu(pn)} \exp\left(K\sum_{ij} s_i s_j\right)\\
&= \frac{1}{Z'} \sum_{\text{paths}:\partial\text{path}=\{\mathrm{p1,\ldots,pn}\}} z^{\text{length(path)}}
\end{aligned}
$$

Sum over mutycomponent paths with ends at measured spins, and $z = \tanh(K)$.

$2d$-Ising model á la GBFT

A factorization of the weight of each path into factors assigned to cells dual to the vertices of the Ising lattice

$$< s(p1) \cdot ... \cdot s(pn) >= \frac{1}{Z'} \#_i R_{J(i)}$$

- Occupied cells are marked as $J(i) = 1$, empty cells as $J(i) = 0$.
- Amplitudes R_0 and R_1 assign weights to boundary states/colorings as follows: $R_0(0,0,0,0) = 1$, $R_0(1,0,0,0) = 0$, $R_0(1,1,0,0) = z$, $R_0(1,1,1,0) = 0$, $R_0(1,1,1,1) = z^2, ...,$ $R_1(0,0,0,0) = 0$, $R_1(1,0,0,0) = z^{1/2}$, $R_1(1,1,0,0) = 0$, $R_1(1,1,1,0) = z^{3/2}$, $R_1(1,1,1,1) = 0, ...$
- Composition/gluing of amplitudes is a sum over the colorings (with 0 or 1) of the bdary links of each cell.
- Multi cell regions are possible.

Spin foam model for QED

euclidian, compact, without fermions; Reisenberger 94, Aroca-Fort-Gambini 96

$$< W_l >= \frac{1}{Z'} \sum_{\text{surfaces:}\partial\text{surface}=l} \exp(-Cn^2 \text{Area(surface)})$$

The sum is over branched colored by integers n.

The weights can be factorized into gluing cell amplitudes with cell types depending on how each cell intersects l

$$< W_l >= \frac{1}{Z'} \#_i R_{J(i)}$$

Objective and interpretation

Approximate, local partial description of a system based on local partial knowledge

- $\mathrm{Available\,Measurements}(\$) \subset \mathrm{Obs}(\mathrm{System})$

In field theory we start with $E \xrightarrow{\pi} M$.

Discrete models may start with $E_\Delta \xrightarrow{\pi} \Delta$.

In E_Δ the location of measurements and their results are recorded; Δ only stores the location of the discrete set of measurements.

As \$ increases more measurements are made, Δ is refined.

Predictions are correlations that need to be corrected

PICTURE

1. Phys. state determined **locally** and **up to some approx.** by: partial knowledge of initial condition and/or bdary conditions
2. Other measurements can be **predicted** up to some error. The error can be corrected increasing $.

Alternative view:
Predictions are correlation functions of available measurements.
The measure predicting such correlations is corrected increasing $.

Wilsonian renormaization:
A sequence of models describing the same system with increasing resolution may correct the predictions up to convergence.

Veselov's discrete time mechanics

Let $q = (q_k)$, $k \in U \subset \mathbb{Z}$, $q_k \in Q$ be a discrete history.
Given the bdary cond. $q|_{\partial U}$ fixed, a motion is an extremum of

$$S(q) = \sum_k L(q_k, q_{k+1}) \quad , \quad \text{with} \quad L : Q_x \times Q_y \to \mathbb{R}$$

$$\begin{aligned} dS(q) \cdot \delta q \;\; &= \;\; \sum_{U^\circ} \Big(\frac{\partial L}{\partial y^i}(q_{k-1}, q_k) + \frac{\partial L}{\partial x^i}(q_k, q_{k+1}) \Big) \delta q^i_k \\ &+ \;\; \frac{\partial L}{\partial x^i}(q_{\partial U^-}, q_{\partial U^- + 1}) \delta q^i_{\partial U^-} + \frac{\partial L}{\partial y^i}(q_{\partial U^+ - 1}, q_{\partial U^+}) \delta q^i_{\partial U^+} \end{aligned}$$

we read the equations of motion and two Lagrange 1-forms

$$\theta^-_L(x,y)(\delta x^i, \delta y^i) = \frac{\partial L}{\partial x^i}(x,y)\delta x^i \, , \quad \theta^+_L(x,y)(\delta x^i, \delta y^i) = \frac{\partial L}{\partial y^i}(x,y)\delta y^i$$

Structure in Veselov's mechanics

- Motions can be param by initial conditions in $Q_x \times Q_y$; there $dS = \theta_L^- + (F^n)^* \theta_L^+$.
  Define $\omega_L = d\theta_L^- = -d\theta_L^+$; then $ddS = 0$ implies
  $$(F^n)^* \omega_L = \omega_L$$
- A G-invariant lagrangian gives $\xi_{Q \times Q} \lrcorner dS = 0 \quad \forall \xi \in \mathrm{Lie}(G)$; thus
  $$(F^n)^* (\xi_{Q \times Q} \lrcorner \theta_L^+) = \xi_{Q \times Q} \lrcorner \theta_L^+$$
- There is a hamiltonian view of the equations of motion in the region where ω_L is not degenerate:
  unit time evolution along X_H for $H(x, y) = -L(x, y) + \int \theta_L^+$

