Non-relativistic quantum theory in the GBF

Asieh Karami

Centro de Ciencias Matematicas UNAM, Morelia

Seminar General Boundary Formulation
May 30, 2013

Reference

This talk is based on section 3 and 4 of

> Oeckl, Robert. "A “general boundary” formulation for quantum mechanics and quantum gravity", Phys. Lett. B. 575.3 (2003): 318-324, [arxiv:hep-th/0306025].

Outline

Holographic quantization

Consider a classical field theory with a given set of fields $\phi(x)$ and an action $\mathcal{S}[\phi]$. Let K_S be the space of field configurations on a hypersurface S as a boundary of a region M. The amount of boundary data encoded in K_S should be such that it essentially uniquely determines a classical solution inside M in a generic situation.

(Q1) The space of states $\mathcal{H}_S$ associated with S is the space of complex valued functions $C(K_S)$ on K_S.

(Q2) The amplitude ρ_M for a state $\psi \in \mathcal{H}_s$ is given by the expression

$$\rho_M(\psi) = \int_{K_S} \mathcal{D}\phi_0 \, \psi(\phi_0) \int_{\phi|_S=\phi_0} \mathcal{D}\phi \, e^{\frac{i}{\hbar}\mathcal{S}[\phi]}.$$

From (Q1)

- There is a vector space $\mathcal{H}_S$ of states associated with S.
- $S = S_1 \cup S_2$ a disjoint union, $K_S = K_{S_1} \times K_{S_2}$ and hence $C(K_S) = C(K_{S_1}) \otimes C(K_{S_2})$.

From (Q2)

- The dualization of boundaries corresponds simply to leaving the evaluation with a state on those boundaries open. Let M have boundaries S_1 and S_2 and consider states $\psi_1 \in \mathcal{H}_{S_1}$ and $\psi_2 \in \mathcal{H}_{S_2}$. Then $\rho_M(\psi_1)$ is an element of $\mathcal{H}^*_{S_2}$, i.e. a linear map $\mathcal{H}_{S_2} \to \mathbb{C}$ by mapping ψ_2 to

$$\int_{K_{S_1} \times K_{S_2}} \mathcal{D}\phi_1 \mathcal{D}\phi_2 \, \psi_1(\phi_1)\psi_2(\phi_2) \int_{\substack{\phi|_{S_1}=\phi_1 \\ \phi|_{S_2}=\phi_2}} \mathcal{D}\phi \, e^{\frac{i}{\hbar}\mathcal{S}[\phi]}.$$

- The composition property is also rather obvious: Consider an integral over all field configurations in two regions with fields fixed on a common boundary and integrate also over the boundary values. Then this is the same as doing the unrestricted integral over field configurations in the union of the regions.

Reformulated QM

Space-time is Euclidean or Minkowski space. A point in space-time can be denoted by coordinates $(\overrightarrow{x}, t)$. The regions R are time intervals $[t_1, t_2]$ extended over all of space. The boundaries S are thus pairs of time-slices $S = S_1 \cup S_2$ with S_1 at t_1 and S_2 at t_2.

$$\mathcal{H}_S = \mathcal{H}_{S_1} \otimes \mathcal{H}_{S_2} \text{ and } \mathcal{H}_{S_2} = \mathcal{H}^*_{S_1}.$$

Let $\mathcal{H}_{S_1} = \mathcal{H}$ be the Hilbert space of quantum mechanics. Then a state in $\mathcal{H}_S$ corresponds to a pair of states $\psi \in \mathcal{H}$ at time t_1 and $\eta \in \mathcal{H}^*$ at time t_2 or a linear combination of such pairs.
Let define $\rho_R : \mathcal{H} \to \mathcal{H}$ to be the time-evolution operator $e^{-i/\hbar H(t_2 - t_1)}$.
The transition amplitude between ψ and η is given by ρ_R via

$$\langle \eta | e^{-\frac{i}{\hbar} H(t_2 - t_1)} | \psi \rangle = \rho_R(\psi \otimes \eta).$$

The composition property of GBF encodes the composition of time evolutions.

