Quantum Field Theory with General Boundaries in Anti de Sitter spacetime

Max Dohse

Centro de Ciencias Matematicas
UNAM Campus Morelia

Outline

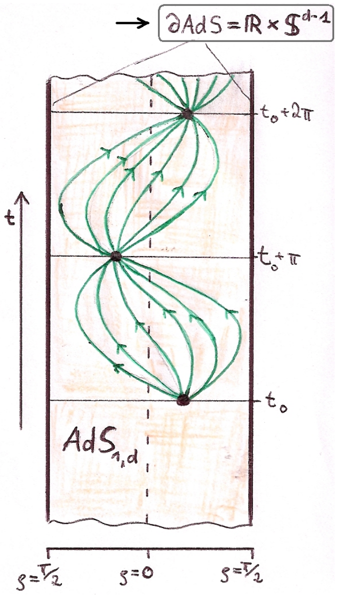
This talk presents some work in progress from my Ph.D. thesis,
done under supervision of Robert Oeckl
and in collaboration with Daniele Colosi
(both CCM-UNAM, Morelia)

1. Motivation
2. Classical Klein-Gordon theory on AdS and Minkowski
3. Schrödinger-Feynman Quantization (SFQ)
4. Holomorphic quantization (HQ)

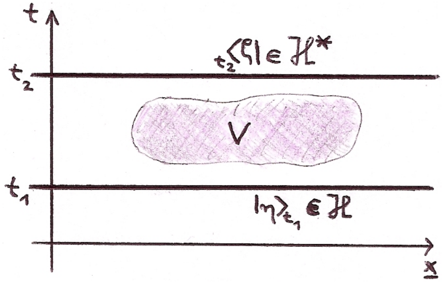
Outline

Anti de Sitter spacetime (AdS)

- constant negative curvature
- global $\mathrm{AdS}_{1,3}$ coordinates:
  time $t \in [-\infty, +\infty]$
  radius $\rho \in [0, \frac{\pi}{2})$
  angles $\Omega = (\theta, \varphi)$ on $\mathbb{S}^2$
- boundary $\partial\mathrm{AdS}$: hypercylinder
  $\mathbb{R} \times \mathbb{S}^2$ at $\rho = \frac{\pi}{2}$ (timelike)
- static metric:
  $\mathrm{d}s^2_{\mathrm{AdS}} = \frac{R^2_{\mathrm{AdS}}}{\cos^2\rho}\left(-\mathrm{d}t^2 + \mathrm{d}\rho^2 + \sin^2\rho\, \mathrm{d}s^2_{\mathbb{S}^2}\right)$
- Penrose diagram
  with timelike geodesics: $\Longrightarrow$
- no (temporally) asymptotically
  free states, no standard $\mathcal{S}$-matrix!



Review: standard $\mathcal{S}$-matrix in Minkowski spacetime



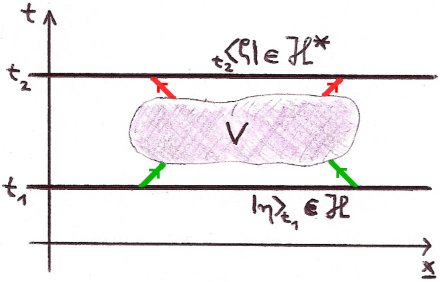
Time-slice region $\mathbb{M}_{[t_1,t_2]}$

- standard QFT in flat spacetime: **one** Hilbert space $\mathcal{H}$ of **free states**
- $\mathcal{S}$-matrix is unitary operator $\mathcal{S}: \ \mathcal{H} \ \to \ \mathcal{H}$ with matrix elements

$$\mathcal{S}_{\eta,\zeta} \ = \ {}_{\text{out}}\langle\,\zeta\,|\,\mathcal{S}\,|\,\eta\,\rangle_{\text{in}}$$
$$\mathcal{S}_{\eta,\zeta} \sim \lim_{t\to\infty} {}_{+t}\langle\,\zeta\,|\,\mathcal{U}_{[-t,+t]}\,|\,\eta\,\rangle_{-t}$$

- usual assumption: interaction switched off for large **times**, states become asymptotically free

Review: standard $\mathcal{S}$-matrix in Minkowski spacetime



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$$\mathcal{S}_{\eta,\zeta} \sim \lim_{t\to\infty} {}_{+t}\langle \zeta | \mathcal{U}_{[-t,+t]} | \eta \rangle_{-t}$$

- usual assumption: interaction switched off for large **times**, states become asymptotically free
- improved assumption: interaction negligible for large **distances**

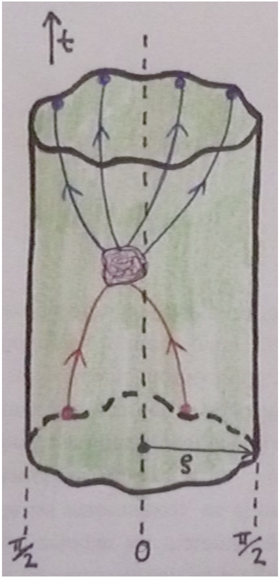
$\Longrightarrow$ **spacetime geometry!**

- Minkowski: large distances for large times (straight geodesics) **AdS: not the case!** (periodically reconverging geodesics)

AdS: hypercylinder

- one solution: use different region!
  natural choice: **rod** hypercylinder region:
  $\mathbb{M}_{\rho_0} = \mathbb{R} \times \mathbb{B}^3_{\rho_0}$
- $ds^2_{\mathrm{AdS}} = \frac{R^2_{\mathrm{AdS}}}{\cos^2\rho}\left(-dt^2 + d\rho^2 + \sin^2\rho \, ds^2_{\mathbb{S}^2}\right)$
  AdS metric causes large distances
  near boundary at $\rho = \frac{\pi}{2}$,
- on hypercylinders $\Sigma_{\rho_0} = \mathbb{R} \times \mathbb{S}^2_{\rho_0}$
  near the boundary $\rho = \frac{\pi}{2}$
  the interaction becomes negligible
  and states become asymptotically free
- How can we construct $\mathcal{S}$-matrix
  for nonstandard regions?

