The positive formalism

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Outline

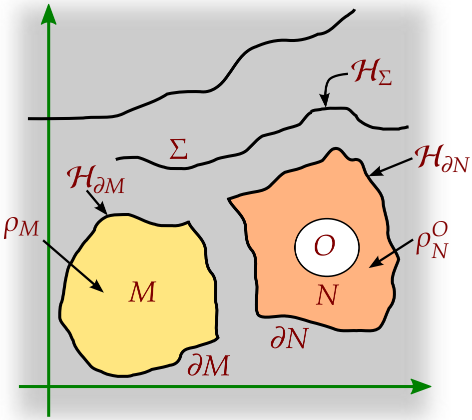
Remark

For simplicity of presentation, we restrict in the following to purely bosonic theory. However, everything generalizes nicely to the case including fermionic fields.

GBF: Spacetime assignments

Amplitude formalism

Spacetime is modeled by a collection of **hypersurfaces** and **regions**.



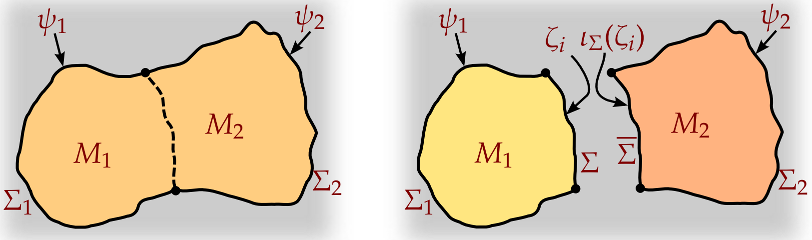
To these geometric structures associate the quantum data,

- per **hypersurface** Σ :
  a **Hilbert space** $\mathcal{H}_\Sigma$,
- per **region** M :
  a linear **amplitude map**
  $\rho_M : \mathcal{H}_{\partial M} \to \mathbb{C}$,
- per **region** M that contains an **observable** O :
  a linear **observable map**
  $\rho_M^O : \mathcal{H}_{\partial M} \to \mathbb{C}$.

Core axioms

Amplitude formalism

- Let $\overline{\Sigma}$ denote Σ with opposite orientation. Then $\mathcal{H}_{\overline{\Sigma}} = \mathcal{H}_{\Sigma}^*$.
- **(Decomposition rule)** Let $\Sigma = \Sigma_1 \cup \Sigma_2$ be a disjoint union of hypersurfaces. Then $\mathcal{H}_\Sigma = \mathcal{H}_{\Sigma_1} \otimes \mathcal{H}_{\Sigma_2}$.
- **(Gluing rule)** If M_1 and M_2 are adjacent regions, then:



$$\rho_{M_1\cup M_2}(\psi_1 \otimes \psi_2) \cdot c_{M_1,M_2} = \sum_{i\in N} \rho_{M_1}(\psi_1 \otimes \zeta_i)\rho_{M_2}(\iota_\Sigma(\zeta_i) \otimes \psi_2)$$

Here, $\psi_1 \in \mathcal{H}_{\Sigma_1}$, $\psi_2 \in \mathcal{H}_{\Sigma_2}$ and $\{\zeta_i\}_{i\in N}$ is an ON-basis of $\mathcal{H}_\Sigma$.

Probabilities

Amplitude formalism

Consider a spacetime region M. The associated amplitude ρ_M allows to extract probabilities for measurements in M.

Probabilities in quantum theory are generally conditional probabilities. They depend on two pieces of information. Here these are:

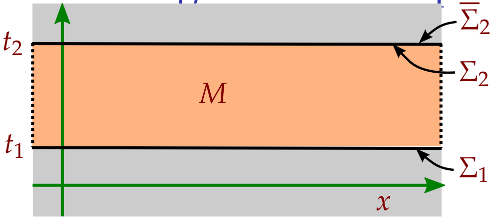
- $\mathcal{S} \subseteq \mathcal{H}_{\partial M}$ representing **preparation** or **knowledge**
- $\mathcal{A} \subseteq \mathcal{H}_{\partial M}$ representing **observation** or the **question**

