Observable currents for effective field theories and their context

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— Outline —

Context

Scale

Coarse graining

Effective Field Theories

- Simplicial first order effective field bundle
- Variational principle, field eqs. and geometric structure

Observable currents

- Definition
- Can observable currents distinguish different solutions?
- Locally hamiltonian vector fields
- Observable currents from LHVFs
- Poisson brackets among observable currents

Summary and remarks

Context

Field theories – for a large class of systems –
GBFT: covariant, local
(spacetime M differentiable manifold, not necessarily metric)

Effective Field Theory (EFT) at a given scale
$\{\mathsf{EFT}_{\mathsf{Sc}}\}_{\mathsf{Scales\ in\ } M} \quad \longrightarrow \quad \mathsf{EFT}_M$

Construct $\mathsf{EFT}_{\mathsf{Sc}}$ as the limit of a correction procedure

$$\mathsf{EFT}_{\mathsf{Sc}} = \lim_{\mathsf{Sc}' \to M} \mathsf{CorrFT}_{\mathsf{Sc}\ (\mathsf{Sc'})}$$

where $\mathsf{CorrFT}(\beta_{\mathsf{Sc}\ (\mathsf{Sc'})}) = RG(\mathsf{PrimeFT}(\beta_{\mathsf{Sc'}}))$

Key concepts:
Scale, coarse graining, $\mathsf{EFT}_{\mathsf{Sc}}$, observables, GBFT

Scale

A history ϕ is a local section of $Y \xrightarrow{\pi} M$.

In a lagrangian formulation, $L = L(x, \phi, D\phi)$, we need **Partial Observables** that talk about $J^1 Y \ni (x, \phi, D\phi)$.

Measuring scale $\longleftrightarrow$ discrete collection of measuring devises

Definition A **scale** is a faithful structure of local subalgebras: to every open set $U \subset M$ corresponds a subalgebra

$$PO_\Delta(U) \sim C^\infty(\pi^{-1} U, \mathbb{R} \text{ or } \mathbb{C}) \subset PO_M$$

such that $\{\mathrm{Eval}_{\phi_\alpha} : PO_\Delta(U_\alpha) \to \mathbb{R} \text{ or } \mathbb{C}\}_{U_\alpha \subset M}$ determines:
(i) the bundle $Y \xrightarrow{\pi} M$ up to equivalence
(ii) each $\phi_\alpha \in \Gamma(Y)$ up to "microscopical details" (homotopy relative to Eval)
–

A notion of kth order agreement of sections leads to $J^k Y_\Delta$

Coarse graining

Definition $\Delta' \geq \Delta$ means that there is a coarse graining map cg consisting of an assignment of a a homomorphism $cg(U)$ to every open set $U \subset M$

$$PO_{\Delta}(U) \xrightarrow{cg(U)} PO_{\Delta'}(U)$$

Scale: topological motivation and implications

Topological motivation:
In discrete approaches to GR, like Regge calculus, we rely on the fact that spacetime's topology can be stored in the discrete structure of a triangulation.
Similarly, the bundle structure of the space where histories live, $J^1 Y$, should be storable in a discrete manner.

Topological implications of this definition of scale:

- "Δ-microscopical" variations of a Δ-history do not tear
- A classical variational problem in a given bundle at scale Δ makes sense
- Coarse graining from scale $\Delta' \geq \Delta$ by summing over Δ-indistinguishable histories is a sum over histories in a well defined bundle

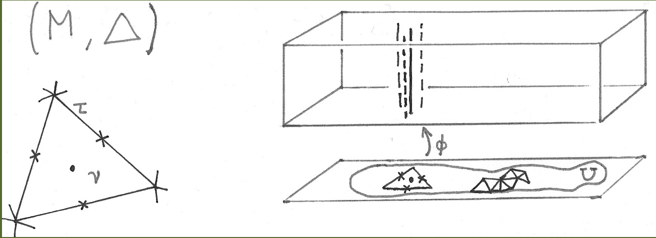
Ex.1) 1st order scalar field theory with $\mathcal{F} = \mathbb{R}^k$