$\Longrightarrow$ **GBF !**



Outline

Minkowski as flat limit of AdS

- scalar curvature of AdS inversely proportional to curvature radius R_{AdS} squared, thus **flat limit** $R_{\mathrm{AdS}} \to \infty$ should give us Minkowski!
  use well known Minkowski results to calibrate corresponding AdS counterparts

Anti de Sitter $\Longrightarrow$ flat limit $\Longrightarrow$ **Minkowski** $(\tau = R_{\mathrm{AdS}} t,\ r = R_{\mathrm{AdS}} \rho)$

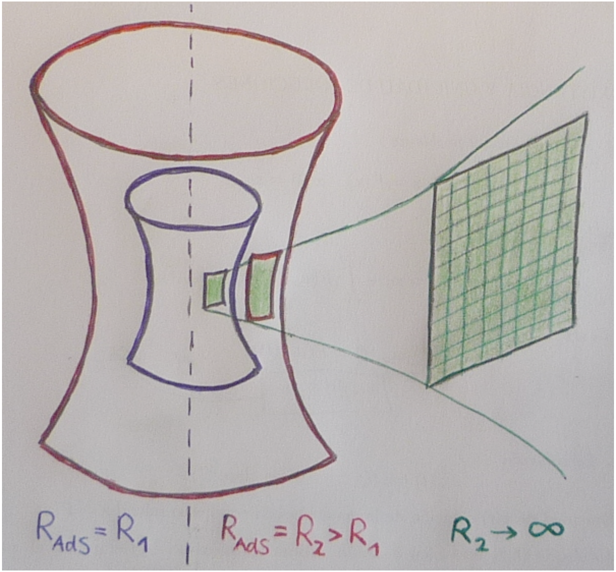
Anti de Sitter

- metric:
  $\mathrm{d}s^2_{\mathrm{AdS}} = \frac{R^2_{\mathrm{AdS}}}{\cos^2\rho}\left(-\mathrm{d}t^2 + \mathrm{d}\rho^2 + \sin^2\rho\,\mathrm{d}s^2_{\mathbb{S}^2}\right)$
- Laplace-Beltrami operator:
  $\square_{\mathrm{AdS}} = R^{-2}_{\mathrm{AdS}} \left\{ -\cos^2\rho\,\partial_t^2 + \tan^{-2}\rho\,\square_{\mathbb{S}^2} + \cos^2\rho\,\partial_\rho^2 + \frac{2}{\tan\rho}\,\partial_\rho \right\}$
- Klein-Gordon equation:
  $(\square_{\mathrm{AdS}} - m^2) = 0$
- 10 Killing vector fields: $(j, k = 1, 2, 3)$
  1 time translation $R^{-1}_{\mathrm{AdS}}\, K_{4,0}$
  3 "4-boosts" $R^{-1}_{\mathrm{AdS}} K_{4,j}$
  3 rotations K_{jk}
  3 "0-boosts" K_{0j}

Minkowski

- metric:
  $\mathrm{d}s^2_{\mathrm{Mink}} = -\mathrm{d}\tau^2 + \mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2_2$
- Laplace-Beltrami:
  $\square_{\mathrm{Mink}} = -\partial_\tau^2 + r^{-2}\,\square_{\mathbb{S}^2} + \partial_r^2 + \frac{2}{r}\,\partial_r$
- Klein-Gordon equation:
  $(\square_{\mathrm{Mink}} - m^2) = 0$
- 10 Killing vector fields:
  1 time translation T_0
  3 spatial translations T_j
  3 rotations K_{jk}
  3 Lorentz boosts K_{0j}

Minkowski as flat limit of AdS

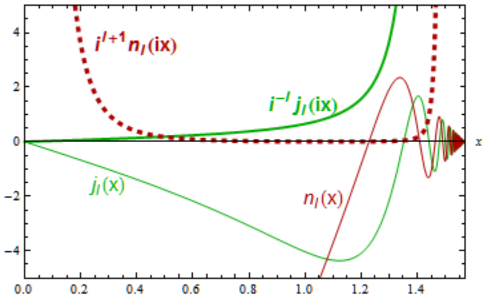


Minkowski: classical Klein-Gordon solutions I

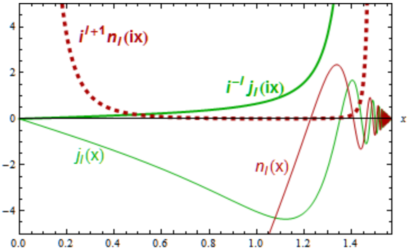
- spherical coordinates: separation of variables gives radial DEQ of 2nd degree, two linear independent solutions $\to$ Bessel modes + Neumann modes
- defining $p_E^{\mathbb{R}} = \sqrt{|E^2 - m^2|}$ the modes write

$$\mu^{(a)}_{Elm_l}(t,r,\Omega) = \frac{p_E^{\mathbb{R}}}{4\pi}\, e^{-iEt} Y_l^{m_l}(\Omega)\, \check{\jmath}_{El}(r) \qquad \check{\jmath}_{El}(r) = \begin{cases} j_l(p_E^{\mathbb{R}} r) & E^2 > m^2 \\ i^{-l}\, j_l(i p_E^{\mathbb{R}} r) & E^2 < m^2 \end{cases}$$

$$\mu^{(b)}_{Elm_l}(t,r,\Omega) = \frac{p_E^{\mathbb{R}}}{4\pi}\, e^{-iEt} Y_l^{m_l}(\Omega)\, \check{n}_{El}(r) \qquad \check{n}_{El}(r) = \begin{cases} n_l(p_E^{\mathbb{R}} r) & E^2 > m^2 \\ i^{l+1}\, n_l(i p_E^{\mathbb{R}} r) & E^2 < m^2 \end{cases}$$



