Algebraic Quantum Field Theory and Category Theory I

Albert Much

UNAM Morelia, CCM,

Seminario de física matemática del CCM

05.04.2017

Outline

Intro to Algebraic Quantum Field Theory
A Few Definitions
General Introduction
Tomita-Takesaki modular theory

AQFT in terms of Category Theory
Definitions
Categories

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Defining a $*$-algebra

Definition

$\mathcal{A}$ is called an **algebra** over $\mathbb{C}$, if $\alpha A + \beta B$ with $A, B \in \mathcal{A}$ and $\alpha, \beta \in \mathbb{C}$, are well defined. In addition, there is a product $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$, which is distributive over addition,

$$A(B + C) = AB + AC, \qquad (A + B)C = AC + BC, \qquad \forall A, B, C \in \mathcal{A}.$$

The algebra $\mathcal{A}$ is called a **unital algebra** if it has a unit $\mathbb{I}$.

Definition

An algebra $\mathcal{A}$ is called a ***-algebra** if it admits an involution $* : \mathcal{A} \to \mathcal{A}$:

$$A^{**} = A, \qquad (AB)^* = B^* A^*, \qquad (\alpha A + \beta B)^* = \overline{\alpha} A^* + \overline{\beta} B^*$$

for any $A, B \in \mathcal{A}$ and $\alpha, \beta \in \mathbb{C}$.

Defining a normed-algebra

Definition

An algebra $\mathcal{A}$ with a norm $\|\cdot\| : \mathcal{A} \to \mathbb{R}$ is called a **normed algebra** if:

$$\|A\| \geq 0, \|A\| = 0 \Leftrightarrow A = 0, \qquad \|\alpha A\| = |\alpha|\|A\|,$$

$$\|A + B\| \leq \|A\| + \|B\|, \qquad \|AB\| \leq \|A\|\|B\|$$

for any A, $B \in \mathcal{A}$ and $\alpha, \in \mathbb{C}$.

Norm topology: The neighborhoods of any $A \in \mathcal{A}$ are given by

$$U(A)_\epsilon = \{B \in \mathcal{A} : ||A - B|| < \epsilon\}, \quad \epsilon > 0.$$

Bounded Operators

Definition

$\mathcal{B}(\mathscr{H}) :=$ set of all bounded, linear operators acting in a Hilbert space $\mathscr{H}$. The norm is given by

$$\|A\| = \sup_{\psi \in \mathscr{H}} \frac{\|A\psi\|}{\|\psi\|} < \infty$$

A $*$-sub-algebra of $\mathcal{B}(\mathscr{H})$ is a subset $S \subset \mathcal{B}(\mathscr{H})$ such that it is also $*$-algebra (i.e. $A \in S, \ A^* \in S$).

Definition

A $*$-sub algebra of $\mathcal{B}(\mathscr{H})$ is called a C^*-**algebra** if it is a normed $*$-algebra which is uniformly closed and whose norm satisfies additionally

$$\|A^* A\| = \|A\|^2, \qquad A \in \mathcal{A}.$$

Von Neumann Algebras

Definition

A weakly closed $*$-sub-algebra of $\mathcal{B}(\mathscr{H})$ containing the unit operator is called a **von Neumann algebra**.

Definition

The **commutant** of an arbitrary subset $S \subset \mathcal{B}(\mathscr{H})$, denoted by $S^{'}$, is the set of all bounded operators that commute with all elements of S.

Theorem

Let $S \subset \mathcal{B}(\mathscr{H})$ be a self-adjoint set. Then

(a) $S^{'}$ is a von Neumann algebra.

(b) $S^{''} \equiv (S^{'})^{'}$ is the smallest von Neumann algebra containing S

(c) $S^{'''} = S^{'}$

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Motivation for AQFT

(a) Incorporate principles of **quantum mechanics and special relativity**

(b) **Mathematical rigorous** QFT relying on fundamental principles

(c) Construct (or solve) **four-dimensional** interacting QFT!

Technical problems that formulation of spaces of states:

(a) In QFT the Stone-von Neumann theorem fails $\Rightarrow$ representation of Weyl-group on the state space non-unique $\Rightarrow$ Requiring choice of representation.

(b) Renormalization theory formulation in terms of states $\Rightarrow$ infrared problems $\Rightarrow$ Absent in Formulation in terms of observables (DuetschFredenhagen).

