(2d-)Yang Mills with corners

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Outline

Axioms for hypersurfaces

Recall that a quantum field theory can be described as rule that assigns:

1. A ("state") topological vector space $\mathcal{H}_\Sigma$ associated to oriented **hypersurfaces** Σ
2. Linear ("propagator") maps $\tilde{\rho}_M$ associated to oriented **regions** M

Certain axioms come from conditions or "physical requirements"
As first approach we consider the case $\partial\Sigma = \emptyset$ for every hypersurface Σ

- Duals:
$$\mathcal{H}_{\overline{\Sigma}} = \mathcal{H}^*_\Sigma$$
where $\overline{\Sigma}$ has the inverse orientation of $\overline{\Sigma}$.
$\overline{\overline{\Sigma}} = \sigma$ implies $(\mathcal{H}^*_\Sigma)^* = \mathcal{H}_\Sigma$.

Hilbert spaces is the desirable category to satisfy this conditions. Nevertheless we have seen that the categorical point of view can not be compatible with this choice of objects with morphisms given by regions. Therefore will avoid considering this axioms in complete generality as a functorial assignement from the cobordism category tho vector spaces

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- Tensor products:
$$\mathcal{H}_{\Sigma_1 \sqcup \Sigma_2} = \mathcal{H}_{\Sigma_1} \otimes \mathcal{H}_{\Sigma_2}$$
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- Empty hypersurfaces: $\mathcal{H}_\emptyset = \mathbb{C}$
- A hermitian structure $\langle \cdot, \cdot \rangle_\Sigma$ on $\mathcal{H}_\Sigma$ and conjugate-linear maps $\iota_{\overline{\Sigma}} : \mathcal{H}_{\overline{\Sigma}} \to \mathcal{H}_\Sigma$

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Time evolution

For a region $M = \overline{\Sigma_1} \sqcup \Sigma_2$, the "time evolution" along the region may be described heuristically as the **"amplitude transition"**)

$$\rho_M(\psi) ='' \int_{K_{\partial M}} \psi \cdot Z_M(\varphi) \mathcal{D}\varphi''_{\partial M}$$

with $''Z_M = e^{iS_M(\varphi)''}$
We would like to describe the "unitary" evolution as a vector

$$\tilde{\rho}_M \in \mathcal{H}^*_{\Sigma_1} \otimes \mathcal{H}_{\Sigma_2} = \mathcal{H}_{\partial M},$$

but this is not possible in general in since the linear map

$$\tilde{\rho}_M : dom\tilde{\rho}_M \subset \mathcal{H}_{\Sigma_1} \to \mathcal{H}_{\Sigma_2}$$

is not necessarily bounded. Instead we take the unbounded dual ("amplitude") linear map $\rho_M : \mathcal{H}_{\partial M} \to \mathbb{C}$

Axioms for regions

- For every region M there exists a hermitian unbounded linear functional

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- Gluing: If M is a manifold with $\partial M = \Sigma_1 \sqcup \Sigma \sqcup \overline{\Sigma}$ while $M_1 = \cup_\Sigma M$ obtained by identifying $\overline{\Sigma}$ with Σ, then for the contraction $\mathcal{H}_\Sigma \otimes \mathcal{H}_{\overline{\Sigma}} \to \mathbb{C}$, the linear map ρ_{M_1} can be defined as

$$\begin{tikzcd}
\mathcal{H}_{\Sigma_1} \otimes \mathcal{H}_\Sigma \otimes \mathcal{H}_{\overline{\Sigma}} \arrow[r, equal] \arrow[d] & \mathcal{H}_{\partial M} \arrow[r, "\rho_M"] \arrow[d] & \mathbb{C} \\
\mathcal{H}_{\Sigma_1} \arrow[r, equal] \arrow[u, dashed, hook, bend left] & \mathcal{H}_{\partial M_1} \arrow[ur, dashed, "\rho_{M_1}"'] &
\end{tikzcd}$$

in particular for $\Sigma_1 = \emptyset$, we have that $\partial M_1 = \emptyset$, therefore $\rho_{M_1} \in \mathbb{C}$. Explicitly we define

$$\rho_{M_1}(\cdot) = \sum_{i \in I} \rho_M \left(\cdot \otimes \xi_i \otimes \iota_\Sigma(\xi_i) \right)$$