Minkowski: classical Klein-Gordon solutions II



- can expand KG solution on time-slice region in propagating Bessel modes

$$\phi(t,r,\Omega) = \int_{E^2>m^2} \mathrm{d}E \sum_{l,m_l} \left\{ \phi^{+}_{Elm_l} \mu^{(a)}_{Elm_l}(t,r,\Omega) + \overline{\phi^{-}_{Elm_l}}\, \overline{\mu^{(a)}_{Elm_l}(t,r,\Omega)} \right\}$$

- on rod region: propagating+evanescent Bessel modes

$$\phi(t,r,\Omega) = \int_{-\infty}^{+\infty} \mathrm{d}E \sum_{l,m_l} \phi^{a}_{Elm_l} \mu^{(a)}_{Elm_l}(t,r,\Omega)$$

- on neighborhood of hypercylinder Σ_{r_0}: prop.+evan. Bessel + Neumann modes

$$\phi(t,r,\Omega) = \int_{-\infty}^{+\infty} \mathrm{d}E \sum_{l,m_l} \left\{ \phi^{a}_{Elm_l} \mu^{(a)}_{Elm_l}(t,r,\Omega) + \phi^{b}_{Elm_l} \mu^{(b)}_{Elm_l}(t,r,\Omega) \right\}$$

AdS: classical Klein-Gordon solutions I

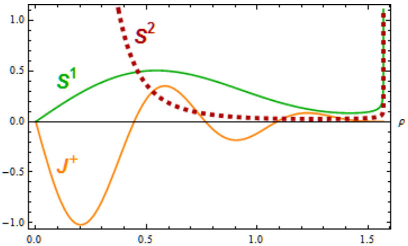
- spherical coordinates: separation of variables gives radial DEQ of 2nd degree, two linear independent solutions → Jacobi + hypergeometric modes
- define **magic frequencies** $\omega^+_{nl} := 2n + l + \tilde{m}_+$, and **mass parameters** $\tilde{m}_\pm = \frac{d}{2} \pm \nu$ wherein $\nu = \sqrt{d^2/4 + m^2 R^2_{\mathrm{AdS}}}$
- the **Jacobi modes** write

$$\mu^{(+)}_{nlm_l}(t,\rho,\Omega) = \mathrm{e}^{-\mathrm{i}\omega^+_{nl} t}\, Y^{m_l}_l(\Omega)\, J^{(+)}_{nl}(\rho) \qquad J^{(+)}_{nl}(\rho) \sim \sin^l\rho\, \cos^{\tilde{m}_+}\rho\, P^{(l+1/2,\nu)}_n(\cos 2\rho)$$

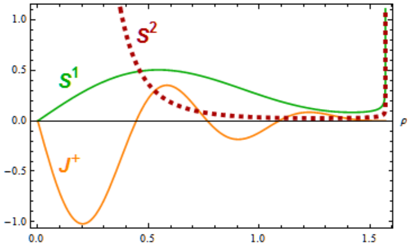
- the **hypergeometric modes** write

$$\mu^{(S,a)}_{\omega l m_l}(t,\rho,\Omega) = \mathrm{e}^{-\mathrm{i}\omega t}\, Y^{m_l}_l(\Omega)\, S^a_{\omega l}(\rho) \qquad S^a_{\omega l}(\rho) = \sin^l\rho\, \cos^{\tilde{m}_+}\rho\, F(\alpha^{S,a},\, \beta^{S,a};\, \gamma^{S,a};\, \sin^2\rho)$$

$$\mu^{(S,b)}_{\omega l m_l}(t,\rho,\Omega) = \mathrm{e}^{-\mathrm{i}\omega t}\, Y^{m_l}_l(\Omega)\, S^b_{\omega l}(\rho) \qquad S^b_{\omega l}(\rho) = \frac{-\cos^{\tilde{m}_+}\rho}{(\sin\rho)^{l+d-2}}\, F(\alpha^{S,b},\, \beta^{S,b};\, \gamma^{S,b};\, \sin^2\rho)$$



AdS: classical Klein-Gordon solutions II



- can expand KG solution on time-slice region in Jacobi modes

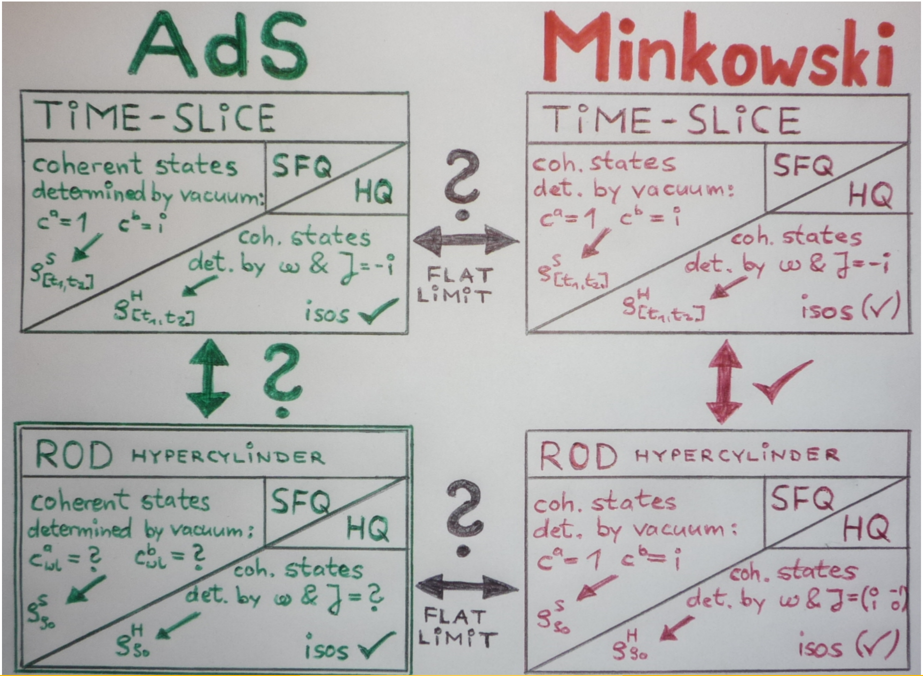
$$\phi(t,\rho,\Omega) = \sum_{nlm_l} \left\{ \phi^+_{nlm_l}\, \mu^{(+)}_{nlm_l}(t,\rho,\Omega) + \overline{\phi^-_{nlm_l}}\, \overline{\mu^{(+)}_{nlm_l}(t,\rho,\Omega)} \right\}$$