The probability that the physics in M is described by $\mathcal{A}$ given that it is described by $\mathcal{S}$ is: (here $\mathcal{A} \subseteq \mathcal{S}$) [RO 2005]

$$\boxed{P(\mathcal{A}|\mathcal{S}) = \frac{\sum_{i \in I} \overline{\rho_M(\xi_i)}\, \rho_M(\mathsf{P}_{\mathcal{A}}(\xi_i))}{\sum_{i \in I} \overline{\rho_M(\xi_i)}\, \rho_M(\mathsf{P}_{\mathcal{S}}(\xi_i))}}$$

$\mathsf{P}_{\mathcal{S}}$ and $\mathsf{P}_{\mathcal{A}}$ are the orthogonal projectors onto the subspaces $\mathcal{S}$ and $\mathcal{A}$; $\{\xi_i\}_{i \in I}$ an ON-basis of $\mathcal{H}_{\partial M}$.

Recovering transition amplitudes and probabilities



- region: $M = [t_1, t_2] \times \mathbb{R}^3$
- boundary: $\partial M = \Sigma_1 \cup \overline{\Sigma}_2$
- state space: $\mathcal{H}_{\partial M} = \mathcal{H}_{\Sigma_1} \otimes \mathcal{H}_{\overline{\Sigma}_2} = \mathcal{H}_{\Sigma_1} \otimes \mathcal{H}^*_{\Sigma_2}$

As before, we identify $\mathcal{H}_{\Sigma_1} \cong \mathcal{H}_{\Sigma_2} \cong \mathcal{H}$. Then,

$$\rho_{[t_1,t_2]}(\psi_1 \otimes \psi_2^*) = \langle \psi_2, U(t_1, t_2)\psi_1 \rangle.$$

To compute the probability of measuring ψ_2 at t_2 given that we prepared ψ_1 at t_1 we set

$$\mathcal{S} = \psi_1 \otimes \mathcal{H}^*, \quad \mathcal{A} = \mathcal{H} \otimes \psi_2^*.$$

The resulting expression recovers precisely the transition probability

$$P(\mathcal{A}|\mathcal{S}) = |\langle \psi_2, U(t_1, t_2)\psi_1 \rangle|^2.$$

Observables and expectation values

Amplitude formalism

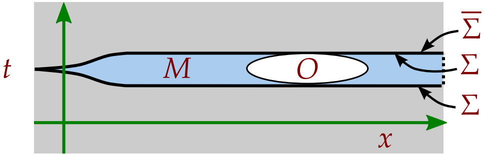
Consider a spacetime region M carrying an observable O. The associated observable map ρ_M^O allows to extract expectation values for measurements in M.

The **expectation value** of the observable O conditional on the system being prepared in the subspace $\mathcal{S} \subseteq \mathcal{H}_{\partial M}$ can be represented as follows: [RO 2010]

$$\boxed{\langle O \rangle_{\mathcal{S}} = \frac{\sum_{i \in I} \overline{\rho_M(\xi_i)}\, \rho_M^O(\mathsf{P}_{\mathcal{S}}(\xi_i))}{\sum_{i \in I} \overline{\rho_M(\xi_i)}\, \rho_M(\mathsf{P}_{\mathcal{S}}(\xi_i))}}$$

$\mathsf{P}_{\mathcal{S}}$ is the orthogonal projector onto the subspace $\mathcal{S}$; $\{\xi_i\}_{i \in I}$ an ON-basis of $\mathcal{H}_{\partial M}$.