NRQM of scalar free particle

The action is $\mathcal{S}[\overrightarrow{x}] = \int \mathrm{d}t\, \frac{1}{2} m \dot{\overrightarrow{x}}^2(t)$ for a path $\overrightarrow{x}(t)$. A classical solution of the equations of motions intersects each time-slice exactly once. For a region R determined by a time interval $[t_1, t_2]$, denote the intersections with the boundaries S_1, S_2 by $\overrightarrow{x}_1$, $\overrightarrow{x}_2$. The configuration space on the boundary K_S associated with $S = S_1 \cup S_2$ which determines a classical solution uniquely is the space of pairs $(\overrightarrow{x}_1, \overrightarrow{x}_2)$, i.e. $\mathbb{R}^3 \times \mathbb{R}^3$.

According to (Q1) we should set $\mathcal{H}_S = C(K_S) = C(\mathbb{R}^3 \times \mathbb{R}^3)$. For the disconnected components we get that $\mathcal{H}_{S_1}$ and $\mathcal{H}_{S_2}$ can be identified with $C(\mathbb{R}^3)$. An element $\Psi(\overrightarrow{x}_1, \overrightarrow{x}_2)$ of $\mathcal{H}_S$ is generally a linear combination of products $\psi_{t_1}(\overrightarrow{x}_1)\eta_{t_2}(\overrightarrow{x}_2)$.

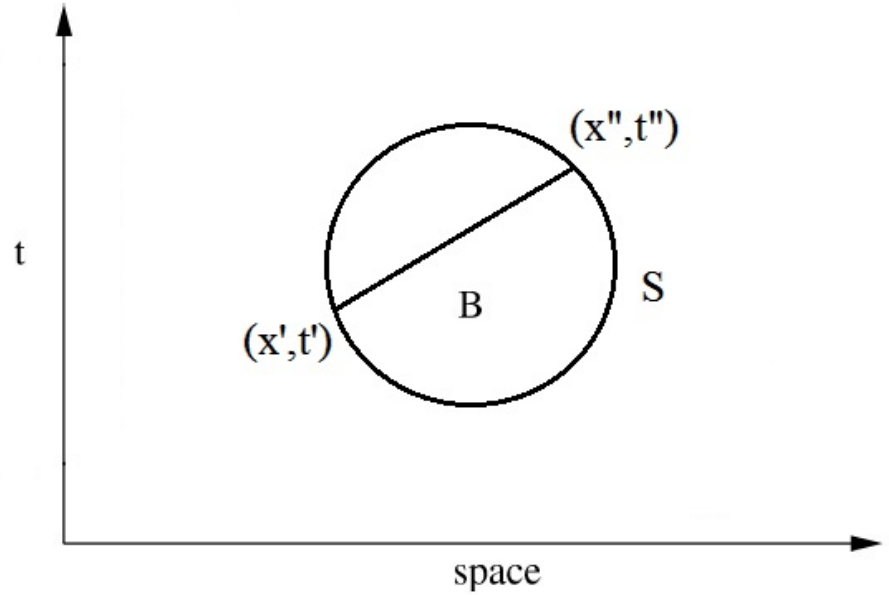
(Q2) tells us that $\rho(\psi \otimes \eta)$ is given by

$$\int_{\mathbb{R}^3 \times \mathbb{R}^3} \mathrm{d}\overrightarrow{x}_1 \mathrm{d}\overrightarrow{x}_2\, \psi(\overrightarrow{x}_1)\eta(\overrightarrow{x}_2) \int_{\substack{\overrightarrow{x}(t_1)=\overrightarrow{x}_1 \\ \overrightarrow{x}(t_2)=\overrightarrow{x}_2}} \mathcal{D}\overrightarrow{x}\, e^{\frac{i}{\hbar}\mathcal{S}[\overrightarrow{x}]}.$$