The general assumptions

(a) Separable Hilbert space $\mathcal{H}$ of state vectors.

(b) Unitary representation $U(a, \Lambda)$ of the Poincaré group $\mathcal{P}_+^\uparrow$ on $\mathcal{H}$

(c) Invariant, normalized state vector $\Omega \in \mathcal{H}$ (vacuum)

(d) A family of $*$-algebras $\mathcal{A}(\mathcal{O})$ of operators on $\mathcal{H}$ (a "field net"), indexed by regions $\mathcal{O} \subset \mathbb{R}^4$

(e) Isotony: $\mathcal{A}(\mathcal{O}_1) \subset \mathcal{A}(\mathcal{O}_2)$ if $\mathcal{O}_1 \subset \mathcal{O}_2$

Assumption: Operators are bounded and algebras are closed in the weak operator topology, i.e. $\Rightarrow$ von Neumann algebras.

Axioms (Haag-Kastler Axioms)

(i) ***Local (anti-)commutativity***: $\mathcal{A}(\mathcal{O}_1)$ (anti-)commutes with $\mathcal{A}(\mathcal{O}_2)$ if $\mathcal{O}_1$ and $\mathcal{O}_2$ space-like separated.

(ii) ***Covariance***: $U(a,\Lambda)\mathcal{A}(\mathcal{O})U(a,\Lambda)^{-1} = \mathcal{A}(\Lambda\mathcal{O} + a)$.

(iii) ***Spectrum condition***: The energy momentum spectrum, i.e. of the generators of the translations $U(a)$ lies in V^+.

(iv) ***Cyclicity of the vacuum***: $\cup_{\mathcal{O}}\mathcal{A}(\mathcal{O})\Omega$ is dense in $\mathscr{H}$.

Example: Free Field

$$(\Box + m^2)\phi = 0,$$

Algebra of observables generated by

$$\mathcal{A}(\mathcal{O}) := \{e^{i\phi(f)}, suppf \subset \mathcal{O}\}''$$

Theorems

(i) Reeh-Schlieder Theorem

(ii) Spin-Statistics Theorem generalized to curved space-times using AQFT (Verch01)

(iii) Bisognano-Wichmann Theorem

Reeh-Schlieder Theorem

Additional assumption on $\mathcal{A}(\mathcal{O})$, weak additivity:
For every fixed open set $\mathcal{O}_0$ the algebra generated by the union of all translates, $\mathcal{A}(\mathcal{O}_0 + x)$, is dense in the union of all $\mathcal{A}(\mathcal{O})$ in the w. o. t.

Theorem

Under the assumption of weak additivity, $\mathcal{A}(\mathcal{O})$ is dense in the Hilbert space $\mathscr{H}$ for all open sets $\mathcal{O} \Rightarrow \Omega$ is cyclic and separating for every local algebra $\mathcal{A}(\mathcal{O})$. (separating $A\Omega = 0 \Rightarrow A = 0$)

Proof.

Pick $\mathcal{O}_0 \subset \mathcal{O}$ such that $\mathcal{O}_0 + x \subset \mathcal{O}$ for all x with $|x| < \epsilon$, for some $\epsilon > 0$.
If $\Psi \perp \mathcal{A}(\mathcal{O})\Omega$ then $\langle \Psi, U(x_1)A_1 U(x_2 - x_1) \cdots U(x_n - x_{n-1})A_n\Omega \rangle = 0$ for all $A_i \in \mathcal{O}_0$ and $|x_i| < \epsilon$. Analyticity of $U(a) \Rightarrow \forall x_i$. Theorem follows by weak additivity. $\square$

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Tomita-Takesaki modular theory

Ingredients : a von Neumann algebra $\mathcal{A}$ together with a cyclic and separating vector Ω. To every such pair

(i) Define an anti-linear operator $S : \mathcal{A}\Omega \to \mathcal{A}\Omega$ by

$$SA\Omega = A^*\Omega.$$

S is well defined on a dense set in $\mathscr{H}$ since Ω is separating and cyclic.

(ii) It has a polar decomposition $S = J\Delta^{1/2} = \Delta^{-1/2}J$ with the modular operator $\Delta = S^*S > 0$ and the anti-unitary modular conjugation J with $J^2 = 1$.