- on rod region in hypergeometric a-modes

$$\phi(t,r,\Omega) = \int_{-\infty}^{+\infty} \mathrm{d}\omega \sum_{l,m_l} \phi^a_{\omega l m_l} \mu^{(a)}_{\omega l m_l}(t,r,\Omega)$$

- on neighborhood of hypercylinder Σ_{r_0} in hypergeometric a and b-modes

$$\phi(t,r,\Omega) = \int_{-\infty}^{+\infty} \mathrm{d}\omega \sum_{l,m_l} \left\{ \phi^a_{\omega l m_l} \mu^{(a)}_{\omega l m_l}(t,r,\Omega) + \phi^b_{\omega l m_l} \mu^{(b)}_{\omega l m_l}(t,r,\Omega) \right\}$$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FLAT LIMIT

Outline

SFQ: coherent states [Colosi:2009]

- consider region $\mathbb{M}$ foliated by hypersurfaces Σ_τ wherein τ is foliation parameter
- Dirac picture **coherent states** determined by characterizing function $\eta(\underline{x})$

$$\psi^{\mathrm{D},\eta}_{\Sigma_\tau}(\varphi) = \exp\left(\int \mathrm{d}^3x\; \varphi(\underline{x})\,\left(\overline{\hat{\Upsilon}(\tau)}\right)^{-1}\eta(\underline{x})\right)\,\psi^{\mathrm{S},0}_{\Sigma_\tau}(\varphi)$$

- **vacuum state** given by $\psi^{\mathrm{S},0}_{\Sigma_\tau}(\varphi) = \exp\left\{-\frac{1}{2}\int_{\Sigma_\tau}\mathrm{d}^3x\;\varphi(\underline{x})\;\left(\hat{A}_{\Sigma_\tau}\varphi\right)(\underline{x})\right\}$

  with vacuum operator $\hat{A}_{\Sigma_\tau} = \mathrm{i}\sqrt{\left|(g^{(3)}g^{\tau\tau})(\tau,\underline{x})\right|}\;\dfrac{\overline{(\partial_\tau\hat{\Upsilon})(\tau)}}{\overline{\hat{\Upsilon}(\tau)}}$

- operator $(\hat{\Upsilon}(\tau)\,U_{\underline{k}})(\underline{x}) = (c^a_{\underline{k}}X^a_{\underline{k}}(\tau)+c^b_{\underline{k}}X^b_{\underline{k}}(\tau))\,U_{\underline{k}}(\underline{x})$
  wherein $c^{a,b}_{\underline{k}}$ are factors determining the vacuum and $U_{\underline{k}}(\underline{x})$ is an ONB on Σ_τ
  Minkowski equal-time plane: $U_{Elm_l}(r,\Omega) = j_l(p_E r)Y^{m_l}_l(\Omega)$ and $\Upsilon_E(t) = \mathrm{e}^{-\mathrm{i}Et}$
  Minkowski hypercylinder: $U_{Elm_l}(t,\Omega) = \mathrm{e}^{-\mathrm{i}Et}Y^{m_l}_l(\Omega)$ and $\Upsilon_{El}(r) = \check{j}_{El}(r)+\mathrm{i}\check{n}_{El}(r)$
  AdS hypercylinder: $U_{Elm_l}(t,\Omega) = \mathrm{e}^{-\mathrm{i}Et}Y^{m_l}_l(\Omega)$ and $\Upsilon_{\omega l}(\rho) = ???\, S^a_{\omega l}(\rho) + ???\, S^b_{\omega l}(\rho)$

SFQ: amplitudes for time-slice $\mathbb{M}_{[t_1,t_2]}$ [ColDo:2010]

- free amplitude with $\hat{\mathcal{B}} = \left(2\left|\hat{\Upsilon}(\tau)\right|^2 \hat{A}^{\mathbb{R}}_{\Sigma_\tau}\right)^{-1}$ is independent of $\tau_{1,2}$

$$\rho^{\mathrm{S},0}_{[\tau_1,\tau_2]}\left(\psi^{\mathrm{D},\eta}_{\Sigma_{\tau_1}} \otimes \overline{\psi^{\mathrm{D},\zeta}_{\Sigma_{\tau_2}}}\right) = \exp \int \mathrm{d}^3x \left(\eta\,\hat{\mathcal{B}}\,\overline{\zeta} - \tfrac{1}{2}\overline{\eta}\,\hat{\mathcal{B}}\,\eta - \tfrac{1}{2}\overline{\zeta}\,\hat{\mathcal{B}}\,\zeta\right)$$

- amplitude with source field $\mu(x)$, Feynman propagator G_{F}

$$\rho^{\mathrm{S},\mu}_{[\tau_1,\tau_2]}\left(\psi^{\mathrm{D},\eta}_{\Sigma_{\tau_1}} \otimes \overline{\psi^{\mathrm{D},\zeta}_{\Sigma_{\tau_2}}}\right) = \rho^{\mathrm{S},0}_{[\tau_1,\tau_2]}\left(\psi^{\mathrm{D},\eta}_{\Sigma_{\tau_1}} \otimes \overline{\psi^{\mathrm{D},\zeta}_{\Sigma_{\tau_2}}}\right) \exp\left(\mathrm{i} \int\limits_{\mathbb{M}_{[\tau_1,\tau_2]}} \mathrm{d}^4x\, \sqrt{|g|}\, \mu(x)\, \phi^{(\eta,\zeta)}(x)\right)$$

$$\exp\left(\frac{\mathrm{i}}{2} \int\limits_{\mathbb{M}_{[\tau_1,\tau_2]}} \mathrm{d}^4x \int\limits_{\mathbb{M}_{[\tau_1,\tau_2]}} \mathrm{d}^4x'\, \sqrt{|g(x)g(x')|}\, \mu(x)\, G_{\mathrm{F}}(x,x')\, \mu(x')\right)$$

$\Longrightarrow$ amplitude with source field is independent of $\tau_{1,2}$, too!

