Free fermions – quantum theory

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Outline

Overview

- So far in this seminar all talks have been essentially limited to the treatment of purely **bosonic** theories. Today (as last week) we shall consider **fermionic** theories. We restrict ourselves to the simplest case of **free field theory**.
- In contrast to the bosonic case we can not directly use the powerful **holomorphic quantization approach** since there is no comparable notion of **coherent state**. Instead we shall use a **Fock space approach**. It turns out that bosonic and fermionic theories can then be treated in a unified way. Moreover, in the bosonic case, both approaches are equivalent.
- As in the bosonic case the basic ingredients in the fermionic case ca be motivated from **geometric quantization**.
- As with holomorphic quantization this leads to a rigorous and functorial quantization scheme.

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A **notion of time** emerges without necessity for a metric.

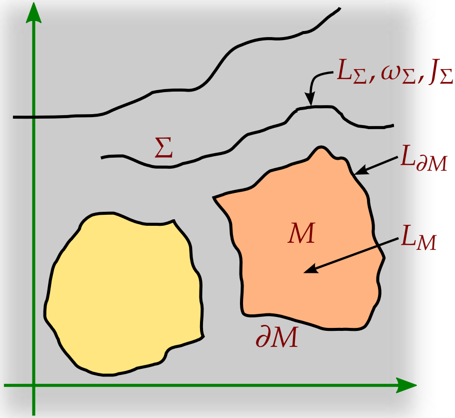
This is true both in the classical and in the quantum theory.

Today: Quantum theory

Last week we dealt with the semiclassical theory. Today we consider the **quantum theory**.

Bosonic semiclassical linear field theory

Spacetime is modeled by a collection of **hypersurfaces** and **regions**.



To these geometric structures associate the classical data,

- per **hypersurface** Σ :
  a symplectic vector space $(L_\Sigma, \omega_\Sigma)$,
- per **region** M :
  a **Lagrangian subspace** $L_M \subseteq L_{\partial M}$.

In addition,

- per **hypersurface** Σ :
  a **complex structure** J_Σ .

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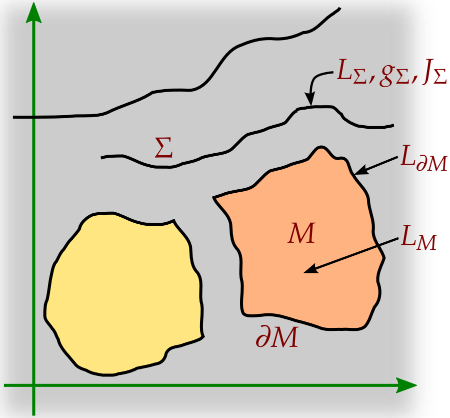


To these geometric structures associate the classical data,

- per **hypersurface** Σ : a **real Krein space** (L_Σ, g_Σ),
- per **region** M : a **hypermaximal neutral subspace** $L_M \subseteq L_{\partial M}$.

In addition,

- per **hypersurface** Σ : a **complex structure** J_Σ .

Krein space

Recall that a **Krein space** V is a complete **indefinite inner product** space with an orthogonal decomposition

$$V = V^+ \oplus V^-.$$

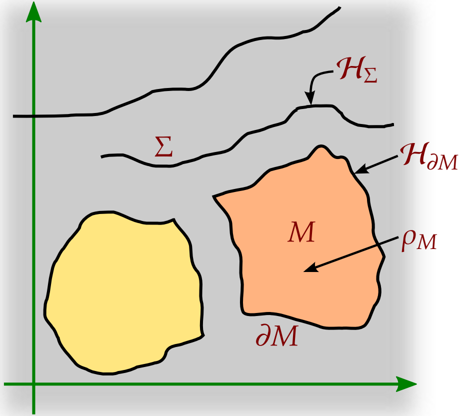
The **positive part** V^+ is **positive definite** and **the negative part** V^- is **negative definite**. Denote by $\overline{V^-}$ the space V^- with minus its inner product. Then, V^+ and $\overline{V^-}$ are both **Hilbert spaces**. For $v \in V$ define the **signature**,

$$[v] := \begin{cases} 0 & \text{if } v \in V^+ \\ 1 & \text{if } v \in V^- \end{cases}.$$

All Krein spaces considered are **separable**. An **ON-basis** of V is the union of an ON-basis of V^+ with an ON-basis of $\overline{V^-}$.

