Observable currents for discrete field theories

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— Outline —

Context

Field theories – for a large class of systems –
Covariant and local (GBFT)
(spacetime M differentiable manifold, NOT necessarily metric)

Effective Field Theory (EFT) at a given scale
$\{\mathsf{EFT}_{\mathsf{Sc}}\}_{\mathsf{Scales\ in}\ M} \quad \longrightarrow \quad \mathsf{EFT}_M$

Construct $\mathsf{EFT}_{\mathsf{Sc}}$ as the limit of a correction procedure

$$\mathsf{EFT}_{\mathsf{Sc}} = \lim_{\mathsf{Sc}' \to M} \mathsf{CorrFT}_{\mathsf{Sc}\ (\mathsf{Sc'})}$$

where $\mathsf{CorrFT}(\beta_{\mathsf{Sc}\ (\mathsf{Sc'})}) = RG(\mathsf{PrimeFT}(\beta_{\mathsf{Sc'}}))$

Key concepts:
Scale, coarse graining, $\mathsf{EFT}_{\mathsf{Sc}}$, <u>observables</u>, spacetime “locality”

Scale

A history ϕ is a local section of $Y \xrightarrow{\pi} M$.

In a lagrangian formulation, $L = L(x, \phi, D\phi)$, we need **Partial Observables** that talk about $J^1 Y \ni (x, \phi, D\phi)$.

Measuring scale $\longleftrightarrow$ discrete collection of measuring devises

Definition A **scale** is a faithful structure of local subalgebras: to every chart $U \in \mathcal{C}$ corresponds a subalgebra

$$PO_\Delta(U) \sim C^\infty(U \times \mathcal{F}, \mathbb{R} \text{ or } \mathbb{C})$$

such that $\{\mathrm{Eval}_{\phi_\alpha} : PO_\Delta(U_\alpha) \to \mathbb{R} \text{ or } \mathbb{C}\}_{U_\alpha \in \mathcal{C}}$ determines:
(i) the bundle $Y \xrightarrow{\pi} M_{\mathcal{C}}$ up to equivalence
(ii) each $\phi_\alpha \in \Gamma(Y)$ up to "microscopical details" (homotopy relative to Eval)
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A notion of kth order agreement of sections leads to $J^k Y_\Delta$

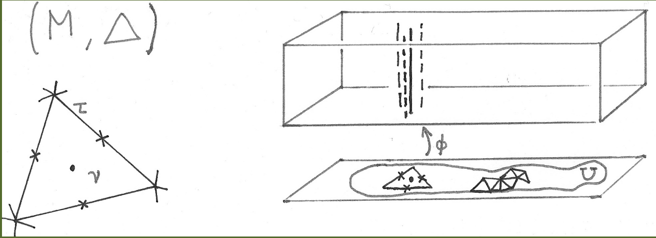
Ex.1) 1st order scalar field theory with $\mathcal{F} = \mathbb{R}^k$

Scale defined with the aide of a triangulation, $U \subset M \to (U, \Delta)$

$x \overset{j^1\phi}{\longmapsto} j^1\phi(x) = (x, \phi, D\phi)$ decimated to

$\nu \overset{\tilde{\phi}}{\longmapsto} \tilde{\phi}(\nu) = (x(C\nu), \phi_\nu \in \mathcal{F}, \{x(C\tau), \phi_\tau \in \mathcal{F}\}_{\tau \subset \partial\nu})$ or

$\nu \overset{\tilde{\phi}}{\longmapsto} \tilde{\phi}(\nu) = (\nu, \phi_\nu \in \mathcal{F}, \{\phi_\tau \in \mathcal{F}\}_{\tau \subset \partial\nu})$



Other classes of examples

Ex.2) Sigma models; scalar fields on a G-principal bundle

Ex.3) Gauge fields on a G-principal bundle

Coarse graining

Definition $\Delta' \geq \Delta$ means that there is a coarse graining map cg consisting of an assignment of a a homomorphism $cg(U)$ to every open set $U \subset M$

$$PO_{\Delta}(U) \xrightarrow{cg(U)} PO_{\Delta'}(U)$$

Modelling physical observables

Strategy:

Operational notion of measuring scale $\longrightarrow$ Algebraic def. $PO_\Delta(U)$

$\{PO_\Delta(U_\alpha)\}_{U_\alpha \in \mathcal{C}} \longrightarrow$ Geometrical framework for PrimeFT$_\Delta$

Physical observables at scale Δ

- from the integration of **observable currents**
- from integration of (equivalence classes of) bulk densities