SFQ: amplitudes for rod hypercylinder $\mathbb{M}_{r_0}$ [ColDo:2010]

- free amplitude with $\hat{\mathcal{B}} = \left(2 \left|\hat{\Upsilon}(r)\right|^2 \hat{A}^{\mathbb{R}}_{\Sigma_r}\right)^{-1}$ is independent of r_0

$$\rho^{\mathrm{S},0}_{\Sigma_{r_0}}\left(\overline{\psi^{\mathrm{D},\xi}_{\Sigma_{r_0}}}\right) = \exp\left(-\frac{1}{2}\int \mathrm{d}t\, \mathrm{d}^2\Omega\, \left\{\overline{\xi}\, \frac{\overline{\hat{c}^b}}{\hat{c}^b}\, \hat{\mathcal{B}}\, \overline{\xi} + \overline{\xi}\, \hat{\mathcal{B}}\, \xi\right\}\right)$$

- amplitude with source field $\mu(x)$, Feynman propagator G_{F}

$$\rho^{\mathrm{S},\mu}_{r_0}\left(\overline{\psi^{\mathrm{S},\xi}_{\Sigma_{r_0}}}\right) = \rho^{\mathrm{S},0}_{r_0}\left(\overline{\psi^{\mathrm{D},\xi}_{\Sigma_{r_0}}}\right)\, \exp\left(\mathrm{i} \int_{\mathbb{M}_{r_0}} \mathrm{d}^4x\, \sqrt{|g|}\, \mu(x)\, \phi^{(\xi)}(x)\right)$$

$$\exp\left(\frac{\mathrm{i}}{2} \int_{\mathbb{M}_{r_0}} \mathrm{d}^4x \int_{\mathbb{M}_{r_0}} \mathrm{d}^4x'\, \sqrt{|g(x)g(x')|}\, \mu(x)\, G_{\mathrm{F}}(x,x')\, \mu(x')\right)$$

$\Longrightarrow$ amplitude with source field is independent of r_0, too!

SFQ: interacting theory [ColOe:2008]

- action for general field interaction with potential V:

$$S_{R,V}(\phi) = S_{R,0}(\phi) + \int\limits_{\rho<R} \mathrm{d}^4 x \, V(x, \phi(x))$$

$$\exp \mathrm{i} S_{R,V}(\phi) = \left[\exp \mathrm{i} \int\limits_{\rho<R} \mathrm{d}^{d+1} x \, \sqrt{|g(x)|} \, \hat{V}\left(x, -\mathrm{i} \frac{\delta}{\delta\mu(x)}\right) \right] \exp \mathrm{i} S_{R,\mu}(\phi) \Big|_{\mu=0}$$

- amplitude:

$$\rho_{R,V}(\psi) = \left[\exp \mathrm{i} \int\limits_{\rho<R} \mathrm{d}^4 x \, \sqrt{|g(x)|} \, \hat{V}\left(x, -\mathrm{i} \frac{\delta}{\delta\mu(x)}\right) \right] \rho_{R,\mu}(\psi) \Big|_{\mu=0}$$

$\Longrightarrow$ **amplitude again independent of hypercylinder's radius R**

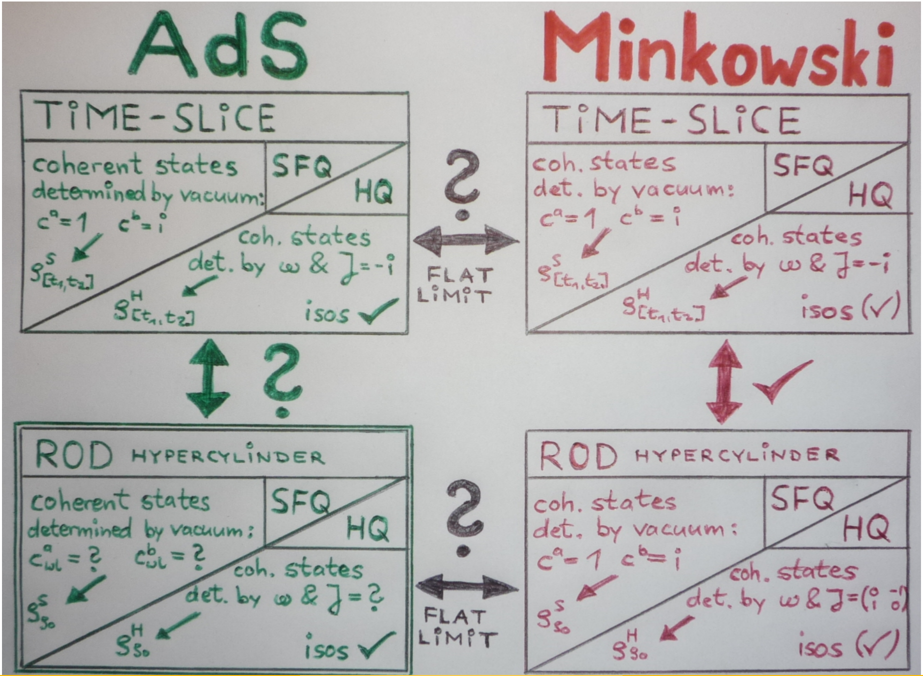
SFQ: rod-slice correspondence [ColOe:2008]

- let rod and time-slice cover all of spacetime
- exponentials quadratic in μ agree in $\rho^{\mathrm{S},\mu}_{[t_1,t_2]}$ and $\rho^{\mathrm{S},\mu}_{r_0}$
- exponentials with coupling of μ and special KG solution agree if $\phi^{(\xi)} = \phi^{(\eta,\zeta)}$
- Minkowski: this induces relation $\xi \Leftrightarrow (\eta,\zeta)$ such that free amplitudes agree:

$$\rho^{\mathrm{S},0}_{[\tau_1,\tau_2]}\left(\psi^{\mathrm{D},\eta}_{\Sigma_{\tau_1}} \otimes \overline{\psi^{\mathrm{D},\zeta}_{\Sigma_{\tau_2}}}\right) = \rho^{\mathrm{S},0}_{\Sigma_{r_0}}\left(\overline{\psi^{\mathrm{D},\xi}_{\Sigma_{r_0}}}\right)$$

- thus in Minkowski rod and time-slice amplitudes with source are equivalent!

