Holomorphic quantization

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Outline

Overview

- Quantum theories are most often obtained through a process of **quantization**, starting from a classical theory.
- We have seen how the **Schrödinger-Feynman** quantization scheme fits naturally into the GBF. However, both of its ingredients, the Schrödinger representation and the path integral are in general not well defined.
- It turns out that a version of **geometric quantization** known as **holomorphic quantization** can be adapted to the GBF.
- In the special case of **linear field theory**, i.e., given a **quadratic Lagrangian**, this leads to a rigorous and functorial quantization scheme.

Review of Schrödinger-Feynman quantization

We consider for each hypersurface Σ the associated space K_Σ of field configurations of the classical field theory. In the **Schrödinger representation** the state space $\mathcal{H}_\Sigma$ is the space of **wave functions** on K_Σ with the inner product

$$\langle\psi',\psi\rangle = \int_{K_\Sigma} \psi(\varphi)\overline{\psi'(\varphi)}\,\mathrm{d}\mu_\Sigma(\varphi).$$

But what is the measure μ_Σ? And, how exactly, is K_Σ defined?

For a spacetime region M the **amplitude map** is given by the **Feynman path integral**,

$$\rho_M(\psi) = \int_{K_M} \psi(\phi|_{\partial M})\exp\left(\mathrm{i}S_M(\phi)\right)\,\mathrm{d}\mu_M(\phi).$$

Here S_M is the **action** in M. The integral is over the space K_M of field configurations in the region M. But what is K_M exactly? And what is the measure μ_M?

Lagrangian field theory (I)

Formulate field theory in terms of first order Lagrangian density $\Lambda(\varphi, \partial\varphi, x)$. For a spacetime region M the **action** of a field ϕ is

$$S_M(\phi) := \int_M \Lambda(\phi(\cdot), \partial\phi(\cdot), \cdot).$$

Classical solutions in M are extremal points of this action. These are obtained by setting to zero the first variation of the action,

$$(\mathrm{d}S_M)_\phi(X) = \int_M X^a \left(\frac{\delta\Lambda}{\delta\varphi^a} - \partial_\mu \frac{\delta\Lambda}{\delta\,\partial_\mu\varphi^a}\right)(\phi) + \int_{\partial M} X^a \partial_\mu \lrcorner \frac{\delta\Lambda}{\delta\partial_\mu\varphi^a}(\phi)$$

under the condition that the infinitesimal field X vanishes on ∂M. This yields the **Euler-Lagrange equations**,

$$\left(\frac{\delta\Lambda}{\delta\varphi^a} - \partial_\mu \frac{\delta\Lambda}{\delta\,\partial_\mu\varphi^a}\right)(\phi) = 0.$$

Lagrangian field theory (II)

The boundary term can be defined for an arbitrary hypersurface Σ.

$$(\theta_\Sigma)_\phi(X) = -\int_\Sigma X^a \partial_\mu \lrcorner \frac{\delta\Lambda}{\delta\partial_\mu\varphi^a}(\phi)$$

This 1-form is called the **symplectic potential**. Its exterior derivative is the **symplectic 2-form**,

$$(\omega_\Sigma)_\phi(X,Y) = (\mathrm{d}\theta_\Sigma)_\phi(X,Y) = -\frac{1}{2}\int_\Sigma \Bigg((X^b Y^a - Y^b X^a)\,\partial_\mu \lrcorner \frac{\delta^2\Lambda}{\delta\varphi^b \delta\,\partial_\mu\varphi^a}(\phi) + (Y^a \partial_\nu X^b - X^a \partial_\nu Y^b)\,\partial_\mu \lrcorner \frac{\delta^2\Lambda}{\delta\,\partial_\nu\varphi^b \delta\,\partial_\mu\varphi^a}(\phi) \Bigg).$$

We denote the space of solutions in M by L_M and the space of germs of solutions on a hypersurface Σ by L_Σ.

Lagrangian field theory (III)

Let M be a region and $\phi \in L_{\partial M}$. Then ϕ may or may not be induced from a solution in M. If ϕ arises from a solution in M and X, Y arise from infinitesimal solutions in M, then,

$$(\omega_{\partial M})_\phi(X, Y) = (\mathrm{d}\theta_{\partial M})_\phi(X, Y) = -(\mathrm{d}\mathrm{d}S_M)_\phi(X, Y) = 0.$$

This means, L_M induces an **isotropic** submanifold of $L_{\partial M}$.