$\longrightarrow$ Algebraic structure for observable currents at scale Δ

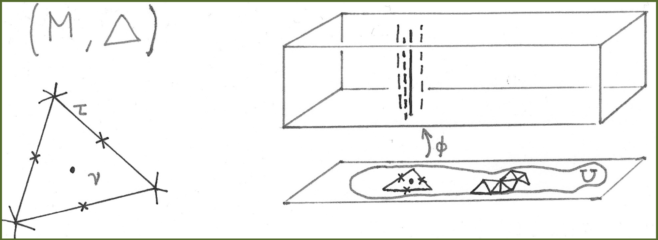
Physical observables in the continuum are defined as physical observables at *some* scale Δ

Geometric framework for classical field theories at scale Δ

Prologue

- The **first order effective field bundle**, $J^1 Y_\Delta$, is a finite dimensional manifold (with the str. of a fiber bundle over a simplicial complex)
- Local objects are defined on $J^1 Y_\Delta$
- Histories are local sections, among them we have "solutions"
- **Geometric structure** emerges as relations among local objects that hold when evaluated on "solutions"

Simplicial first order effective field bundle (scalar field)



Decimated local record of a history in 1st order format

$$\nu \xmapsto{\tilde{\phi}} \tilde{\phi}(\nu) = (\nu, \phi_\nu \in \mathcal{F}, \{\phi_\tau \in \mathcal{F}\}_{\tau \subset \partial\nu})$$

A variation $\delta\tilde{\phi}(\nu) = \tilde{v}(\nu) = (v_\nu \in T_{\phi_\nu}\mathcal{F}, \{v_\tau \in T_{\phi_\tau}\mathcal{F}\}_{\tau \subset \partial\nu})$

Notation: (M, Δ), $\nu \in U^n_\Delta$, $\tau \in (\partial U)^{n-1}_\Delta$, or $\tau \in U^{n-1}_\Delta$, $\tilde{\phi}(\nu) \in J^1 Y_\Delta$, $\tilde{\phi} \in \mathrm{Hists}_U$, $\tilde{v} \in T_{\tilde{\phi}}\mathrm{Hists}_U$, or $\tilde{v} \in \mathfrak{X}(J^1 Y_\Delta|_U)$

Variational principle, field eqs. and geometric structure

$$S(\tilde{\phi}) = \sum_{\nu \in U^n_\Delta} L(\tilde{\phi}(\nu))$$

$\Rightarrow$

$$dS(\tilde{\phi})[\tilde{v}] = \sum_{U - \partial U} \tilde{\phi}^* \; i_{\tilde{v}} \mathrm{E}_L + \sum_{\partial U} \tilde{\phi}^* \; i_{\tilde{v}} \Theta_L$$

where

$$\Theta_L(\cdot, \tilde{\phi}(\tau_\nu)) = \frac{\partial L}{\partial \phi_\tau}(\tilde{\phi}(\nu)) d\phi_\tau \quad \text{[1 form, n-1 cochain] on } J^1 Y_\Delta,$$

$$\mathrm{E}_L(\cdot, \tilde{\phi}(\nu)) = \frac{\partial L}{\partial \phi}(\tilde{\phi}(\nu)) d\phi_\nu + \sum_{\tau \in (\partial \nu)^{n-1}} \frac{\partial L}{\partial \phi_\tau}(\tilde{\phi}(\nu)) d\phi_\tau$$

Hamilton's principle: **(i)** field equations, **(ii)** geometric str.
Field eqs: **(i.a)** internal to each ν,
(i.b) gluing (momentum matching) at each $\tau = \nu \cap \nu'$

† Sigma models and gauge theory also available

The (pre)multisymplectic form

$$\Omega_L(\tilde{v}(\nu), \tilde{w}(\nu), \tilde{\phi}(\tau_\nu)) \doteq -d(\Theta_L|_{\tilde{\phi}(\tau_\nu)})(\tilde{v}(\nu), \tilde{w}(\nu))$$

assigns (pre)symplectic structures to spaces of data over codimension 1 domains $\Sigma \mapsto \Omega_\Sigma$

$$\Omega_{\Sigma, \tilde{\phi}}(\tilde{v}, \tilde{w}) = \sum_\Sigma \tilde{\phi}^* \ i_{\tilde{w}} i_{\tilde{v}} \Omega_L$$

E.g. scalar field Σ spacelike $\Omega_{\Sigma, \tilde{\phi}}(\tilde{v}, \tilde{w}) = \frac{2k}{h} \sum_\Sigma d\phi_\nu \wedge d\phi_\tau(\tilde{v}, \tilde{w})$

Given any $\tilde{\phi} \in \mathrm{Sols}_U$, $\tilde{v}, \tilde{w} \in T_{\tilde{\phi}}\mathrm{Sols}_U$ and $U' \subset U$
the multisymplectic formula holds:

$$\sum_{\partial U'} \tilde{\phi}^* \ i_{\tilde{w}} i_{\tilde{v}} \Omega_L = 0$$

Proof.
$0 = -ddS = -d(\sum_{\partial U} \tilde{\phi}^* \ \Theta_L) = \sum_{\partial U} \tilde{\phi}^* \ \Omega_L$

The space of first variations

Consider $\tilde{\phi} \in \mathrm{Sols}_U$.
First variations of $\tilde{\phi}$ are elements of

$$T_{\tilde{\phi}}\mathrm{Sols}_U \subset T_{\tilde{\phi}}\mathrm{Hists}_U,$$

and they may be induced by vector fields on $J^1 Y_\Delta$.