Theorem: Modular group and KMS-condition

$$\Delta^{it}\mathcal{A}\Delta^{-it} = \mathcal{A}, \qquad \forall t \in \mathbb{R}, \qquad\qquad J\mathcal{A}J = \mathcal{A}^{'}$$

Moreover, for $A, B \in \mathcal{A}$

$$\langle \Omega, AB\Omega \rangle = \langle \Omega, B\Delta^{-1}A\Omega \rangle$$

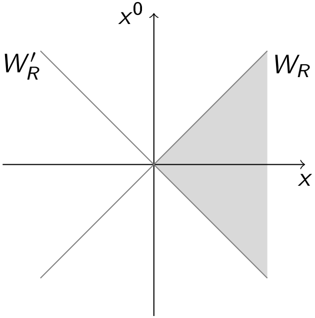
$\Rightarrow$ Equivalent to the Kubo-Martin Schwinger (KMS) condition that characterizes thermal equilibrium states w.r.t. "time" evolution

$$A \rightarrow \alpha_t(A) := \Delta^{it} A \Delta^{-it} = A$$

on $\mathcal{A}$.

Wedge

A space-like wedge W is, by definition, a Poincaré transform of the standard wedge $W_R = \{x \in \mathbb{R} : |x_0| < x_1\}$, i.e.



To W is associated a one-parameter family $\Lambda_W(s)$ of Lorentz boosts that leave W invariant and a reflection j_W, that maps W into the opposite wedge : product of space-time inversion θ and a rotation $R(\pi)$ around the 1-axis.

Bisognano-Wichmann Theorem

Consider algebras $\mathcal{A}(W)$ with vacuum Ω as cyclic and separating vector. The modular objects Δ and J associated with $(\mathcal{A}(W), \Omega)$ depend on W but it is sufficient to consider W_R.

BW75 discovered Δ and J are related to the representation U of the Lorentz group and the PCT operator θ as:

Theorem Bisognano-Wichmann

$$J = \theta\, U(R(\pi)), \qquad \Delta^{it} = U(\Lambda_{W_R}(2\pi t))$$

$\Longrightarrow$ Modular localization associates a localization structure with any (anti-)unitary representation of $P_+^\uparrow$ satisfying the spectrum condition:

Weyl quantization generates naturally a local net satisfying all the axioms of a AQFT!

Example: Free Bosonic Field

Let U be an (anti-)unitary representation of $P_+^\uparrow$ satisfying the spectrum condition on $\mathscr{H}_1$. For a wedge W, let Δ_W be

$$\Delta_W^{it} = U(\Lambda_W(2\pi t))$$

and let J_W be the anti–unitary involution representing j_W, define:

$$S_W := J_W \Delta_W^{1/2}.$$

The space

$$K(W) := \{\phi \in \mathrm{domain}\, \Delta_W^{1/2} : S_W\phi = \phi\} \subset \mathscr{H}_1$$

satisfies:

(i) $K(W)$ is a closed real subspace of $\mathscr{H}_1$ in the real sp

(ii) $K(W) \cap iK(W) = \{0\}$ and $K(W) + iK(W)$ is dense in $\mathscr{H}_1$.

(iii) $K(W)^\perp := \{\psi \in \mathscr{H}_1 : Im\langle\psi, \phi\rangle = 0,\ \forall\phi \in K(W)\} = K(W^{'})$

Weyl-Quantization

The functorial procedure of Weyl (second-) quantization leads for any $\psi \in \bigcap_W K(W)$ to an (unbounded) field operator $\Psi(\phi)$ on the Fock space

$$\mathscr{H} = \oplus_{n=0}^{\infty} \mathscr{H}_1^{\otimes_s ymm}$$

such that

$$[\Psi(\psi), \Psi(\phi)] = iIm\langle\psi, \phi\rangle.$$

In particular,

$$[\Psi(\psi), \Psi(\phi)] = 0, \qquad \psi \in K(W),\, \phi \in K(W^{'})$$

Finally, a net of algebras $\mathscr{A}$ satisfying the axioms is defined by $\mathcal{A}(\mathcal{O}) := \{exp(i\Psi(\phi)) : \psi \in \bigcap_{\mathcal{O}\subset W} K(W)\}^{''}$.