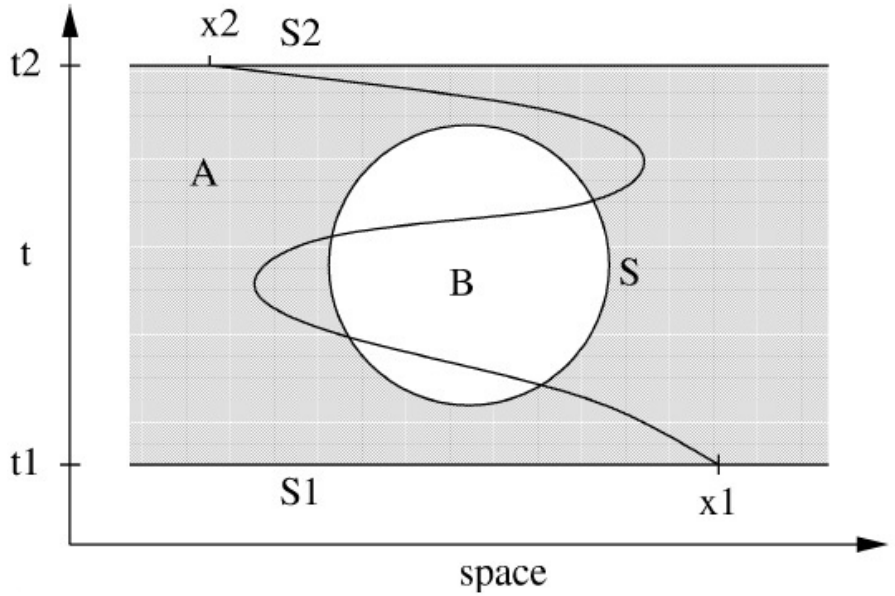
Note that we can easily generalize the above amplitude to include a potential in the action. Also, the extension to several particles is rather obvious. For example, for two distinguishable particles, K_S would be the space of quadruples $(\overrightarrow{x}_1, \overrightarrow{y}_1, \overrightarrow{x}_2, \overrightarrow{y}_2)$, while K_{S_1} would be given by pairs $(\overrightarrow{x}_1, \overrightarrow{y}_1)$ etc. $\mathcal{H}_{S_1}$ would be given by $C(K_{S_1}) = C(\mathbb{R}^3 \times \mathbb{R}^3)$, i.e. fixed-time wave functions $\psi(\overrightarrow{x}, \overrightarrow{y})$ of two particles. The amplitude is also generalized in the obvious way with the path integral now over one path for each particle.
For identical (and bosonic) particles we have to take for $\mathcal{H}_{S_1}$ the subspace of symmetric functions in $(\overrightarrow{x}, \overrightarrow{y})$. A different way to look at this is to replace the space K_{S_1} of ordered pairs by the space of unordered pairs. Of course this is not something coming out of the quantization prescription sketched above, but compatible with it.

Extending NRQM of particles to GBF

Consider a 4-ball shaped region B in space-time with boundary S. As with the spatial slices, a classical particle trajectory intersects S exactly twice. Thus, the configuration space is essentially $K_S = S \times S$. However, the entry time of the particle into B is necessarily earlier than the exit time. Thus K_S is really the subspace of $S \times S$ where one point (say the first one) has a smaller time coordinate.