$\Longrightarrow$ **Can we contruct the same for AdS?**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FLAT LIMIT

Outline

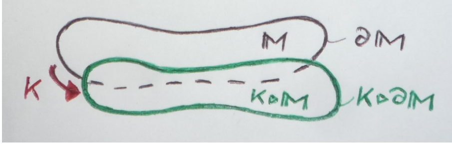
Holomorphic quantization [Oeckl:2012]

- associate to any hypersurface Σ space L_Σ of solutions near Σ
- symplectic structure: $\omega_\Sigma : \; L_\Sigma \times L_\Sigma \; \to \; \mathbb{R}$
- complex structure: $J_\Sigma : \; L_\Sigma \; \to \; L_\Sigma$ with $J_\Sigma^2 = -\mathbb{1}$ and $\omega_\Sigma(\cdot,\cdot) = \omega_\Sigma(J_\Sigma \cdot, J_\Sigma \cdot)$
- field metric: $g_\Sigma(\cdot,\cdot) \; = \; 2\omega_\Sigma(\cdot, J_\Sigma \cdot)$
- inner product: $\{\cdot,\cdot\}_\Sigma \; = \; g_\Sigma(\cdot,\cdot) + 2i\,\omega_\Sigma(\cdot,\cdot)$
- states are holomorphic function(al)s: $\psi^{H}_\Sigma : \; L_\Sigma \; \to \; \mathbb{C}$
- coherent states determined by characteristic solution $\phi \in L_\Sigma$ via $\psi^{H,\phi}_\Sigma(\lambda) \; = \; \exp \tfrac{1}{2} \{\phi, \lambda\}_\Sigma$
- amplitude for region M with boundary ∂M (rigorous path integral) $\rho^{H,0}_M(\psi^{H,\phi}_{\partial M}) = \exp\left(-\tfrac{i}{2} g_{\partial M}(\phi^{\mathbb{R}}, \phi^{\mathbb{I}}) - \tfrac{1}{2} g_{\partial M}(\phi^{\mathbb{I}}, \phi^{\mathbb{I}})\right)$

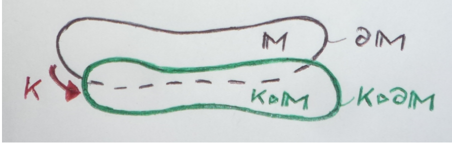
Invariance under isometry actions

- isometry K:
  $K: \quad \mathrm{M} \to K \triangleright \mathrm{M}$
  $K: \quad \partial\mathrm{M} \to K \triangleright \partial\mathrm{M} = \partial(K \triangleright \mathrm{M})$
- isometry invariance of amplitude requires two properties:

1. symplectic structure K-invariant:
   $\omega_{K\triangleright\partial\mathrm{M}}(K \triangleright \lambda, K \triangleright \phi) \stackrel{!}{=} \omega_{\partial\mathrm{M}}(\lambda, \phi)$
2. complex structure commutes with K:
   $\mathrm{J}_{K\triangleright\partial\mathrm{M}}\,(K \triangleright \lambda) \stackrel{!}{=} K \triangleright (\mathrm{J}_{\partial\mathrm{M}}\,\lambda)$
   for all $\lambda, \phi \in \mathrm{L}_{\partial\mathrm{M}}$



Invariance under isometry actions



- isometry K:
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- isometry invariance of amplitude requires two properties:

1. symplectic structure K-invariant:
   $\omega_{K\triangleright\partial\mathrm{M}}(K \triangleright \lambda, K \triangleright \phi) \overset{!}{=} \omega_{\partial\mathrm{M}}(\lambda, \phi)$
2. complex structure commutes with K:
   $\mathrm{J}_{K\triangleright\partial\mathrm{M}}\,(K \triangleright \lambda) \overset{!}{=} K \triangleright (\mathrm{J}_{\partial\mathrm{M}}\,\lambda)$
   for all $\lambda, \phi \in \mathrm{L}_{\partial\mathrm{M}}$

- then we have:

$$
\begin{aligned}
&\mathrm{g}_{K\triangleright\partial\mathrm{M}}(K \triangleright \lambda, K \triangleright \lambda) \\
&= \omega_{K\triangleright\partial\mathrm{M}}\big(K \triangleright \lambda, \mathrm{J}_{K\triangleright\partial\mathrm{M}}(K \triangleright \lambda)\big) \\
&= \omega_{K\triangleright\partial\mathrm{M}}\big(K \triangleright \lambda, K \triangleright (\mathrm{J}_{\partial\mathrm{M}}\lambda)\big) \\
&= \omega_{\partial\mathrm{M}}(\lambda, \mathrm{J}_{\partial\mathrm{M}}\lambda) \\
&= \mathrm{g}_{\partial\mathrm{M}}(\lambda, \lambda)
\end{aligned}
$$

Rod hypercylinder $\mathbb{M}_{\rho_0}$ in AdS: Symplectic structure [Dohse:2013]