Examples

- For solved systems $L(q_k, q_{k+1}) = S_H(q_k, q_{k+1})$, "perfect".
- Particle in $\mathbb{R}^n$ with $L = T - V$ modeled by

$$L(q_k, q_{k+1}) = (\frac{q_{k+1} - q_k}{\Delta_t} - V(\frac{q_k + q_{k+1}}{2}))\Delta_t\,, \quad \text{"approx."}.$$

- **Veselov's top:** Let $q_k \in SO(n)$ be the top's orientation,

$$L(q_k, q_{k+1}) = \mathrm{tr}(q_k\, I\, q_{k+1}^T)\,, \quad \text{approx. in a sense, but integrable.}$$

On the classical continuum limit

- Continuum histories q_{cont} induce discrete histories $q_{\text{cont}}|_{\$}$.
- In a continuum limit

$$dS_{\text{disc},\$}(q_{\text{cont}}|_{\$}) \cdot \delta q_{\text{cont}}|_{\$} \to dS_{\text{cont}}(q_{\text{cont}}) \cdot \delta q_{\text{cont}}.$$

  This implies convergence of
  (i) the equations of motion,
  (ii) the symplectic structure and
  (iii) the conserved quantities.

Geometric quantization

- $Q_x \times Q_y|_{\omega_L \mathrm{n-deg}}$, is a symplectic manifold preserved by evolution and wave functions of the form $\psi(x)$, $\psi(y)$ both come from a polarization.
  However, gluing neighboring intervals is not always simple.
- A discretization à la Reisenberger confining the potential to centers of the 1-cells solves this problem.
  Gluing is done as mandated by the free theory.
- The propagator is constructed as a product of unit time propagators in the same fashion as the Transfer Matrix.
- A continuum limit **may yield** a definition of the path integral in phase space.

Multisymplectic discrete field theory

A version of the framework of Marsden and collaborators developed using a discretization à la Reisenberger $M \to D(\Delta) \sim (\Delta, \Delta^*)$.
Let $\phi = (\phi_\nu, \phi_\tau)$ be a discrete section/history,
where $\nu \in U^n \subset \Delta^n, \tau \in U^{n-1} \subset \Delta^{n-1}$, $\phi_\nu \in F$, $\phi_\tau \in F$.

Given the bdary cond. $\phi|_{\partial U}$ fixed, a motion is an extremum of

$$S(\phi) = \sum_{c \in D(U)} L(\phi|_c)\,, \quad \text{with} \quad L(\phi|_c) = L(\phi_\nu; \phi_{\tau 1}, ..., \phi_{\tau n})$$

$$\begin{aligned} dS(\phi)\cdot\delta\phi &= \sum_{U^\circ} \left\{ \sum_{c\cap\nu} \frac{\partial L}{\partial \phi^A_\nu}(\phi|_c)\delta\phi^A_\nu + \sum_{c\cap\tau} \frac{\partial L}{\partial \phi^A_\tau}(\phi|_c)\delta\phi^A_\tau \right\} \\ &+ \sum_{\partial U} \frac{\partial L}{\partial \phi^A_\tau}(\phi|_c)\delta\phi^A_\tau \end{aligned}$$

Two types of eqs of motion, Lagrange n-form "integrated" in ∂U:

Structure in multisymplectic discrete field theory

$$\text{1-forms:}\quad \Theta_L^\tau(\phi|_c)\cdot(\phi_\nu^A, \delta\phi_{\tau 1}^A, \ldots, \delta\phi_{\tau n}^A) = \frac{\partial L}{\partial \phi_\tau^A}(\phi|_c)\delta\phi_\tau^A,$$

$$\text{corresponding 2-forms:}\quad \Omega_L^\tau(\phi|_c) = d\Theta_L^\tau(\phi|_c)$$

Multisymplectic formula:
(ϕ solves the eqs of motion and V, W are a 1st variations of it)

$$V \lrcorner W \lrcorner \left(\sum_{\partial U} \pi^* \Omega_L^\tau \right) = 0$$

A G-invariant lagrangian implies a G action that preserves the space of solutions. For any $\xi \in \mathrm{Lie}(G)$, we denote by ξ_q its corresponding 1st variation. Then

$$\sum_{\partial U} \xi_q \lrcorner \Theta_L^\tau(\phi|_c) = 0$$

Etc

- Examples: 1d case = Veselov's discrete mechanics,
  nonlinear PDEs (Marsden et al),
  classical electrodynamics (Mona?), 2d gravity (Mona),
  hamiltonian LGT revisited (?)
- Classical continuum limit follows the same principle.
  Case study of convergence in Marsden et al.
- Geometric quantization à la GBFT (???)
  May arrive to spin foam models or similar.
  USE 1d CASE AS A GUIDE

References

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