It is natural to require that the symplectic form is **non-degenerate**. We are then led to the converse statement: If given X we have $(\omega_{\partial M})_\phi(X, Y) = 0$ for all induced Y, then X itself must be induced. This means, L_M induces a **coisotropic** submanifold of $L_{\partial M}$.

Taking both statements together yields,

L_M induces a **Lagrangian** submanifold of $L_{\partial M}$.

Encoding classical field theory

A classical field theory may be encoded by "general boundary" data:

- For each hypersurface Σ there is a manifold L_Σ (of germs of classical solutions near Σ). L_Σ carries a non-degenerate symplectic form ω_Σ (from Lagrangian field theory).
- For each region M there is a manifold L_M (of classical solutions in M) and an embedding $r_M : L_M \to L_{\partial M}$.
- The submanifold $r_M(L_M) \subseteq L_{\partial M}$ is Lagrangian with respect to $\omega_{\partial M}$.
- These structures are compatible with orientation change, decomposition of hypersurfaces and gluing of regions.

This defines in (various different ways) a **category** of classical field theories.

Geometric quantization: Prequantization

Geometric quantization is designed to output the structures of the **standard formulation** of quantum theory, i.e., a Hilbert space of states and an operator algebra of observables acting on it. Its main input is the space L of solutions of the classical theory in spacetime with its symplectic structure ω. It proceeds roughly in two steps:

1. We consider a hermitian line bundle B over L with a connection ∇ that has curvature 2-form ω. Define the **prequantum** Hilbert space H as the space of square-integrable sections with inner product

$$\langle s', s\rangle = \int (s'(\eta), s(\eta))_\eta \, \mathrm{d}\mu(\eta).$$

   Here the measure $\mathrm{d}\mu$ is given by the $2n$-form $\omega \wedge \cdots \wedge \omega$ if L has dimension $2n$. Classical observables, i.e., functions on L, act naturally as operators on H with the "correct" commutation relations.

Geometric quantization: Polarization

2. This Hilbert space is too large. Choose in each complexified tangent space $(T_\phi L)^{\mathbb{C}}$ a Lagrangian subspace P_ϕ with respect to ω_ϕ. We then restrict H to those sections s of B such that

$$\nabla_{\overline{X}} s = 0,$$

if $X_\phi \in P_\phi$ for all $\phi \in L$. This is called a **polarization**. The subspace $\mathcal{H}$ of H obtained in this way is the Hilbert space of states. Not all observables are well defined on it as they might not leave the subspace $\mathcal{H} \subseteq H$ invariant.

Kähler polarization

We are interested in a **Kähler polarization**. Then P_ϕ is determined by a complex structure J_ϕ in $T_\phi L$ that is compatible with ω_ϕ. J_ϕ satisfies $J_\phi \circ J_\phi = -1$ and $\omega_\phi(J_\phi X, J_\phi Y) = \omega_\phi(X, Y)$. Then

$$P_\phi = \{X \in (T_\phi L)^{\mathbb{C}} : \mathrm{i}X = J_\phi X\}.$$

J_ϕ yields a real inner product on $T_\phi L$:

$$g_\phi(X_\phi, Y_\phi) := 2\omega_\Sigma(X_\phi, J_\phi Y_\phi).$$

We shall require g_ϕ to be positive definite. We also obtain a complex inner product on $T_\phi L$ viewed as a complex vector space:

$$\{X_\phi, Y_\phi\}_\phi := g_\phi(X_\phi, Y_\phi) + 2\mathrm{i}\omega_\phi(X_\phi, Y_\phi).$$

The Hilbert space $\mathcal{H}$ obtained from H through a Kähler polarization is also called the **holomorphic representation**.

Linear field theory

To be able to deal with the field theory case where L is generically infinite-dimensional we restrict ourselves to the simplest setting of linear field theory. That is, we take L to be a real vector space and the symplectic form ω to be invariant under translations in L. Not much is known beyond this setting.