Recovering standard expectation values



- region: $M = [t, t] \times \mathbb{R}^3$
- boundary: $\partial M = \Sigma \cup \overline{\Sigma}$
- state space: $\mathcal{H}_{\partial M} = \mathcal{H}_\Sigma \otimes \mathcal{H}_{\overline{\Sigma}} = \mathcal{H}_\Sigma \otimes \mathcal{H}_\Sigma^*$

To compute the expectation value of observable O at time t given by

$$\rho^{O}_{[t,t]}(\psi_1 \otimes \psi_2^*) = \langle \psi_2, \hat{O}\psi_1 \rangle$$

in the state ψ we set

$$\mathcal{S} = \psi \otimes \mathcal{H}_\Sigma^*.$$

The standard expectation value is then correctly recovered as

$$\langle O \rangle_{\mathcal{S}} = \langle \psi, \hat{O}\psi \rangle.$$

Applications of the amplitude formalism (AF)

- By restricting to spacetimes with spacelike foliations the standard formulation **is reproduced exactly**. [RO 2005; 2010]
- **Three dimensional quantum gravity** is already formulated as a TQFT and fits thus "automatically" into the AF.
- (Part of) the AF is extensively used in **spin foam quantum gravity**. [C. Rovelli et al.]
- A natural testing ground for the GBF is **quantum field theory**.
  - State spaces on timelike hypersurfaces and "evolution" in spacelike directions. [RO 2005]
  - New **S-matrix** type asymptotic amplitudes in Minkowski space, deSitter space, Anti-deSitter space. [D. Colosi, RO 2008; D. Colosi 2009; M. Dohse 2011; 2012]
  - **Quantum Yang-Mills theory** in 2 dimensions for arbitrary regions and hypersurfaces with corners. [RO 2006]
  - Rigorous and functorial quantization of linear and affine field theories without metric background. [RO 2010; 2011; 2012]
  - **Unruh effect**. [D. Colosi, D. Rätzel 2012]

A critical look at the operational core

The operational core of the GBF lies in the prediction of probabilities and expectation values for measurements. Looking closely at the extraction of these quantities in the amplitude formalism we notice the following:

- The relevant objects on boundaries are not "states" (elements of the Hilbert space), but subspaces or, equivalently, projection operators.
- The formulas for probabilities and expectation values look somewhat complicated and unnatural. But they suggest a common element.

Ingredients

Positive formalism

- For each hypersurface Σ we consider the algebra $\mathcal{D}_\Sigma$ of operators on $\mathcal{H}_\Sigma$.
- For each region M we define the linear **probability map** $A_M : \mathcal{D}_{\partial M} \to \mathbb{C}$ by

$$A_M(\sigma) := \sum_{i \in I} \overline{\rho_M(\xi_i)}\, \rho_M(\sigma(\xi_i))$$

- For each region M carrying an observable O we define the linear **expectation map** $A_M^O : \mathcal{D}_{\partial M} \to \mathbb{C}$ by

$$A_M^O(\sigma) := \sum_{i \in I} \overline{\rho_M(\xi_i)}\, \rho_M^O(\sigma(\xi_i))$$

Probabilities and expectation values

Positive formalism

Given a region M and subspaces $\mathcal{A} \subseteq \mathcal{S} \subseteq \mathcal{H}_{\partial M}$ we have $\mathsf{P}_{\mathcal{A}}, \mathsf{P}_{\mathcal{S}} \in \mathcal{D}_{\partial M}$. The probability for measuring $\mathcal{A}$ given $\mathcal{S}$ is,

$$\boxed{P(\mathcal{A}|\mathcal{S}) = \frac{A_M(\mathsf{P}_{\mathcal{A}})}{A_M(\mathsf{P}_{\mathcal{S}})}}$$

Given a region M carrying an observable O and given a subspace $\mathcal{S} \subseteq \mathcal{H}_{\partial M}$, the corresponding expectation value is,

$$\boxed{\langle O \rangle_{\mathcal{S}} = \frac{A_M^O(\mathsf{P}_{\mathcal{S}})}{A_M(\mathsf{P}_{\mathcal{S}})}}$$

This looks much simpler than in the amplitude formalism. . .

Realness and positivity

Positive formalism

...but it is also more natural!