Scale defined with the aide of a triangulation, $M \to (M, \Delta)$

$x \xmapsto{j^1\phi} j^1\phi(x) = (x, \phi, D\phi)$ decimated to

$\nu \xmapsto{\tilde{\phi}} \tilde{\phi}(\nu) = (x(C\nu), \phi_\nu \in \mathcal{F}, \{x(C\tau), \phi_\tau \in \mathcal{F}\}_{\tau \subset \partial\nu})$ or

$\nu \xmapsto{\tilde{\phi}} \tilde{\phi}(\nu) = (\nu, \phi_\nu \in \mathcal{F}, \{\phi_\tau \in \mathcal{F}\}_{\tau \subset \partial\nu})$



Ex.2) Sigma models; scalar fields on a G-principal bundle

A family of local sections to local trivializations

$$\{\phi_\alpha : U_\alpha \to U_\alpha \times G\}_{U_\alpha \subset M}$$

determines the transition functions $g_{\alpha\beta}(x) = (g_\alpha(x))^{-1} g_\beta(x)$

Decimated local sections give **partial information** about the transition functions

$$\nu \xmapsto{\tilde{\phi}} (x(C\nu), g_\nu \in G; \{x(C\tau^{n-1}), g_{\tau^{n-1}} \in G\}_{\tau^{n-1} \subset (\partial\nu)^{n-1}};$$
$$...; \{x(C\tau^0), g_{\tau^0} \in G\}_{\tau^0 \subset (\partial\nu)^0};$$
$$W = \{ \text{ h. type of } (\phi_\nu^{-1}\phi_\tau)|_\tau\}_{\tau \subset \partial\nu})$$

determines the bundle up to equivalence
and the history up to homotopy

Ex.3) gauge fields on a G-principal bundle

A connection on a G-bundle determines a holonomy homom.

$$H_A : \mathcal{L}_{\star,b} \to G$$

A homom. H satisfying certain smoothness conditions determines
(i) a principal bundle (up to equivalence) and
(ii) a connection up to gauge [Barrett 1991]

A decimated parallel transport (semigroup) homom.

$$\nu \overset{\tilde{A}}{\longmapsto}$$

$$(x(C\nu), \{h_l \in G\}_{l\subset\nu}; \{x(C\tau^{n-1}), \{k_{r_{n-1}} \in G\}_{r\subset\tau^{n-1}}\}_{\tau^{n-1}\subset(\partial\nu)^{n-1}};$$
$$...; \{x(C\tau^0)\}_{\tau^0\subset(\partial\nu)^0}; W = \{\text{ h. type of gluing } \nu \text{ and } \tau \text{ loc. triv. }\}_{\tau\subset\partial\nu})$$

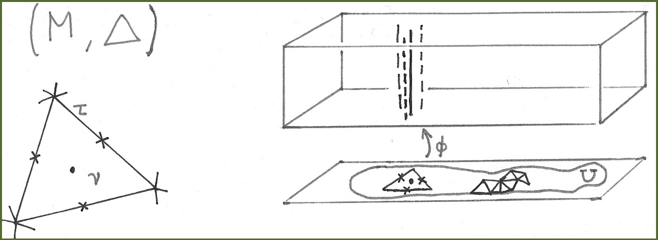
determines a principal bundle (up to equivalence)
and a connection up to homotopy and gauge ["Local gauge theory and coarse graining", Z. 2011]

Geometric framework for classical field theories at scale Δ

Prologue

- The **first order effective field bundle**, $J^1 Y_\Delta$, is a finite dimensional manifold (with the str. of a fiber bundle over a simplicial complex)
- Local objects are defined on $J^1 Y_\Delta$
- Histories are local sections, among them we have "solutions"
- **Geometric structure** emerges as relations among local objects that hold when evaluated on "solutions"