- boundary $\partial\mathbb{M}_{\rho_0}$ is hypercylinder Σ_{ρ_0}, KG solutions near boundary: $\mathrm{L}_{\Sigma_{\rho_0}}$

$$\phi(t,r,\Omega) = \int\!\mathrm{d}\omega \sum_{l,m_l} \left\{ \phi^{S,a}_{\omega l m_l}\, \mathrm{e}^{-\mathrm{i}\omega t}\, Y^{m_l}_l(\Omega)\, S^a_{\omega l}(\rho) + \phi^{S,b}_{\omega l m_l}\, \mathrm{e}^{-\mathrm{i}\omega t}\, Y^{m_l}_l(\Omega)\, S^b_{\omega l}(\rho) \right\}$$

- symplectic structure induced by Lagrange density turns out to be:

$$\omega_{\Sigma_\rho}(\eta,\zeta) = \tfrac{1}{2}\int\!\mathrm{d}t\, \mathrm{d}^2\Omega\; R^2_{\mathrm{AdS}} \tan^2\rho\; \big(\eta\,\partial_\rho\zeta - \zeta\,\partial_\rho\eta\big)$$

$$= \pi R^2_{\mathrm{AdS}} \int\!\mathrm{d}\omega \sum_{l,m_l} (2l+1) \left\{ \eta^{S,a}_{\omega l m_l}\, \zeta^{S,b}_{-\omega,l,-m_l} - \eta^{S,b}_{\omega l m_l}\, \zeta^{S,a}_{-\omega,l,-m_l} \right\}$$

- isometry actions: $\big(K \triangleright \omega\big)(\eta,\zeta) = \omega(K^{-1}\triangleright\eta,\, K^{-1}\triangleright\zeta)$ with $(K^{-1}\triangleright\eta)(x) = \eta(Kx)$
- to show isometry invariance of ω, we translate action of K on coordinates into action in solution space: $K:\ \eta^{S,a}_{\omega l m_l} \to (K\triangleright\eta)^{S,a}_{\omega l m_l}$ which gives

$$\big(K\triangleright\omega_{\Sigma_\rho}\big)(\eta,\zeta) = \pi R^2_{\mathrm{AdS}} \int\!\mathrm{d}\omega \sum_{l,m_l} (2l+1) \Big\{ (K\triangleright\eta)^{S,a}_{\omega l m_l}\, (K\triangleright\zeta)^{S,b}_{-\omega,l,-m_l} - (K\triangleright\eta)^{S,b}_{\omega l m_l}\, (K\triangleright\zeta)^{S,a}_{-\omega,l,-m_l} \Big\}$$

$$= \omega_{\Sigma_\rho}(\eta,\zeta) \text{ for all isometries of AdS}$$

Rod hypercylinder $\mathbb{M}_{\rho_0}$ in AdS: Complex structure J

- KG solutions $\mathrm{L}_{\Sigma_{\rho_0}}$

$$\phi(t,r,\Omega) \;=\; \int\!\mathrm{d}\omega \sum_{l,m_l} \left\{ \phi^{S,a}_{\omega l m_l}\, \mathrm{e}^{-\mathrm{i}\omega t}\, Y_l^{m_l}(\Omega)\, S^a_{\omega l}(\rho) + \phi^{S,b}_{\omega l m_l}\, \mathrm{e}^{-\mathrm{i}\omega t}\, Y_l^{m_l}(\Omega)\, S^b_{\omega l}(\rho) \right\}$$

$$\big(\mathrm{J}\phi\big)(t,r,\Omega) \;=\; \int\!\mathrm{d}\omega \sum_{l,m_l} \left\{ \big(\mathrm{J}\phi\big)^{S,a}_{\omega l m_l}\, \mu^{(S,a)}_{\omega l m_l}(t,\rho,\Omega) + \big(\mathrm{J}\phi\big)^{S,b}_{\omega l m_l}\, \mu^{(S,b)}_{\omega l m_l}(t,\rho,\Omega) \right\}$$

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$$\big(\mathrm{J}\phi\big)(t,r,\Omega) \;=\; \int\!\mathrm{d}\omega \sum_{l,m_l} \left\{ \big(\mathrm{J}\phi\big)^{S,a}_{\omega l m_l}\, \mu^{(S,a)}_{\omega l m_l}(t,\rho,\Omega) + \big(\mathrm{J}\phi\big)^{S,b}_{\omega l m_l}\, \mu^{(S,b)}_{\omega l m_l}(t,\rho,\Omega) \right\}$$

- most general ansatz for action of J:

$$\big(\mathrm{J}\phi\big)^{S,a}_{\omega l m_l} \;=\; \int\!\mathrm{d}\omega' \sum_{l',m_l'} \Bigg\{ j^{S,aa}\!\begin{pmatrix} \omega & l & m_l \\ \omega' & \underline{l}' & m_l' \end{pmatrix} \phi^{S,a}_{\omega' l' m_l'} + j^{S,ab}\!\begin{pmatrix} \omega & l & m_l \\ \omega' & \underline{l}' & m_l' \end{pmatrix} \phi^{S,b}_{\omega' l' m_l'}$$
$$+\, \tilde{j}^{S,aa}\!\begin{pmatrix} \omega & l & m_l \\ \omega' & \underline{l}' & m_l' \end{pmatrix} \overline{\phi^{S,a}_{\omega' l' m_l'}} + \tilde{j}^{S,ab}\!\begin{pmatrix} \omega & l & m_l \\ \omega' & \underline{l}' & m_l' \end{pmatrix} \overline{\phi^{S,b}_{\omega' l' m_l'}} \Bigg\}$$

$$\big(\mathrm{J}\phi\big)^{S,b}_{\omega l m_l} \;=\; \int\!\mathrm{d}\omega' \sum_{l',m_l'} \Bigg\{ j^{S,ba}\!\begin{pmatrix} \omega & l & m_l \\ \omega' & \underline{l}' & m_l' \end{pmatrix} \phi^{S,a}_{\omega' l' m_l'} + j^{S,bb}\!\begin{pmatrix} \omega & l & m_l \\ \omega' & \underline{l}' & m_l' \end{pmatrix} \phi^{S,b}_{\omega' l' m_l'}$$
$$+\, \tilde{j}^{S,ba}\!\begin{pmatrix} \omega & l & m_l \\ \omega' & \underline{l}' & m_l' \end{pmatrix} \overline{\phi^{S,a}_{\omega' l' m_l'}} + \tilde{j}^{S,bb}\!\begin{pmatrix} \omega & l & m_l \\ \omega' & \underline{l}' & m_l' \end{pmatrix} \overline{\phi^{S,b}_{\omega' l' m_l'}} \Bigg\}$$