where $\{\xi_i\}_{i \in I}$ is an orthonormal basis of $\mathcal{H}_\Sigma$

Another gluing axiom

Let $M = M' \sqcup M''$ be a manifold such that $\partial M' = \Sigma' \sqcup \Sigma, \partial M'' = \overline{\Sigma} \sqcup \Sigma''$ and let $M_1 = M' \cup_\Sigma M''$ be the gluing along Σ, then

$$\begin{tikzcd}
\mathcal{H}_{\Sigma'} \otimes \mathcal{H}_{\Sigma} \otimes \mathcal{H}_{\overline{\Sigma}} \otimes \mathcal{H}_{\Sigma''} \arrow[r, equal] \arrow[d, ->>] & \mathcal{H}_{\partial M} \arrow[r, "\rho_M"] \arrow[d, ->>] & \mathbb{C} \\
\mathcal{H}_{\Sigma'} \otimes \mathcal{H}_{\Sigma''} \arrow[r, equal] \arrow[u, hook, dashed, bend left] & \mathcal{H}_{\partial M_1} \arrow[ur, dashed, "\rho_{M_1}"'] &
\end{tikzcd}$$

Explicitly we define

$$\rho_{M_1}(\cdot, \cdot) = \sum_{i \in I} \rho_{M'}\left(\cdot \otimes \xi_i\right) \cdot \rho_{M''}(\iota_\Sigma(\xi_i) \otimes \cdot)$$

Axiom for topological field theories

- For any cylindrical region $\Sigma \times [0,1]$,

  $$\tilde{\rho}_{\Sigma\times[0,1]} = id \in \mathcal{H}_{\overline{\Sigma}} \otimes \mathcal{H}_{\Sigma},$$

  where $\partial(\Sigma \times [0,1]) = \overline{\Sigma} \sqcup \Sigma$. This implies that

  $$\rho_{\Sigma\times[0,1]} : \mathcal{H}_{\Sigma}^{*} \otimes \mathcal{H}_{\Sigma} \to \mathbb{C}$$

  equals the bilinear pairing given by the contraction

  $$\mathcal{H}_{\Sigma}^{*} \otimes \mathcal{H}_{\Sigma} \to \mathbb{C}$$

This axiom is optional for topological theories. It follows from the fact that the diffeomorphism type of a cylinder remains the same for concatenation of two cylinders

Manifolds with corners

In order to consider **smooth structure with corners** on an $n-$dimensional manifold M we need a smooth atlas with corners $\{(x_\nu, U\nu)\}$ where the coordinate are continuous maps

$$x_\nu : U_\nu \subset M \to \mathbb{R}^n_+$$

and where the change of ("curved") coordinates

$$x_\mu \circ x_\nu^{-1} : x_\nu(U_\nu \cap U_\mu) \to x_\mu(U_\nu \cap U_\mu)$$

are restrictions to open of $\mathbb{R}^n_+$ of $\mathbb{R}^n-$diffeomorphisms.
The "corner subset" of the manifolds correspond to the points $p \in M$ such that $x_\nu(p) \in \mathbb{R}^n$ has two or more null coordinates ν.
This "corner subset" has a stratified structure an decomposes as a union of corner manifolds

Presentation of hypersurfaces

Let $\Sigma \subset \partial M$ be a connected component contained in the *topological* boundary of a manifold with corners. We present it as a union of hypersurfaces *with boundary*

$$\Sigma = \Sigma_1 \cup \Sigma_2 \cup \cdots \cup \Sigma_n$$

We suppose that Σ_i, Σ_j may intersect on $(n-2)$-manifolds contained in their boundary, i.e.

$$\Sigma_i \cap \Sigma_j = \partial\Sigma_1 \cap \partial\Sigma_2 \neq \emptyset$$

All this topological manifolds $M, \Sigma_i, \Sigma_i \cap \Sigma_j$, should be considered as manifolds with corners.

Example: 2d-manifolds with corners

When $\dim M = 2$ hypersurfaces Σ_i in the presentation of a component $\Sigma \subset \partial M$ are diffeomorphic to:

1. Segments O ("open strings")
2. Closed circles C ("closed strings")

on the other hand nonempty intersections $\Sigma_i \cap \Sigma_j = \{p\}$ are just "corner" points

Axioms for corners

Besides of the Hilbert spaces $\mathcal{H}_\Sigma$ associated to boundary components of $\Sigma \subset \partial M$, we also consider Hilbert spaces associated to every hypersurfaces Σ_i of the decomposition $\mathcal{H}_{\Sigma_i}$.