- Consider the subset $\mathcal{D}_\Sigma^\mathbb{R} \subseteq \mathcal{D}_\Sigma$ of **self-adjoint** operators. This is a real vector space and $\mathcal{D}_\Sigma$ is its complexification.
- Consider the subset $\mathcal{D}_\Sigma^+ \subseteq \mathcal{D}_\Sigma^\mathbb{R}$ of **positive** operators. This forms a generating proper cone in the real vector space $\mathcal{D}_\Sigma^\mathbb{R}$ making it into an **ordered vector space**.
- The orthogonal projection operators form a **lattice** in $\mathcal{D}_\Sigma^\mathbb{R}$. This is equivalent the lattice of closed subspaces of $\mathcal{H}_\Sigma$. That is,

$$\mathsf{P}_{\mathcal{A}_1} \le \mathsf{P}_{\mathcal{A}_2} \iff \mathcal{A}_1 \subseteq \mathcal{A}_2$$

- The probability map is **positive**, i.e.,

$$A_M(\sigma) \in \mathbb{R} \text{ if } \sigma \in \mathcal{D}_{\partial M}^\mathbb{R} \quad \text{and} \quad A_M(\sigma) \ge 0 \text{ if } \sigma \in \mathcal{D}_{\partial M}^+$$

  This implies $0 \le P(\mathcal{A}|\mathcal{S}) \le 1$.

A formalism in its own right

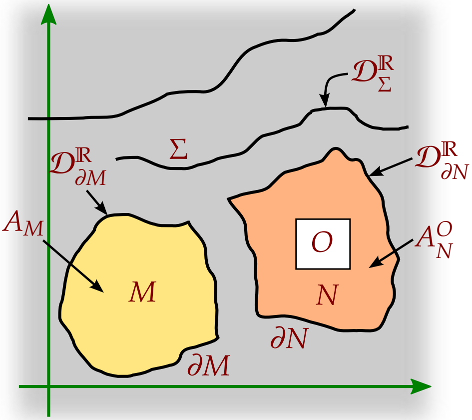
Positive formalism

- Remarkably, the new structures $\mathcal{D}_\Sigma$, A_M and A_M^O satisfy axioms quite similar to those satisfied by $\mathcal{H}_\Sigma$, ρ_M and ρ_M^O.
- This suggests to postulate the new structures as objects in their own right, rather than to derive them from the amplitude formalism. This gives rise to the **positive formalism** [RO 2012].
- Positivity and normalization of probabilities now derive directly from the positivity of the probability map.
- We may restrict to the real vector spaces $\mathcal{D}_\Sigma^\mathbb{R}$, even forgetting $\mathcal{D}_\Sigma$.
- The latter step provokes a transition from an **oriented** to an **unoriented** formalism.
- We can generalize the expectation maps to not only represent observables, but more general **quantum operations**. We call these then **operation maps**.

Spacetime assignments

Positive formalism

Spacetime is modeled by a collection of **hypersurfaces** and **regions**.



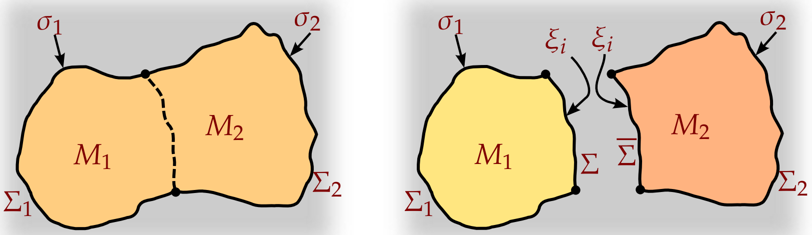
To these geometric structures associate the quantum data,

- per **hypersurface** Σ : an **ordered vector space** $\mathcal{D}^{\mathbb{R}}_{\Sigma}$,
- per **region** M : a positive **probability map** $A_M : \mathcal{D}^{\mathbb{R}}_{\partial M} \to \mathbb{R}$,
- per **region** M that contains an **operation** O : an **operation map** $A^O_M : \mathcal{D}^{\mathbb{R}}_{\partial M} \to \mathbb{C}$.