Simplicial first order effective field bundle



Decimated local record of a history in 1st order format

$$\nu \xmapsto{\tilde{\phi}} \tilde{\phi}(\nu) = (\nu, \phi_\nu \in \mathcal{F}, \{\phi_\tau \in \mathcal{F}\}_{\tau \subset \partial\nu})$$

A variation $\delta\tilde{\phi}(\nu) = \tilde{v}(\nu) = (v_\nu \in T_{\phi_\nu}\mathcal{F}, \{v_\tau \in T_{\phi_\tau}\mathcal{F}\}_{\tau \subset \partial\nu})$

Notation: (M, Δ), $\nu \in U^n_\Delta$, $\tau \in (\partial U)^{n-1}_\Delta$, or $\tau \in U^{n-1}_\Delta$,
$\tilde{\phi}(\nu) \in J^1 Y_\Delta$, $\tilde{\phi} \in \mathrm{Hists}_U$, $\tilde{v} \in T_{\tilde{\phi}}\mathrm{Hists}_U$, or $\tilde{v} \in \mathfrak{X}(J^1 Y_\Delta|_U)$

Variational principle, field eqs. and geometric structure

$$S(\tilde{\phi}) = \sum_{\nu \in U^n_\Delta} L(\tilde{\phi}(\nu))$$

$\Rightarrow$

$$dS(\tilde{\phi})[\tilde{v}] = \sum_{U - \partial U} \tilde{\phi}^* \; i_{\tilde{v}} \mathrm{E}_L + \sum_{\partial U} \tilde{\phi}^* \; i_{\tilde{v}} \Theta_L$$

where

$$\Theta_L(\cdot, \tilde{\phi}(\tau_\nu)) = \frac{\partial L}{\partial \phi_\tau}(\tilde{\phi}(\nu)) d\phi_\tau \quad \text{[1 form, n-1 cochain] on } J^1 Y_\Delta,$$

$$\mathrm{E}_L(\cdot, \tilde{\phi}(\nu)) = \frac{\partial L}{\partial \phi}(\tilde{\phi}(\nu)) d\phi_\nu + \sum_{\tau \in (\partial \nu)^{n-1}} \frac{\partial L}{\partial \phi_\tau}(\tilde{\phi}(\nu)) d\phi_\tau$$

Hamilton's principle: **(i)** field equations, **(ii)** geometric str.
Field eqs: **(i.a)** internal to each ν,
(i.b) gluing (momentum matching) at each $\tau = \nu \cap \nu'$

† Sigma models and gauge theory also available

The (pre)multisymplectic form

$$\Omega_L(\tilde{v}(\nu), \tilde{w}(\nu), \tilde{\phi}(\tau_\nu)) \doteq -d(\Theta_L|_{\tilde{\phi}(\tau_\nu)})(\tilde{v}(\nu), \tilde{w}(\nu))$$

assigns (pre)symplectic structures to spaces of data over codimension 1 domains $\Sigma \mapsto \Omega_\Sigma$

$$\Omega_{\Sigma,\tilde{\phi}}(\tilde{v}, \tilde{w}) = \sum_{\Sigma} \tilde{\phi}^* \; i_{\tilde{w}} i_{\tilde{v}} \Omega_L$$

E.g. scalar field Σ spacelike $\Omega_{\Sigma,\tilde{\phi}}(\tilde{v}, \tilde{w}) = \frac{2k}{h} \sum_{\Sigma} d\phi_\nu \wedge d\phi_\tau(\tilde{v}, \tilde{w})$

Given any $\tilde{\phi} \in \mathrm{Sols}_U$, $\tilde{v}, \tilde{w} \in T_{\tilde{\phi}}\mathrm{Sols}_U$ and $U' \subset U$
the multisymplectic formula holds:

$$\sum_{\partial U'} \tilde{\phi}^* \; i_{\tilde{w}} i_{\tilde{v}} \Omega_L = 0$$

Proof.
$0 = -ddS = -d(\sum_{\partial U} \tilde{\phi}^* \; \Theta_L) = \sum_{\partial U} \tilde{\phi}^* \; \Omega_L$

The space of first variations

Consider $\tilde{\phi} \in \mathrm{Sols}_U$.
First variations of $\tilde{\phi}$ are elements of $T_{\tilde{\phi}}\mathrm{Sols}_U \subset T_{\tilde{\phi}}\mathrm{Hists}_U$, and may be induced by vector fields on $J^1 Y_\Delta$.