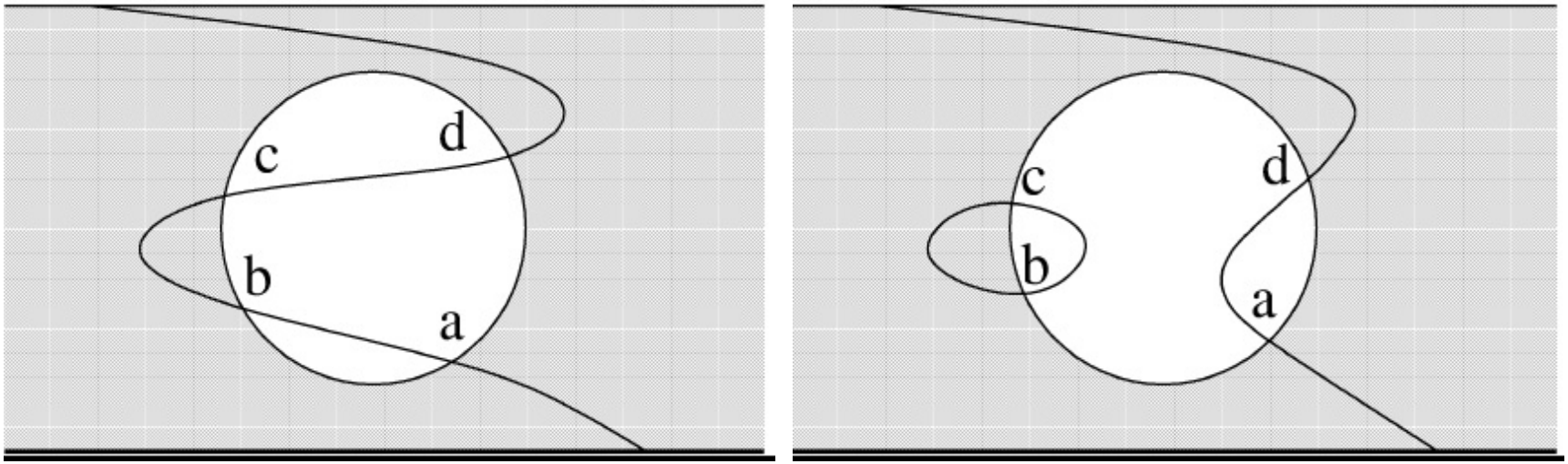
By (Q1) then $\mathcal{H}_S$ is the space of functions $\psi(z_{\text{in}}, z_{\text{out}})$ on $S \times S$ with this restriction. Here z denotes a parameterization of the hypersurface S. By (Q2) then we have a function ρ_B that associates amplitudes with such a generalized wave function ψ. The physical interpretation is that of the amplitude of a particle being sent into the region B at z_{in} (x', t') and being observed emerging from B at time and place $z_{\text{out}}(x", t")$.



A problem!

Choose a time interval $[t_1, t_2]$ containing B and the region of space-time R defined by it with boundaries S_1 at t_1 and S_2 at t_2. Since $R = A \cup B$ the composition rule requires that ρ_R equals the composition of ρ_A and ρ_B. Consider a one-particle state on $S_1 \cup S_2$. Then ρ_R contains an integral over paths from S_1 to S_2. Such a path may cross the inner region B an arbitrary number of times. However, we have taken the state space $\mathcal{H}_S$ associated with the boundary S the one for one particle. This only accounts for the paths in the integral that cross B exactly once and the composition rule seems to be violated.

The path integral in the expression for ρ_B only constrains paths at their starting point and end point. There is no a priori restriction for them to lie entirely inside B. However, we only want to allow to integrate inside B. Thus, how do we deal with paths that would leave B in between? The answer is rather obvious now. This corresponds to states with several particles on B. We need to let $\mathcal{H}_S$ be a direct sum of state spaces **for any number of particles**, i.e. $\mathcal{H}_S = \mathcal{H}_S^0 \oplus \mathcal{H}_S^1 \oplus \mathcal{H}_S^2 \oplus \cdots$. Now one may restrict the path integral to paths inside B. The occurrence of all paths in a composition (such as with A) is ensured by the summation over all numbers of particles on S. This way we do obtain a consistent formalism, as guaranteed by the composition rule.



In the configuration spaces for the multi-particle states, one has to keep track which classical endpoint is connected to which other one by paths in the integral. If we remove this restriction we introduce pair creation and annihilation of particles.

Physical reason to keep those paths

An observer on S, sees a particle coming into B at a and c and a particle emerging at b and d. If the particles are identical s/he has no way to say if it was **the same** particle that crossed S twice or two different particles.