Then, L can be naturally identified with its tangent space. Moreover, the symplectic form ω, the complex structure J, the real and complex inner products g, $\{\cdot,\cdot\}$ all become structures on the vector space L. The line bundle B becomes trivial and its section (the elements of H) can be identified with complex functions on L. For a Kähler polarization the elements of the subspace $\mathcal{H} \subseteq H$ are precisely the **holomorphic** functions on L. Moreover, the inner product formula simplifies,

$$\langle\psi',\psi\rangle = \int \overline{\psi'(\eta)}\psi(\eta)\exp\left(-\frac{1}{2}g(\eta,\eta)\right)\mathrm{d}\mu(\eta).$$

The measure

What is the measure $\mathrm{d}\mu$?

It turns out that on an infinite-dimensional vector space L no translation-invariant measure exists. Instead, we should look for a **Gaussian measure**

$$\mathrm{d}\nu \approx \exp\left(-\frac{1}{2}g(\eta,\eta)\right)\mathrm{d}\mu.$$

However, not even that exists on the Hilbert space L. The measure does exist if we extend L to a larger vector space $\hat{L}$. Concretely ν and $\hat{L}$ can be constructed as an inductive limit of finite-dimensional quotient spaces of L. It turns out that $\hat{L}$ can also be identified with the algebraic dual of the topological dual of L.

A priori, wave functions are thus really functions of $\hat{L}$ rather than on L. But, a function that is square-integrable on $\hat{L}$ and holomorphic is completely determined by its values on L. This allows us to "forget" about $\hat{L}$ to some extent.

Encoding linear (semi)classical field theory

A semiclassical linear field theory is encoded as:

- For each hypersurface Σ there is a real vector space L_Σ (of classical solutions near Σ). L_Σ carries a non-degenerate symplectic form ω_Σ (from Lagrangian field theory). Moreover, L_Σ carries a compatible complex structure J_Σ (for geometric quantization). L_Σ is a real Hilbert space with g_Σ and a complex Hilbert space with $\{\cdot,\cdot\}_\Sigma$.
- For each region M there is a real vector space L_M (of classical solutions in M) and a real linear map $r_M : L_M \to L_{\partial M}$.
- The subspace $r_M(L_M) \subseteq L_{\partial M}$ is Lagrangian with respect to $\omega_{\partial M}$.
- These structures are compatible with orientation change, decomposition of hypersurfaces and gluing of regions.

It follows: $L_{\partial M} = r_M(L_M) \oplus_{\mathbb{R}} J_{\partial M} r_M(L_M)$ is an orthogonal sum.
Define: $u_M : L_{\partial M} \to L_{\partial M}$ as $u_M(r_M(v) + J_{\partial M} r_M(w)) = r_M(v) - J_{\partial M} r_M(w)$.

State spaces

For each hypersurface Σ we define a Hilbert space of states $\mathcal{H}_\Sigma$ by using the geometric quantization prescription. Thus, $\mathcal{H}_\Sigma$ is a space of holomorphic functions on L_Σ with the inner product,

$$\langle \psi', \psi \rangle_\Sigma := \int_{\hat{L}_\Sigma} \overline{\psi'(\phi)} \psi(\phi) \, \mathrm{d}\nu_\Sigma(\phi).$$

Amplitudes

For each region M we define the linear amplitude map $\rho_M : \mathcal{H}_{\partial M} \to \mathbb{C}$ by

$$\rho_M(\psi) := \int_{\hat{L}_M} \psi(r(\phi))\, \mathrm{d}\nu_M(\phi).$$

Here $\hat{L}_M$ is an extension of L_M and ν_M is a Gaussian measure on $\hat{L}_M$, depending on $g_{\partial M}$ that heuristically takes the form

$$\mathrm{d}\nu_M \approx \exp\left(-\frac{1}{4} g_M(\eta, \eta)\right) \mathrm{d}\mu$$

with μ a (fictitious) translation-invariant measure.

It can be shown that this prescription is here equivalent to the Feynman path integral prescription.

Main Result

We obtain a quantum theory in terms of the data of the GBF.

Theorem

With an additional integrability assumption, the GBF core axioms are satisfied by this quantization prescription.

The quantization prescription may be viewed (in various ways) as a **functor** from classical field theories to general boundary quantum field theories.