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- KG solutions $\mathrm{L}_{\Sigma_{\rho_0}}$

$$\phi(t,r,\Omega) = \int\!\mathrm{d}\omega \sum_{l,m_l} \left\{ \phi^{S,a}_{\omega l m_l}\, \mathrm{e}^{-\mathrm{i}\omega t}\, Y_l^{m_l}(\Omega)\, S^a_{\omega l}(\rho) + \phi^{S,b}_{\omega l m_l}\, \mathrm{e}^{-\mathrm{i}\omega t}\, Y_l^{m_l}(\Omega)\, S^b_{\omega l}(\rho) \right\}$$

$$\big(\mathrm{J}\phi\big)(t,r,\Omega) = \int\!\mathrm{d}\omega \sum_{l,m_l} \left\{ \big(\mathrm{J}\phi\big)^{S,a}_{\omega l m_l}\, \mu^{(S,a)}_{\omega l m_l}(t,\rho,\Omega) + \big(\mathrm{J}\phi\big)^{S,b}_{\omega l m_l}\, \mu^{(S,b)}_{\omega l m_l}(t,\rho,\Omega) \right\}$$

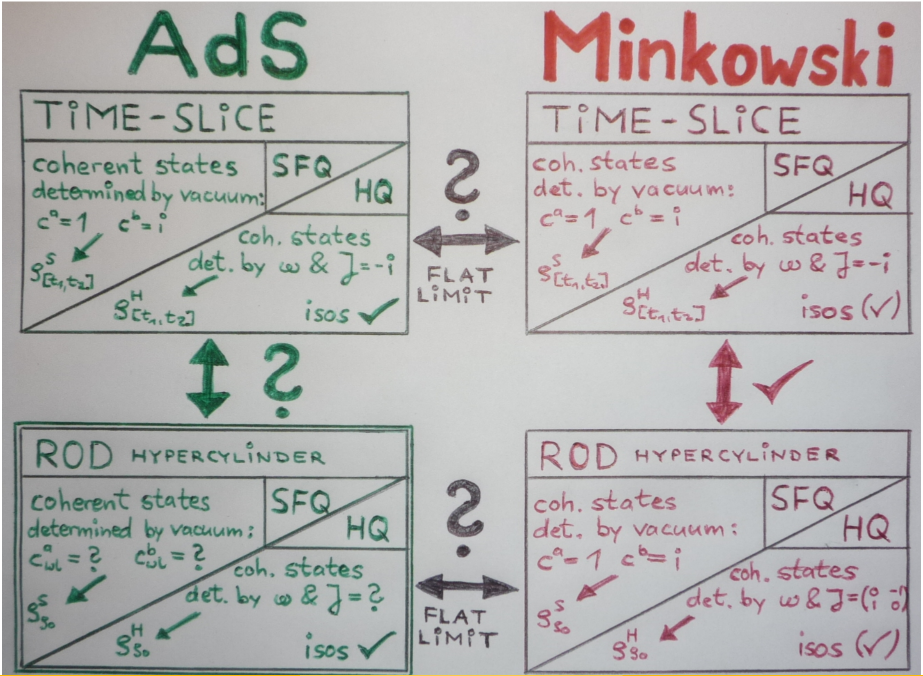
- $[J,K]=0$ with $J^2=-1$ and $\omega_\Sigma(\cdot,\cdot)=\omega_\Sigma(\mathrm{J}_\Sigma\,\cdot,\, \mathrm{J}_\Sigma\,\cdot)$ imply

$$\big(\mathrm{J}\phi\big)^{S,a}_{\omega \underline{l} m_l} = j^S_{\omega l}\, \phi^{S,b}_{\omega \underline{l} m_l} \qquad\qquad \big(\mathrm{J}\phi\big)^{S,b}_{\omega \underline{l} m_l} = -(j^S_{\omega l})^{-1}\, \phi^{S,a}_{\omega \underline{l} m_l}$$

wherein $j^S_{\omega l}$ must fulfill

$$j^S_{\omega-1,l+1} = -j^S_{\omega l}\, \frac{(\widetilde{m}_+ + \omega - l - 3)(\widetilde{m}_+ - \omega + l)}{(2l+3)\,(2l+1)} \qquad j^{S,ab}_{\omega+1,l+1} = -j^S_{\omega l}\, \frac{(\widetilde{m}_+ - \omega - l - 3)(\widetilde{m}_+ + \omega + l)}{(2l+3)\,(2l+1)}$$

candidate: $j^S_{\omega l} = (-1)^l \dfrac{\Gamma(\alpha^{S,a})\,\Gamma(\beta^{S,a})}{\Gamma(\alpha^{S,b})\,\Gamma(\beta^{S,b})\;\Gamma(\gamma^{S,a})\,\Gamma(\gamma^{S,a}-1)}$ we have many more...


AdS
Minkowski
TIME-SLICE
coherent states determined by vacuum:
SFQ
HQ
coh. states det. by ω & J=-i
isos ✓
FLAT LIMIT
TIME-SLICE
coh. states det. by vacuum:
SFQ
HQ
coh. states det. by ω & J=-i
isos (✓)
ROD HYPERCYLINDER
coherent states determined by vacuum:
SFQ
HQ
coh. states det. by ω & J=?
isos ✓
FLAT LIMIT
ROD HYPERCYLINDER
coh. states det. by vacuum:
SFQ
HQ
coh. states det. by ω & J=
isos (✓)

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