- They are characterized by satisfying $\mathcal{L}_{\tilde{v}}\mathrm{E}_L = 0$
  (Recall $dS(\tilde{\phi}) = \sum_{U-\partial U} \tilde{\phi}^* \, \mathrm{E}_L + \sum_{\partial U} \tilde{\phi}^* \, \Theta_L$)
- They define a lagrangian[2] subspace of $\Omega_{\partial U', \tilde{\phi}}$ for all $U' \subset U$

[2] qua qua qua, la la la, blu blu blu qua qua qua, la la la, blu blu blu

Observable currents

$F \in \mathrm{OC}_U$ iff it is an n-1 cochain on $J^1 Y_\Delta$ ·st· $\forall\ \tilde{\phi} \in \mathrm{Sols}_U$

$$F(\tilde{\phi}(\tau_\nu)) \doteq F(\tau, \phi_\tau, \phi_\nu) = -F(\tilde{\phi}(\tau_{\nu'})) = F(-\tau, \phi_\tau, \phi_{\nu'}),$$

$$\sum_{\partial U'} \tilde{\phi}^*\ F = 0 \qquad \forall\ \ U' \subset U$$

Observables

$$Q_{F,\Sigma}(\tilde{\phi}) \doteq \sum_{\Sigma} \tilde{\phi}^*\ F$$

Notice that if Σ' is homologous to Σ and $\tilde{\phi} \in \mathrm{Sols}_U$

$$Q_{F,\Sigma'}(\tilde{\phi}) - Q_{F,\Sigma}(\tilde{\phi}) = Q_{F,\Sigma'-\Sigma}(\tilde{\phi}) = Q_{F,\partial U'}(\tilde{\phi}) = 0$$

Notice that OC_U is a vector space.

† Sigma models and gauge theory also available

Can observable currents distinguish neighboring solutions?

Consider a curve of solutions $\gamma(s) \in \mathrm{Sols}_U$ with

$$\gamma(0) = \tilde{\phi} \in \mathrm{Sols}_U \ , \quad \dot{\gamma}(0) = \tilde{w} \in T_{\tilde{\phi}}\mathrm{Sols}_U .$$

** Is OC_U large enough to resolve $T_{\tilde{\phi}}\mathrm{Sols}_U$? **
$Q_{F,\Sigma}$ separates $\tilde{\phi}$ from nearby solutions in γ if

$$\frac{d}{ds}|_{s=0} Q_{F,\Sigma}(\gamma(s)) = \sum_{\Sigma} \tilde{\phi}^* \ dF[\tilde{w}] \neq 0$$

If the observable current has an associated hamiltonian vector field

$$dF = -i_{\tilde{v}}\Omega_L$$

(let us call such an OC a hamiltonian OC, $F \in \mathrm{HOC}_U$)
the separability condition reads

$$\sum_{\Sigma} \tilde{\phi}^* \ i_{\tilde{w}} i_{\tilde{v}} \Omega_L \neq 0.$$

Observable currents distinguish neighboring solutions

Separability measuring in the bulk

Assume Ω_L is non degenerate. Then for any $\tilde{\phi} \in \mathrm{Sols}_U$ there is a hamiltonian OC F that can be used to separate $\tilde{\phi}$ from any neighboring solution.

Sketch of proof.

Given any non constant curve $\gamma(s) \in \mathrm{Sols}_U$ as above,
Ω_L non deg. $\Rightarrow \exists\ \tilde{v}$ and $\tau \subset U$ ·st·
$\Omega_L(\tilde{v}, \tilde{w} = \dot{\gamma}(0), \tilde{\phi}(\tau)) \neq 0$. Construct F from $\tilde{v}$.