- For every decomposition $\Sigma = \Sigma_1 \cup \cdots \cup \Sigma_n$ there exists vector spaces $\mathcal{H}_\Sigma, \mathcal{H}_{\Sigma_i}$ and *surjective* linear maps

  $$\tau : \mathcal{H}_{\Sigma_1} \otimes \cdots \otimes \mathcal{H}_{\Sigma_n} \to \mathcal{H}_\Sigma$$

  For the case without corners τ is an isomorphism.
- τ is compatible with the hermitian structure $\langle \cdot, \cdot \rangle_\Sigma$, i.e. the orthogonal complement of $\ker \tau$ is isomorphic via τ to its image $\mathcal{H}_\Sigma$.
- τ is compatible with the conjugation ι_Σ, i.e. $\tau \circ (\iota_{\Sigma_1} \otimes \cdots \otimes \iota_{\Sigma_n}) = \iota_\Sigma \circ \tau$.

Gluing axiom for corners

- Let $M = M' \sqcup M''$ be a manifold such that we have decompositions $\partial M' = \Sigma' \cup \Sigma, \partial M'' = \overline{\Sigma} \cup \Sigma''$ and let $M_1 = M' \cup_\Sigma M''$ be the gluing along Σ, then

$$
\begin{tikzcd}
\mathcal{H}_{\Sigma'} \otimes \mathcal{H}_{\Sigma} \otimes \mathcal{H}_{\overline{\Sigma}} \otimes \mathcal{H}_{\Sigma''} \arrow[r, two heads, "\tau"] \arrow[d, two heads, shift left] & \mathcal{H}_{\partial M} \arrow[r, "\rho_M"] \arrow[d] & \mathbb{C} \\
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\end{tikzcd}
$$

  Explicitly we define

$$\rho_{M_1} \circ \tau(\cdot, \cdot) = \sum_{i \in I} \rho_{M'} \circ \tau_{\Sigma'}\left(\cdot \otimes \xi_i\right) \rho_{M''} \circ \tau_{\Sigma''}(\iota_\Sigma(\xi_i) \otimes \cdot)$$

- Suppose that for $\partial M = \overline{\Sigma_1} \cup \Sigma_2$ we have that $\rho_M \circ \tau : \mathcal{H}^*_{\Sigma_1} \otimes \mathcal{H}_{\Sigma_2} \to \mathbb{C}$ gives rise to a linear isomorphism $\tilde{\rho}_M : \mathcal{H}_{\Sigma_1} \to \mathcal{H}_{\Sigma_2}$ then $\tilde{\rho}_M$ is unitary.

Axiom for empty regions

For non necessarily topological field theories it is useful to consider an instantaneous time evolution, i.e. we consider **empty regions** with boundary $\Sigma \cup \overline{\Sigma}$ with

$$\rho_\emptyset \circ \tau : \mathcal{H}_\Sigma^* \otimes \mathcal{H}_\Sigma \to \mathbb{C}$$

and we demand that

$$\rho_\emptyset \circ \tau(\iota_\Sigma(\cdot), \cdot) = \langle \cdot, \cdot \rangle_\Sigma$$

We think of $\Sigma, \overline{\Sigma}$ glued along its boundaries

For the two kinds of hypersurfaces in dimension 2, there are two **basic decompositions**:

$$\tau_{OO} : \mathcal{H}_O \otimes \mathcal{H}_O \to \mathcal{H}_O$$

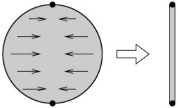
and

$$\tau_{OC} : \mathcal{H}_O \to \mathcal{H}_C$$

And there is just one fundamental region D (cf. the case of tqft where there are 3 fundamental regions: disc, cylinder and pants)

Amplitude for the disc

$$\rho_D : \mathcal{H}_C \to \mathbb{C}$$



Since $C = \partial D$ may be decomposed into two open strings and two open strings and also for the emty region we have this decomposition then, we may be squeeze the disc onto one open string, i.e.