Core axioms

Positive formalism

- **(Decomposition rule)** Let $\Sigma = \Sigma_1 \cup \Sigma_2$ be a disjoint union of hypersurfaces. Then $\mathcal{D}^{\mathbb{R}}_{\Sigma} = \mathcal{D}^{\mathbb{R}}_{\Sigma_1} \otimes \mathcal{D}^{\mathbb{R}}_{\Sigma_2}$.
- **(Gluing rule)** If M_1 and M_2 are adjacent regions, then:



$$A_{M_1 \cup M_2}(\sigma_1 \otimes \sigma_2) \cdot |c_{M_1,M_2}|^2 = \sum_{i \in N} A_{M_1}(\psi_1 \otimes \xi_i) A_{M_2}(\xi_i \otimes \psi_2)$$

Here, $\sigma_1 \in \mathcal{D}^{\mathbb{R}}_{\Sigma_1}$, $\sigma_2 \in \mathcal{D}^{\mathbb{R}}_{\Sigma_2}$ and $\{\xi_i\}_{i \in N}$ is an ON-basis of $\mathcal{D}^{\mathbb{R}}_{\Sigma}$.

First summary

Positive formalism

The positive formalism is intriguing for a number of reasons:

- its spacetime locality and metric background independence (as an incarnation of the GBF)
- its wide applicability inherited from the amplitude formalism
- its potential applicability beyond the amplitude formalism
- its operationalism with a simple and elegant way to predict probabilities and expectation values
- its amenability to quantum information theory

At the same time it immediately invites many further questions. . .

No states, no collapse, but. . . ?

Question 1

- As becomes particularly clear in the positive formalism, the traditional concept of "state" as a specification of the reality of a system is untenable in the GBF. This also kills "collapse" interpretations and any model of the "collapse" as a physical event.
- Instead, the relevant mathematical objects entering the probability interpretation are the elements of the spaces $\mathcal{D}^+_{\partial M}$. We tentatively call them **quantum boundary conditions**. Only the "atomic" elements (one-dimensional projectors) correspond to elements in a Hilbert space. In turn, these coincide only in special circumstances with the traditional quantum states.
- But can we say anything more about the physical interpretation of the elements of $\mathcal{D}^+_{\partial M}$? Do only special elements of $\mathcal{D}^+_{\partial M}$ have a physical interpretation (e.g. the projectors)?

Spaces of quantum boundary conditions

Question 2

There are also mathematical questions about the spaces $\mathcal{D}_\Sigma$.

- Is the structure of ordered vector spaces sufficient? Do we need e.g., a Jordan product or even the "full" operator product? (In [RO 2012] I have also given them a Hilbert space structure.)
- What is the right "size" and topology for these spaces? In this talk I have assumed that these contain all bounded operators. In [RO 2012] I have assumed that these are only the Hilbert-Schmidt operators.

A related remark: The probability map A_M is actually not defined on $\mathcal{D}_{\partial M}$, but on a "dense" subspace $\mathcal{D}^\circ_{\partial M}$. Positivity suggests a solution to this problem. First, restrict A_M to $\mathcal{D}^{+\circ}_{\partial M}$. Second, extend the range of A_M from $[0,\infty]$ to $[0,\infty)$ to obtain a map $A_M : \mathcal{D}^+_{\partial M} \to [0,\infty)$.

The new freedom

Question 3

- The transition from Hilbert spaces $\mathcal{H}_\Sigma$ to spaces of quantum boundary conditions $\mathcal{D}_\Sigma$ gets rid of operationally irrelevant information (mostly phases). What is more, the structural requirements on $\mathcal{D}_\Sigma$ are weaker than those coming from $\mathcal{H}_\Sigma$. This gives us **new freedom** in the construction of quantum theories.
- What can we do with this freedom? I am hopeful in particular concerning solving the "state locality problem" in QFT...

Quantum information theory
Question 4

- The positive formalism enables us in particular to do within the GBF everything that can be done with the mixed state formalism of the standard formulation. We can implement arbitrary quantum operations, compose them, define notions of entropy, etc.
- [wild speculation] Can this help us to work towards a general relativistic (and quantum) framework for statistical physics, thermodynamics etc.?

Main reference

Positive formalism (Christmas paper):

R. O., *A positive formalism for quantum theory in the general boundary formulation*. arXiv:1212.5571.