Separability measuring in the boundary

Assume Ω_L satisfies a non deg. condition. Then for any $\tilde{\phi} \in \mathrm{Sols}_U$ there is $F \in \mathrm{HOC}_U$ that separates $\tilde{\phi}$ from any neighboring solution measuring at $\Sigma \subset \partial U$.

Sketch of proof.

Ω_L non deg.' $\Rightarrow \exists\ \tilde{v}$ and $\Sigma' \subset U$ with $\partial\Sigma' \subset \partial U$ ·st·
$\frac{d}{ds}|_{s=0} Q_{F,\Sigma'}(\gamma(s)) = -\sum'_{\Sigma} \tilde{\phi}^*\ i_{\tilde{w}} i_{\tilde{v}} \Omega_L \neq 0$.
F may be measured at $\Sigma \subset \partial U$ ·st· $\Sigma' - \Sigma = \partial U'$.

Locally hamiltonian vector fields

We will investigate the space of hamiltonian observable currents.

Hamiltonian (or exact) vector fields

If
$$- i_{\tilde{v}}\Omega_L = dF$$

$\tilde{v}$ is said to be a **hamiltonian vector field** for F.
$\tilde{v} \in \mathrm{Ha}(J^1 Y_\Delta|_U) \subset \mathfrak{X}(J^1 Y_\Delta|_U)$ and $F \in \mathrm{HOC}_U \subset \mathrm{OC}_U$.

Locally hamiltonian (or closed) vector fields

If $\; -i_{\tilde{v}}\Omega_L \doteq \sigma_{\tilde{v}} \;$ with

$$d\sigma_{\tilde{v}} = 0 \quad \text{and} \quad \sum_{\partial U'} \tilde{\phi}^* \; i_{\tilde{w}}\sigma_{\tilde{v}} = 0$$

for all $U' \subset U$ and $(\tilde{w}, \tilde{\phi}) \in T\mathrm{Sols}_U$,
$\tilde{v}$ is said to be a **locally hamiltonian vector field**.
$\tilde{v} \in \mathrm{LHa}(J^1 Y_\Delta|_U) \subset \mathfrak{X}(J^1 Y_\Delta|_U)$.

Conditions for a vector field to be locally hamiltonian

$$
\begin{array}{rcl}
d\sigma_{\tilde{v}} = 0 & \Longleftrightarrow & \mathcal{L}_{\tilde{v}}\Omega_L = 0 \\
\sum_{\partial U'} \tilde{\phi}^* \, i_{\tilde{w}}\sigma_{\tilde{v}} = 0 \quad \forall U', \tilde{w} & \Longleftarrow & \mathcal{L}_{\tilde{v}}\mathrm{E}_L = 0 \\
\sum_{\partial U'} \tilde{\phi}^* \, i_{\tilde{w}}\sigma_{\tilde{v}} = 0 \quad \forall U', \tilde{w} & \Longrightarrow^{\dagger} & \mathcal{L}_{\tilde{v}}\mathrm{E}_L = 0
\end{array}
$$

All evaluated at a $\tilde{\phi} \in \mathrm{Sols}_U$.

Notice that if $\mathcal{L}_{\tilde{v}}\Omega_L = 0$ holds at Σ,
the multisymplectic formula implies that it also holds at any $\Sigma' = \Sigma + \partial U'$ if $\mathcal{L}_{\tilde{v}}\mathrm{E}_L = 0$ holds inside U'.