$$\rho_D \circ \tau_{OC} \circ \tau_{OO} : \mathcal{H}_O \otimes \mathcal{H}_O \to \mathbb{C}$$

may be identified with the bilinear mapping

$$\langle \cdot, \cdot \rangle_C = \rho_D \circ \tau_{OC} \circ \tau_{OO} (\iota_C(\cdot), \cdot)$$

Amplitude for the cylinder

We decompose the boundary ∂D into two open strings and then into four and glue opposite open strings. Then, by the gluing axiom

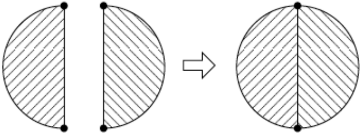
$$\rho_{cylinder} \circ (\tau_{OC} \otimes \tau_{OC}) : \mathcal{H}_O \otimes \mathcal{H}_O \to \mathbb{C}$$

equals

$$\sum_i \rho_D \circ \tau_{OC} \circ \tau_{OO} \circ (\tau_{OO} \otimes \tau_{OO})(\psi \otimes \xi_i \otimes \iota_O(\xi_i) \otimes \eta)$$

Gluing of two discs

We can also consider the gluing of two discs along an open string contained in the boundary



$$\rho_D \circ \tau_{OC} \circ \tau_{OO}(\psi \otimes \eta) = \sum_i \rho_{D_1} \circ \tau_{OC} \circ \tau_{OO}(\psi \otimes \xi_i) \rho_{D_2} \circ \tau_{OC} \circ \tau_{OO}(\iota_O(\xi_i) \otimes \eta)$$

2d Yang Mills

Consider G a compact Lie group and A a connection on a $G-$principal bundle $\pi : P \to M_{\mathrm{g},n}$ over a surface $M_{\mathrm{g},n}$ of genus g and n boundary components. Take the area form $\omega \in \Omega^2(M_{\mathrm{g},n}, \mathbb{R})$

$$S_{YM}(A) = -\int_{M_{\mathrm{g},n}} tr(F^A \wedge *F^A) = -\int_{M_{\mathrm{g},n}} tr(F^2)\omega, \ A \in \Omega^1(M_{\mathrm{g},n}, \mathfrak{g})$$

Then for every area preserving diffeomorphism $f \in Diff_{0,\omega} M_{\mathrm{g},n}$, we have $S_{YM}(f^* A) = S_{YM}(A)$
Hence by a result of Moser $S_{YM}(A)$ depends just on the total area s and the topology of the region $M_{\mathrm{g},n}$. Thus it is useful to adopt the notation

$$\rho^s_{\mathrm{g},n} = \rho_{M_{\mathrm{g},n}}$$

Yang Mills (II)

- A connection A of the bundle $\pi : P\,|_O \to O$ restricted to an open string $O \subset \partial M_{\mathfrak{g},n}$ can be modified by gauge transformations and made locally constant. Therefore gauge class may be completely defined by its holonomy. The holonomy along the open string O, $\exp \int_O A$, may be identified with an element of G and encodes all the information of the connection O modlo gauge

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- Therefore the Hilbert space describing the state space $\mathcal{H}_O$ is the L^2 class functions $\mathcal{C}(G)$, with the inner product

$$\langle \psi, \eta \rangle_O = \int_G \overline{\psi}(g)\eta(g)\, dg$$

where dg is the Haar measure on G.

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- In order to describe the holonomy along a closed string $C \subset M_{g,n}$ we require conjugation classes on G, i.e. $\mathcal{H}_C$ is the completion of the class functions

$$C_{Class}(G) = \{f : G \to \mathbb{C} \mid f(g^{-1} \cdot x \cdot g) = f(x)\}$$

Representation theory

- ON basis: By Peter Weil theorem there exists an ON basis $\{\chi^V\}$ on $C_{Class}(G) \subset L^2(G, dg)$ given by characters $\chi^V(g) = tr(V(g))$ associated to irreducible representations V. Any state in $C_{Class}(G)$ may be written as $\psi = \sum_V \psi^V \chi^V$.
  For $C(G) = L^2(G, dg)$ an orthogonal basis is given by the entries t^V_{ij} of representation matrices. There is a product given by convolution, characters are nilpotent mod coefficients. Recall that $\chi^V = \sum_i t^V_{ii}$
- Inner products:

$$\langle \chi^V, \chi^W \rangle_C = \delta_{V,W},\ \langle t^V_{ij}, t^W_{mn} \rangle_O = \delta_{V,W}\delta_{i,m}\delta_{j,n}\frac{1}{\dim V}$$