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([BFV03]) : A local covariant quantum field theory is a functor from the category of globally hyperbolic spacetimes, with isometric hyperbolic embeddings as arrows, to the category of $*-$algebras, with monomorphisms as arrows.

AQFT in terms of Category Theory

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What the heck???

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Definitions I

Morphism

A structure-preserving map from one mathematical structure to another.

Homomorphism

A structure-preserving map between two algebraic structures of the same type.

Monomorphism

An injective homomorphism or a left-cancellative morphism, that is, an arrow $f : X \to Y$ such that, for all morphisms $g_1, g_2 : Z \to X$,

$$f \circ g_1 = f \circ g_2 \Rightarrow g_1 = g_2.$$

Definitions II

Category C, (ob(C), arrows)

(i) A class of objects denoted by ob(C)

(ii) A class hom(C) of morphisms, s.t. $\forall f$ has a source a and a target object b where $a, b \in$ ob(C), i.e. $f : a \to b$

(iii) For $a, b, c \in$ ob(C), $\exists$ a binary operation hom$(a, b) \times$ hom$(b, c) \to$ hom(a, c) (composition); s.t.

  (i) (associativity) if $f : a \to b, g : b \to c$ and $h : c \to d$ then $h \circ (g \circ f) = (h \circ g) \circ f$,

  (ii) (identity) for every object x, $\exists$ morphism $1_x : x \to x$ called identity morphism for x

Definitions III

Functor

Let C and D be categories. A **functor** F from C to D is a mapping that associates to each object X in C an object $F(X)$ in D and associates to each morphism $f : X \to Y$ in C a morphism $F(f) : F(X) \to F(Y)$ in D s.t:

(i) $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$ for every object X in C,

(ii) $F(g \circ f) = F(g) \circ F(f)$ for all morphisms $f : X \to Y$ and $g : Y \to Z$ in C.

Functors must preserve identity morphisms and composition of morphisms.

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Globally Hyperbolic Spacetimes???

Definitions IV

Globally hyperbolic spacetime (M, g)

M a smooth, four-dimensional, orientable and time-orientable MF!

Time-orientability: $\exists\, C^\infty$-VF u on M s.t. $g(u, u) > 0$.

A smooth curve $\gamma : I \to M$, I being a connected subset of $\mathbb{R}$, is **causal** if $g(\dot\gamma, \dot\gamma) \geq 0$. A CC is future directed if $g(\dot\gamma, u) > 0$ and past directed if $g(\dot\gamma, u) < 0$. For any point $x \in M$, $J^\pm(x)$ denotes the set of all points in M which can be connected to x by a future$(+)$/past $(-)$-directed causal curve. M is **globally hyperbolic** if for $x, y \in M$ the set $J^-(x) \cap J^+(y)$ is compact if non-empty.

Intuitively: The spacetime has a Cauchy surface!

Advantage of GHST: Cauchy-problem for linear hyperbolic wave-equation is well-posed.

Isometric Embedding

Let (M_1, g_1) and (M_2, g_2) be two globally hyperbolic spacetimes. A map $\psi : M_1 \to M_2$ is called an **isometric** embedding if ψ is a diffeomorphism onto its range $\psi(M)$, i.e. $\bar{\psi} : M_1 \to \psi(M_1) \subset M_2$ is a diffeomorphism and if ψ is an isometry, that is, $\psi_* g_1 = g_2 \upharpoonright \psi(M_1)$.

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Man: Class of all objects Obj(Man) formed by globally hyperbolic spacetimes (M, g). Given two such objects (M_1, g_1) and (M_2, g_2), the morphisms $\psi \in \mathrm{hom}_{Man}((M_1, g_1), (M_2, g_2))$ are taken to be the isometric embeddings $\psi : (M_1, g_1) \to (M_2, g_2)$ of (M_1, g_1) into (M_2, g_2) as defined above, but with constraint :

> The isometric embedding preserves orientation and time-orientation of the embedded spacetime.

Alg: Category class of objects Obj(Alg) formed by all C^* -algebras possessing unit elements, and the morphisms are faithful (injective) unit-preserving $*$-homomorphisms. For $\alpha \in \mathrm{hom}_{Alg}(\mathcal{A}_1, \mathcal{A}_2)$ and $\alpha' \in \mathrm{hom}_{Alg}(\mathcal{A}_2, \mathcal{A}_3)$ the composition $\alpha \circ \alpha' \in \mathrm{hom}_{Alg}(\mathcal{A}_1, \mathcal{A}_3)$.