$\Longrightarrow$ The bulk condition is $\mathcal{L}_{\tilde{v}}\mathrm{E}_L = 0$ (i.e. $\tilde{v} \in T_{\tilde{\phi}}\mathrm{Sols}_U$)

† If $T_{\tilde{\phi}}\mathrm{Sols}_U$ defines a lagrangian subspace of $\Omega_{\partial U', \tilde{\phi}}$ for all $U' \subset U$

Observable currents and locally hamiltonian vector fields

- Some closed 1-forms may be integrated, revealing that they are exact. This is the subject of the next slide.

$$\mathrm{LHa}(J^1 Y_\Delta|_U) \supset \mathrm{Ha}(J^1 Y_\Delta|_U)$$

- If $\Omega_L(\cdot, \cdot; \tilde{\phi}(\tau_\nu))$ is non degenerate $\forall \tau_\nu \in U$

$$0 \longrightarrow \mathrm{OC}_U \xrightarrow{\Omega_L^{-1}} \mathrm{Ha}(J^1 Y_\Delta|_U)$$

This contrasts with Multisymplectic Field Theory in the continuum, where the n+1 form Ω_L is not invertible.
The situation is closer to initial data formulations of field theory where the symplectic form is invertible.
The difference arises from the fact that in the discrete setting there is a predetermined set of codimension 1 faces on which Ω_L may be evaluated to induce a (collection of) 2 forms.

Observable currents from LHVFs

$\tilde{v} \in \mathrm{LHa}(J^1 Y_\Delta|_U)$ induces $\sigma_{\tilde{v}}$,
integration on the fibers may lead to $F_{\tilde{v},K} \in \mathrm{OC}_U$.
Integration requires the choice of
a system of integration constants K;
an allowed choice of integration constants implies

$$\sum_{\partial U'} \tilde{\phi}^* \, F_{\tilde{v},K} = 0 \qquad \forall \, \tilde{\phi} \in \mathrm{Sols}_U, \;\; U' \subset U$$

Adding a closed n-1 cochain C in U to a system of allowed integration constants K yields a new system of allowed integration constants $K' = K + C$.
$F_{\tilde{v},K} \in \mathrm{OC}_U$ and its physical meaning are determined by $\tilde{v}$ and K.

OC_U is in correspondence with $T\mathrm{Sols}_U$;
when Ω_L is non deg. the corresp. is roughly 1 to 1
making OCs capable of separating neighboring solutions.

Poisson brackets among observable currents

Given two observable currents $F_{\tilde{v},K}, G_{\tilde{w},L} \in \mathrm{OC}_U$
their Poisson bracket is another observable current

$$\{F_{\tilde{v},K}, G_{\tilde{w},L}\}(\tilde{\phi}(\tau_\nu)) = \Omega_L(\tilde{w}, \tilde{v}, \tilde{\phi}(\tau_\nu))$$

whose hamiltonian vector field is $[\tilde{v}, \tilde{w}]$.

Recall $\Omega_L = -d\Theta_L$ and $dS|_{\mathsf{Sols}} = \sum_{\partial U} \Theta_L$
Then $\{F, \cdot\}$ is related to the variation of a solution $\tilde{\phi}$ induced by
$S_U(\tilde{\phi}) \to S_{U,\lambda}(\tilde{\phi}) = S(\tilde{\phi}) + \lambda \sum_{\partial U} F\tilde{\phi}$.

Similar considerations for bulk obs. lead to Peierls' bracket.
Peierls' bracket defines an equivalence relation among bulk obs.
Using our bracket, the equivalence relation extends to bdary obs.
In fact, bdary obs. may be used to label equiv. classes of bulk obs.

Summary and remarks

- The concepts of:
  scale,
  coarse graining,
  field theory (at a given scale) and
  observable currents (at a given scale)
  were studied in the GBFT spirit for **classical theories**

A path integral quantization takes these concepts to quantum GBFT – spin foam models –. There are some caveats:

- For theories with gauge symmetries Ω_L is degenerate
  derived structures like measures, inner products, etc do have a kernel
  (if an appropriate quotient is taken nondegeneracy is restored)
- Observable currents in general lead to

$$\hat{Q}_{F,\Sigma} = \hat{Q}_{F,\Sigma'} + \hat{R}_{F,B} \quad \text{for } \Sigma - \Sigma' = \partial B$$

  (the classical property holds only when eval. on solutions)

Thank you for your attention!