Structures of quantum field theory in the GBF

Spacetime is modeled by a collection of **hypersurfaces** and **regions**.



To these geometric structures associate the quantum data,

- per **hypersurface** Σ : an **f-graded Krein space** $\mathcal{H}_\Sigma$,
- per **region** M : a linear **f-graded amplitude map** $\rho_M : \mathcal{H}_{\partial M} \to \mathbb{C}$.

Compared to the purely bosonic case we have a $\mathbb{Z}_2$-grading called **f-grading** on all structures. Moreover, instead of **Hilbert spaces** we have **Krein spaces**.

Core axioms (I)

The generalization of the core axioms to include the fermionic case is now relatively straightforward.

T1b There is an antilinear f-graded involutive isometry $\iota_\Sigma : \mathcal{H}_\Sigma \to \mathcal{H}_{\overline{\Sigma}}$.

T2 **(Decomposition rule)** Let $\Sigma = \Sigma_1 \cup \Sigma_2$ be a disjoint union. There is an isometric isomorphism $\tau_{\Sigma_1,\Sigma_2;\Sigma} : \mathcal{H}_{\Sigma_1} \otimes \mathcal{H}_{\Sigma_2} \to \mathcal{H}_\Sigma$. Moreover, $\tau^{-1}_{\Sigma_2,\Sigma_1;\Sigma} \circ \tau_{\Sigma_1,\Sigma_2;\Sigma} : \mathcal{H}_{\Sigma_1} \otimes \mathcal{H}_{\Sigma_2} \to \mathcal{H}_{\Sigma_2} \otimes \mathcal{H}_{\Sigma_1}$ is the f-graded transposition

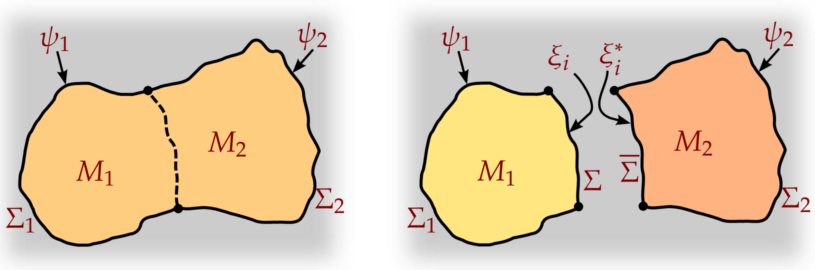
$$\psi_1 \otimes \psi_2 \mapsto (-1)^{|\psi_1|+|\psi_2|} \psi_2 \otimes \psi_1.$$

T3x The inner product for a hypersurface Σ is determined by the amplitude for the slice region $\hat{\Sigma}$ with $\partial\hat{\Sigma} = \overline{\Sigma} \cup \Sigma'$:

$$\langle \psi, \psi' \rangle_\Sigma = \rho_{\hat{\Sigma}}(\iota_\Sigma(\psi) \otimes \psi').$$

Core axioms (II)

T5 **(Gluing rule)** If M_1 and M_2 are adjacent regions, then:



$$\rho_{M_1 \cup M_2}(\psi_1 \otimes \psi_2) \cdot c_{M_1,M_2} = \sum_{i \in N} (-1)^{[\xi_i]} \rho_{M_1}(\psi_1 \otimes \xi_i) \rho_{M_2}(\iota_\Sigma(\xi_i) \otimes \psi_2)$$

Here, $\psi_1 \in \mathcal{H}_{\Sigma_1}$, $\psi_2 \in \mathcal{H}_{\Sigma_2}$ and $\{\xi_i\}_{i \in N}$ is an ON-basis of $\mathcal{H}_\Sigma$.
c_{M_1,M_2} is the **gluing anomaly factor**.