Locally covariant quantum field theory

(i) LCQFT is a covariant functor $\mathcal{A}$ between the two categories *Man* and *Alg*, i.e., writing α_ψ for $\mathcal{A}(\psi)$:

$$\begin{tikzcd}
(M,g) \arrow[r, "\psi"] \arrow[d, "\mathscr{A}"'] & (M',g') \arrow[d, "\mathscr{A}"] \\
\mathscr{A}(M,g) \arrow[r, "\alpha_\psi"] & \mathscr{A}(M',g')
\end{tikzcd}$$

together with the covariance properties

$$\alpha_{\psi'} \circ \alpha_\psi = \alpha_{\psi' \circ \psi}, \qquad \alpha_{\mathrm{id}_M} = \mathrm{id}_{\mathcal{A}(M,g)},$$

for all morphisms $\psi \in \hom_{Man}((M_1, g_1), (M_2, g_2))$, all morphisms $\psi' \in \hom_{Man}((M_2, g_2), (M_3, g_3))$ and all $(M,g) \in \mathrm{Obj}(Man)$.

(ii) A LCQFT described by a covariant functor $\mathcal{A}$ is called causal if: There are morphisms $\psi_j \in \mathrm{hom}_{Man}((M_j, g_j), (M, g)), j = 1, 2$, so that $\psi_1(M_1)$ and $\psi_2(M_2)$ are causally separated in (M, g), then

$$[\alpha_{\psi_1}(\mathcal{A}(M_1, g_1)), \alpha_{\psi_2}(\mathcal{A}(M_2, g_2))] = 0,$$

(iii) We say that a locally covariant quantum field theory given by the functor $\mathcal{A}$ obeys the **time-slice axiom** if

$$\alpha_\psi(\mathcal{A}(M, g)) = \mathcal{A}(M', g')$$

holds for all $\psi \in \mathrm{hom}_{Man}((M, g), (M', g'))$ such that $\psi(M)$ contains a Cauchysurface for (M', g').

Example KG-field

Global hyperbolicity entails the well-posedness of the Cauchy-problem for the scalar Klein-Gordon equation on (M, g),

$$(\nabla^a \nabla_a + m^2 + \xi R)\varphi = 0$$

Let $E = E_{adv} - E_{ret}$ be the causal propagator of the Klein-Gordon equation and the range of $E(C_0^\infty(M, \mathbb{R}))$ is denoted by $\mathcal{R}$. By defining

$$\sigma(Ef, Eh) = \int_M f(Eh) d\mu_g, \qquad f, h \in C_0^\infty(M, \mathbb{R})$$

it endows $\mathcal{R}$ with a symplectic form, and thus $(\mathcal{R}, \sigma)$ is a symplectic space. $\Rightarrow$ Weyl-algebra $\mathcal{W}(\mathcal{R}, \sigma)$, generated by $W(\phi)$, $\phi \in \mathcal{R}$ satisfying

$$W(\phi)W(\psi) = e^{-i\sigma(\phi,\psi)} W(\phi + \psi).$$

Theorem

If one defines for each $(M,g) \in Obj(Man)$ the C^* -algebra $\mathscr{A}(M,g)$ as the CCR-algebra $\mathcal{W}(\mathcal{R}(M,g),\sigma(M,g))$ of the Klein-Gordon equation and for each $\psi \in hom_{Man}(M,M^{'})$ the C^*-algebraic endomorphism $\alpha_\psi = \tilde{\alpha}_{\iota_\psi} \circ \tilde{\alpha}_\psi : \mathscr{A}(M,g) \to \mathscr{A}(M',g')$ according to (1) and (2), then one obtains in this way a covariant functor $\mathscr{A}$ with the properties of the definitions above. Moreover, this functor is causal and fulfills the time-slice axiom.

In this sense, the free Klein-Gordon FT is a locally covariant QFT.

Thus, a locally covariant quantum field theory is an assignment of C^*-algebras to (all) globally hyperbolic spacetimes so that the algebras are identifiable when the spacetimes are isometric, in the indicated way.