Coherent States

The Hilbert spaces $\mathcal{H}_\Sigma$ are reproducing kernel Hilbert spaces and contain coherent states of the form

$$K_\xi(\phi) = \exp\left(\frac{1}{2}\{\xi, \phi\}_\Sigma\right)$$

associated to classical solutions $\xi \in L_\Sigma$. They have the reproducing property,

$$\langle K_\xi, \psi \rangle_\Sigma = \psi(\xi),$$

and satisfy the completeness relation

$$\langle \psi', \psi \rangle_\Sigma = \int_{\hat{L}_\Sigma} \langle \psi', K_\xi \rangle_\Sigma \langle K_\xi, \psi \rangle_\Sigma \, \mathrm{d}\nu_\Sigma(\xi).$$

They can be thought of as representing quantum states that approximate specific classical solutions.

Complex conjugation and evolution (I)

Given a region M we recall the map $u_M : L_{\partial M} \to L_{\partial M}$. It has a quantum counter part $U_M : \mathcal{H}_{\partial M} \to \mathcal{H}_{\partial M}$ given by

$$(U_M \psi)(\phi) := \overline{\psi(u_M \phi)}.$$

This is compatible with coherent states in the sense

$$U_M K_\xi = K_{u_M \xi}.$$

The maps u_M and U_M have remarkable properties. They are involutive (i.e, square to the identity), conjugate linear and isometric. In fact, they act like a **complex conjugation** in the classical respectively quantum setting. In particular,

$$\rho_M(U_M(\psi)) = \overline{\rho_M(\psi)}.$$

Complex conjugation and evolution (II)

At the same time the maps u_M and U_M play the role of generalized **evolution maps**. Suppose M is a region with boundary $\partial M = \Sigma_1 \cup \Sigma_2$. The classical dynamics of the theory in M can be described as an evolution between the hypersurfaces Σ_1 and Σ_2 precisely if u_M restricted to $L_{\Sigma_1} \subseteq L_{\partial M} = L_{\Sigma_1} \oplus L_{\Sigma_2}$ has image $L_{\Sigma_2} \subseteq L_{\partial M}$. In this case

$$u_M(\phi_1 + \phi_2) = t^{-1}(\phi_2) + t(\phi_1) \quad \text{where} \quad \phi_1 \in L_{\Sigma_1}, \phi_2 \in L_{\Sigma_2}$$

with $t : L_{\Sigma_1} \to L_{\Sigma_2}$ the classical evolution map.

U_M plays a similar role in the quantum theory. Under the same conditions we have

$$U_M(\psi_1 \otimes \iota(\psi_2)) = T^{-1}\psi_2 \otimes \iota(T\psi_1) \quad \text{where} \quad \psi_1 \in \mathcal{H}_{\Sigma_1}, \psi_2 \in \mathcal{H}_{\overline{\Sigma_2}}.$$

Here $T : \mathcal{H}_{\Sigma_1} \to \mathcal{H}_{\overline{\Sigma_2}}$ is the unitary quantum evolution map with

$$\rho_M(\psi_1 \otimes \iota(\psi_2)) = \langle \psi_2, T\psi_1 \rangle_{\overline{\Sigma_2}}.$$

Amplitude formula

Remarkably, the amplitude of a coherent state can be calculated explicitly. Let $\xi \in L_{\partial M}$. Then,

$$\rho_M(K_\xi) = \exp\left(\frac{1}{4}\{\xi, u_M \xi\}_{\partial M}\right).$$

This has a simple and compelling physical interpretation. It becomes more evident in a slightly different presentation. Set $\xi = \xi^{\mathrm{R}} + J_{\partial M}\xi^{\mathrm{I}} \in r_M(L_M) \oplus_R J_{\partial M} r_M(L_M)$. Let $\tilde{K}_\xi$ denote the normalized coherent state associated with ξ. Then,

$$\rho_M(\tilde{K}_\xi) = \exp\left(-\frac{1}{2} g_{\partial M}(\xi^{\mathrm{I}}, \xi^{\mathrm{I}}) - \frac{\mathrm{i}}{2} g_{\partial M}(\xi^{\mathrm{R}}, \xi^{\mathrm{I}})\right).$$

Klein-Gordon Theory

Spacelike Hypersurfaces

Parametrization of solutions near hypersurface at time t:

$$\phi(t,x) = \int \frac{\mathrm{d}^3 k}{(2\pi)^3 2E} \left(\phi(k) e^{-\mathrm{i}(Et-kx)} + \overline{\phi(k)} e^{\mathrm{i}(Et-kx)}\right),$$