Fock space (I)

We distinguish bosonic and fermionic case via

$$\kappa := 1 \quad \text{in the bosonic case,} \qquad \kappa := -1 \quad \text{in the fermionic case.}$$

Given a Krein space L, the **Fock space** $\mathcal{F}(L)$ over L is the completion of an $\mathbb{N}_0$-graded Krein space,

$$\mathcal{F}(L) = \widehat{\bigoplus_{n=0}^{\infty}} \mathcal{F}_n(L),$$

$$\mathcal{F}_n(L) := \{\psi : L \times \cdots \times L \to \mathbb{C} \; n\text{-lin. cont.} \; : \psi \circ \sigma = \kappa^{|\sigma|}\psi, \forall \sigma \in S^n\}.$$

There is a natural $\mathbb{Z}_2$-grading. In the bosonic case it is trivial, i.e., $|\psi| = 0$ for all $\psi \in \mathcal{F}(L)$. In the fermionic case it is,

$$|\psi| := \begin{cases} 0 & \text{if } \psi \in \mathcal{F}_n(L), n \text{ even} \\ 1 & \text{if } \psi \in \mathcal{F}_n(L), n \text{ odd.} \end{cases}$$

Fock space (II)

Given $\xi_1, \ldots, \xi_n \in L$ define a **generating state** in $\mathcal{F}_n(L)$ as

$$\psi[\xi_1, \ldots, \xi_n](\eta_1, \ldots, \eta_n) := \frac{1}{n!} \sum_{\sigma \in S^n} \kappa^{|\sigma|} \prod_{i=1}^{n} \{\xi_i, \eta_{\sigma(i)}\}.$$

The inner product in Fock space is determined by the inner product of generating states,

$$\langle \psi[\eta_1, \ldots, \eta_n], \psi[\xi_1, \ldots, \xi_n] \rangle := 2^n \sum_{\sigma \in S^n} \kappa^{|\sigma|} \prod_{i=1}^{n} \{\xi_i, \eta_{\sigma(i)}\}.$$

This makes $\mathcal{F}(L)$ into a **Krein space** as well.

Quantization: State spaces

For each hypersurface Σ we define the corresponding **state space** $\mathcal{H}_\Sigma$ to be the Fock space $\mathcal{F}(L_\Sigma)$.

For all $n \in \mathbb{N}_0$ define $\iota_{\Sigma,n} : \mathcal{F}_n(L_\Sigma) \to \mathcal{F}_n(L_{\overline{\Sigma}})$ by,

$$(\iota_{\Sigma,n}(\psi))(\xi_1, \ldots, \xi_n) := \overline{\psi(\xi_n, \ldots, \xi_1)}.$$

Taking these maps together for all $n \in \mathbb{N}_0$ defines $\iota_\Sigma : \mathcal{F}(L_\Sigma) \to \mathcal{F}(L_{\overline{\Sigma}})$.

A decomposition $\Sigma = \Sigma_1 \cup \Sigma_2$ induces a direct sum of Krein spaces $L_\Sigma = L_{\Sigma_1} \oplus L_{\Sigma_2}$. This induces an isomorphism of Fock spaces

$$\tau_{\Sigma_1,\Sigma_2;\Sigma} : \mathcal{F}(L_{\Sigma_1}) \otimes \mathcal{F}(L_{\Sigma_2}) \to \mathcal{F}(L_\Sigma).$$

This also yields the f-graded transposition,

$$\mathcal{F}(L_{\Sigma_1}) \otimes \mathcal{F}(L_{\Sigma_2}) \to \mathcal{F}(L_{\Sigma_2}) \otimes \mathcal{F}(L_{\Sigma_1}) \quad : \quad \psi_1 \otimes \psi_2 \mapsto (-1)^{|\psi_1|+|\psi_2|} \psi_2 \otimes \psi_1.$$