- Conjugation:

$$\overline{t^V_{ij}}(g) = t^V_{ji}(g^{-1}),\ \overline{\chi^V} = \chi^V$$

- Notice that the projection $\tau_{OC} : \mathcal{H}_O \to \mathcal{H}_C$ may be written as $\tau_{OC}(\psi)(g) = \int_{h\in[g]} \psi(hgh^{-1})dh$ or with these bases

$$\tau_{OC}(t^V_{ij}) = \delta_{V,W}\delta_{i,j}\frac{1}{(\dim V)^2}\chi^V$$

- Also for the decomposition of one open string into two strings $\tau_{OO} : \mathcal{H}_O \otimes \mathcal{H}_O \to \mathcal{H}_O$

$$\tau_{OO}(t^V_{ij} \otimes t^W_{mn}) = \delta_{V,W}\delta_{j,m}\frac{1}{\dim V}t^V_{in}$$

Amplitude for the disc

With these tools we are now able to give explicit calculations for the amplitudes $\rho^s_{g,n}$,

- For the disc $\rho^s_{0,1} \in \mathcal{H}^*_C$ therefore

$$\rho^s_{0,1} = \sum a_V(s)\chi^V,\ \rho^s_{0,1}(\chi^V) = a_V(s)$$

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- By "shrinking the disc" and making the are area $s = 0$

$$\rho^0_{0,1} \circ \tau_{OC} \circ \tau_{OO}(\iota_O(\cdot), \cdot) = \langle \cdot, \cdot \rangle_O$$

hence

$$\delta_{V,W}\delta_{j,m}\delta_{i,n}\frac{a_V(s)}{(\dim V)^2} = \rho^0_{0,1} \circ \tau_{OC} \circ \tau_{OO}(t^V_{ij} \otimes t^W_{mn}) = \delta_{V,W}\delta_{j,m}\delta_{i,n}\frac{1}{\dim V}$$

here we use the fact that $\iota_O(t^V_{ij}) = t^V_{ji}$ and $\iota_O(\chi^V) = \chi^V$
Therefore $a_V(0) = \dim V$

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- Define $\alpha_V(s)\dim V = a_V(s)$, then $\alpha_V(0) = 1$

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Therefore $a_V(0) = \dim V$
- Define $\alpha_V(s)\dim V = a_V(s)$, then $\alpha_V(0) = 1$
- By the gluing of two discs of areas s_1 and s_2 by taking $\{\dim V\, t^V_{ij}\}$ as a basis of $\mathcal{H}^*_O$ we have
$$\alpha_V(s_1 + s_2) = \alpha(s_2)\alpha(s_2)$$
hence we may write $\alpha(s) = \exp(\beta_V s)$ and the propagator between two open strings may be written as
$$\tilde{\rho}^s_{0,1}(t_{i,j}) = \exp(\beta_V s)t^V_{ij}$$

The cylinder
By the gluing axiom applied to the disc with four boundary open strings:

$$\rho^s_{0,2} \circ (\tau_{OC} \otimes \tau_{OC})(\cdot \otimes \cdot) = \sum_{U,i,j} (\dim U) \rho^s_{0,1} \circ \tau_{OC} \circ \tau_{OO} \circ (\tau_{OO} \otimes \tau_{OO})(\cdot \otimes t^U_{ij} \otimes t^U_{ji} \otimes \cdot)$$

introduce t^V_{ij} and t^W_{ij} then

$$\rho^s_{0,2}(\chi^V \otimes \chi^W) = \delta_{V,W} \exp(\beta_V s)$$

The closed surface
Decompose the disc as an 2g$-$gon and gluing properly the sides

$$\rho^s_{g,0} = \sum_V \exp(\beta_V)(\dim V)^{2-2g} \in \mathbb{C}$$

notice that the sum is formal and that may diverge in some cases
Any surface

$$\rho^s_{g,n}(\chi^{V_1} \otimes \cdots \otimes \chi^{V_n}) = \delta_{V_1,V_2,\ldots,V_n} \exp(\beta_{V_1})(\dim V_1)^{2-2g-n}$$

if constants β_V are real we consider the heat propagator end the euclidian field theory, if they are imaginary the complete calculation requires representation theory to get the right value $iC_V/4$ where C_V is the quadratic Casimir of the representation.

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