Additional structures:

$$\omega_t(\phi_1,\phi_2) = \frac{\mathrm{i}}{2} \int \frac{\mathrm{d}^3 k}{(2\pi)^3 2E} \left(\phi_2(k)\overline{\phi_1(k)} - \phi_1(k)\overline{\phi_2(k)}\right),$$

$$(J(\phi))(k) = -\mathrm{i}\phi(k),$$

$$g_t(\phi_1,\phi_2) = \int \frac{\mathrm{d}^3 k}{(2\pi)^3 2E} \left(\phi_1(k)\overline{\phi_2(k)} + \phi_2(k)\overline{\phi_1(k)}\right),$$

$$\{\phi_1,\phi_2\}_t = 2\int \frac{\mathrm{d}^3 k}{(2\pi)^3 2E} \phi_1(k)\overline{\phi_2(k)}.$$

Klein-Gordon Theory

Timelike Hypersurfaces I

Parametrize solution near constant x_1 hypersurface,

$$\eta(t,x_1,\tilde{x}) = \int \frac{\mathrm{d}E\,\mathrm{d}^2\tilde{k}}{(2\pi)^3 2k_1}\Big(\eta(E,\tilde{k})f(E,\tilde{k},x_1)e^{-\mathrm{i}(Et-\tilde{k}\tilde{x})} + \overline{\eta(E,\tilde{k})}\,\overline{f(E,\tilde{k},x_1)}e^{\mathrm{i}(Et-\tilde{k}\tilde{x})}\Big),$$

where $\tilde{x} := (x_2,x_3)$, $\tilde{k} := (k_2,k_3)$, $k_1 := \sqrt{|E^2-\tilde{k}^2-m^2|}$ and

$$f(E,\tilde{k},x_1) := \begin{cases} e^{\mathrm{i}k_1x_1} = \cos(k_1x_1) + \mathrm{i}\sin(k_1x_1) & \text{if } E^2-\tilde{k}^2-m^2 > 0 \\ \cosh(k_1x_1) + \mathrm{i}\sinh(k_1x_1) & \text{if } E^2-\tilde{k}^2-m^2 < 0. \end{cases}$$

Two classes of solutions:

- **Propagating waves**: $E^2-\tilde{k}^2-m^2 > 0$, oscillate in space
- **Evanescent waves**: $E^2-\tilde{k}^2-m^2 < 0$, exponential in space

Klein-Gordon Theory

Timelike Hypersurfaces II

Additional structures:

$$\omega_{x_1}(\eta_1,\eta_2) = \frac{\mathrm{i}}{2}\int \frac{\mathrm{d}E\,\mathrm{d}^2\tilde{k}}{(2\pi)^3 2k_1}\left(\eta_2(E,\tilde{k})\overline{\eta_1(E,\tilde{k})} - \overline{\eta_2(E,\tilde{k})}\eta_1(E,\tilde{k})\right),$$

$$(J(\eta))(E,\tilde{k}) = -\mathrm{i}\eta(E,\tilde{k}),$$

$$g_{x_1}(\eta_1,\eta_2) = \int \frac{\mathrm{d}E\,\mathrm{d}^2\tilde{k}}{(2\pi)^3 2k_1}\left(\eta_1(E,\tilde{k})\overline{\eta_2(E,\tilde{k})} + \eta_2(E,\tilde{k})\overline{\eta_1(E,\tilde{k})}\right)$$

$$\{\eta_1,\eta_2\}_{x_1} = 2\int \frac{\mathrm{d}E\,\mathrm{d}^2\tilde{k}}{(2\pi)^3 2k_1}\,\eta_1(E,\tilde{k})\overline{\eta_2(E,\tilde{k})}.$$

Space of solutions decomposes $L_{x_1} = L^{\mathrm{p}}_{x_1} \oplus L^{\mathrm{e}}_{x_1}$.
Hilbert space $\mathcal{H}_{x_1}$ has subspaces $\mathcal{H}^{\mathrm{p}}_{x_1}$ and $\mathcal{H}^{\mathrm{e}}_{x_1}$ such that $\mathcal{H}_{x_1} = \mathcal{H}^{\mathrm{p}}_{x_1} \otimes \mathcal{H}^{\mathrm{e}}_{x_1}$.

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