Quantization: Amplitudes

Given a region M we recall the real orthogonal decomposition $L_{\partial M} = L_M \oplus J_{\partial M} L_M$ giving rise to the map $u_M : L_{\partial M} \to L_{\partial M}$,

$$u_M(\xi + J_{\partial M}\eta) = \xi - J_{\partial M}\eta, \quad \forall \xi, \eta \in L_M.$$

The **amplitude** for a generating state is now defined as,

$$\rho_M(\psi[\xi_1, \ldots, \xi_{2n}]) := \frac{1}{n!} \sum_{\sigma \in S^{2n}} \kappa^{|\sigma|} \prod_{j=1}^{n} \{\xi_{\sigma(j)}, u_M(\xi_{\sigma(2n+1-j)})\}_{\partial M}.$$

The amplitude vanishes for states with odd particle number.

Main result

This **quantization scheme** yields the data of a quantum theory in terms of the GBF.

Theorem

With an additional integrability assumption, the GBF core axioms as well as the vacuum axioms are satisfied.

The quantization prescription may be viewed (in various ways) as a **functor** from semiclassical field theories to general boundary quantum field theories.

The integrability assumptions amounts to requiring the finiteness of the **gluing anomaly factor**. Without it, **gluing axiom** T5b may be violated.

Comparison to holomorphic quantization

Theorem

In the bosonic case this quantization scheme is equivalent to **holomorphic quantization**.

The underlying isomorphism $T : \mathcal{F}(L_\Sigma) \to H^2(\hat{L}_\Sigma, \nu_\Sigma)$ of state spaces is given by,

$$(T(\psi))(\xi) := \psi(\xi, \ldots, \xi).$$

For a coherent state we have,

$$K_\xi = T\left(\sum_{n=0}^{\infty} \frac{1}{n!2^n} \psi[\xi, \ldots, \xi]\right).$$

Complex conjugation

Given a region M we recall the map $u_M : L_{\partial M} \to L_{\partial M}$. It has a quantum counter part $U_M : \mathcal{H}_{\partial M} \to \mathcal{H}_{\partial M}$ given by

$$(U_M\psi)(\xi_1, \ldots, \xi_n) := \overline{\psi(u_M\xi_n, \ldots u_M\xi_1)}.$$

The maps u_M and U_M have remarkable properties. They are involutive (i.e, square to the identity), conjugate linear and f-graded isometric. In fact, they act like a **complex conjugation** in the classical respectively quantum setting. In particular,

$$\rho_M(U_M(\psi)) = \overline{\rho_M(\psi)}.$$

Algebraic time

Recall that u_M also plays the role of a **generalized evolution map** in the classical theory. Moreover, remember from last week that its gives rise in the fermionic case to an **algebraic notion of time**. Given the canonical decomposition $L_{\partial M} = L^+_{\partial M} \oplus L^-_{\partial M}$, we obtain an "evolution" from "initial data" $L^+_{\partial M}$ to "final data" $L^-_{\partial M}$ via u_M restricted to

$$\tilde{u}_M : L^+_{\partial M} \to L^-_{\partial M}.$$

We saw that in the Dirac field theory, this coincides with the **geometric notion of time** for the time-interval geometry.

In the quantum theory we have a decomposition

$$\mathcal{H}_\Sigma = \mathcal{F}(L^+_{\partial M} \oplus L^-_{\partial M}) = \mathcal{F}(L^+_{\partial M}) \otimes \mathcal{F}(L^-_{\partial M}).$$

This induces from U_M an **f-graded** isometric isomorphism, representing the **quantum version** of the algebraic time evolution,

$$\tilde{U}_M : \mathcal{F}(L^+_{\partial M}) \to \mathcal{F}(L^-_{\partial M}).$$

Probabilities and superselection (I)

We recall the probability rule for the bosonic GBF, where all state spaces are **Hilbert spaces**. A measurement is determined by two subspaces of $\mathcal{H}_{\partial M}$,

- $\mathcal{S}$, representing the **preparation** and
- $\mathcal{A} \subseteq \mathcal{S}$, representing the **question** asked .

The probability for an affirmative answer is then,

$$P(\mathcal{A}|\mathcal{S}) = \frac{\sum_{i\in J} |\rho_M(\xi_i)|^2}{\sum_{i\in I} |\rho_M(\xi_i)|^2}.$$

Here $\{\xi\}_{i\in I}$ is an ON-basis of $\mathcal{S}$ and $\{\xi\}_{i\in J}$ is an ON-basis of $\mathcal{A}$, with $J \subseteq I$.

Probabilities and superselection (II)

The very same formula works for **Krein spaces**. But there are some differences.

The notion of ON-basis is more restrictive in the Krein space case. It implies that the subspaces $\mathcal{S}$ and $\mathcal{A}$ must decompose as direct sums $\mathcal{S} = \mathcal{S}^+ \oplus \mathcal{S}^-$ and $\mathcal{A} = \mathcal{A}^+ \oplus \mathcal{A}^-$, where $\mathcal{S}^\pm, \mathcal{A}^\pm \subseteq \mathcal{H}_{\partial M}^\pm$. This amounts to a **signature superselection rule**.

In the fermionic case this superselection rule is not invariant under **orientation change**. But the physics should be. But for fermionic theories there is also the **fermionic superselection rule** [Wick, Wightman, Wigner 1952] that forbids the mixing of states with even and odd fermion number. This amounts to requiring decompositions $\mathcal{S} = \mathcal{S}_0 \oplus \mathcal{S}_1$ and $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$ in terms of the **f-grading** of $\mathcal{H}_{\partial M}$.

In combination with the fermionic superselection rule, the signature superselection rule becomes **orientation invariant**.

Renormalizing the gluing anomaly (I)

Recall the main **gluing identity** of the **gluing axiom** T5b:

$$\rho_{M_1}(\psi) \cdot c = \sum_{i \in N} (-1)^{[\xi_i]} \rho_M(\psi \otimes \xi_i \otimes \iota_\Sigma(\xi_i))$$

If all state spaces are **finite-dimensional** the sum on the right hand side is finite. The axiom is then satisfied without any additional **integrability condition** with finite **gluing anomaly factor** c (Theorem). This can only happen in the fermionic case. There, if L_Σ is finite-dimensional so is the Fock space $\mathcal{F}(L_\Sigma)$.

Consider now the set $\{L_{\Sigma,\alpha}\}_{\alpha \in A}$ of all **finite-dimensional** subspaces of L_Σ. This is an **injective system** with the inclusion. Moreover, it induces an **projective system** $\{\mathcal{F}(L_{\Sigma,\alpha})\}_{\alpha \in A}$ of the corresponding Fock spaces. Define P_α as the orthogonal projector $\mathcal{F}(L_\Sigma) \to \mathcal{F}(L_{\Sigma,\alpha})$.

Renormalizing the gluing anomaly (II)

Consider a "reduced version" of the gluing identity,

$$\rho_{M_1}(\psi) \cdot c_\alpha = \sum_{i \in N} (-1)^{[\xi_i]} \rho_M(\psi \otimes P_\alpha \xi_i \otimes \iota_\Sigma(P_\alpha \xi_i)). \qquad (1)$$

This of course will not hold for arbitrary states ψ if we fix α.

But, (Theorem) there exists a set $\{c_\alpha\}_{\alpha \in A}$ such that for any state ψ there is $\beta \in A$ such that for all $\gamma \geq \beta$ the identity (1) holds. The limit $\varinjlim_\alpha c_\alpha$ does not exist. But the following limit does,

$$\varinjlim_\alpha \left(\rho_{M_1}(\psi) \cdot c_\alpha - \sum_{i \in N} (-1)^{[\xi_i]} \rho_M(\psi \otimes P_\alpha \xi_i \otimes \iota_\Sigma(P_\alpha \xi_i)) \right) = 0.$$

This is the **renormalized gluing identity**. It is satisfied in the fermionic theory without any **integrability condition**.

References

Main reference:

R. O., *Free Fermi and Bose Fields in TQFT and GBF*, SIGMA **9** (2013) 028. arXiv:1208.5038.