- They are characterized by satisfying $\mathcal{L}_{\tilde{v}}\mathrm{E}_L = 0$
  (Recall $dS(\tilde{\phi}) = \sum_{U-\partial U} \tilde{\phi}^* \mathrm{E}_L + \sum_{\partial U} \tilde{\phi}^* \Theta_L$)
- They define a lagrangian[2] subspace of $\Omega_{\partial U', \tilde{\phi}}$ for all $U' \subset U$

2 qua qua qua, la la la, blu blu blu qua qua qua, la la la, blu blu blu

Observable currents

$F \in \mathrm{OC}_U$ iff it is an n-1 cochain on $J^1 Y_\Delta$ ·st· $\forall\ \tilde{\phi} \in \mathrm{Sols}_U$

$$F(\tilde{\phi}(\tau_\nu)) \doteq F(\tau, \phi_\tau, \phi_\nu) = -F(\tilde{\phi}(\tau_{\nu'})) = F(-\tau, \phi_\tau, \phi_{\nu'}),$$

$$\sum_{\partial U'} \tilde{\phi}^* F = \sum_{\nu \subset \partial U'} F(\tilde{\phi}(\nu)) = 0 \qquad \forall \ \ U' \subset U$$

Observables

$$Q_{F,\Sigma}(\tilde{\phi}) \doteq \sum_{\nu \subset \Sigma} F(\tilde{\phi}(\nu))$$

Notice that if Σ' is homologous to Σ and $\tilde{\phi} \in \mathrm{Sols}_U$

$$Q_{F,\Sigma'}(\tilde{\phi}) - Q_{F,\Sigma}(\tilde{\phi}) = Q_{F,\Sigma'-\Sigma}(\tilde{\phi}) = Q_{F,\partial U'}(\tilde{\phi}) = 0$$

Notice that OC_U is a vector space.

† Sigma models and gauge theory also available

Family of examples: “pointwise” measurement

Given a place in the region of interest, say $\nu \subset U$
we can construct an observable current F that gives an estimate for either the evaluation of the field at that place ϕ_ν or any of its derivatives (ϕ_ν, ϕ_τ)

$$F = F_{\phi_n u} \quad \text{or} \quad F = F_{(\phi_\nu, \phi_\tau)}.$$

In a few slides we will see how this can be done.

Family of examples: Noether currents *100th anniversary*

The Lie group $\mathcal{G}$ acts on $J^1 Y_\Delta$ and on histories in 1st order format

$$(\tilde{g\phi})(\nu) = (\nu, g_\nu(\phi_\nu), \{g_\tau(\phi_\tau)\}_{\tau \subset \partial\nu})$$

If $L(g\tilde{\phi}(\nu)) = L(\tilde{\phi}(\nu)) \quad \forall \quad \nu, \phi, g \implies S$ and Sols_U are $\mathcal{G}$ inv.

Thus, $\xi \in \mathfrak{g}$ induces a first variation $\tilde{v}_\xi$ of any solution $\tilde{\phi}$.
We associate to it a **Noether current**

$$N_\xi = -i_{\tilde{v}_\xi} \Theta_L$$

• Thm. (Noether)
$N_\xi \in \mathrm{OC}_U$, $\quad dN_\xi = -i_{\tilde{v}_\xi} \Omega_L$, $\quad \{N_\xi, N_\xi\} = N_{[\xi,\eta]}$

Proof. Let $\tilde{\phi} \in \mathrm{Sols}_U$ and $U' \subset U$, then

$$0 = dS|_{U'}(\tilde{\phi})[\tilde{v}_\xi] = -\sum_{\partial U'} \tilde{\phi}^* \; N_\xi \quad , \ldots$$

Can observable currents distinguish neighboring solutions?

Consider a curve of solutions $\gamma(s) \in \mathrm{Sols}_U$ with

$$\gamma(0) = \tilde{\phi} \in \mathrm{Sols}_U \ , \quad \dot{\gamma}(0) = \tilde{w} \in T_{\tilde{\phi}}\mathrm{Sols}_U .$$

** Is OC_U large enough to resolve $T_{\tilde{\phi}}\mathrm{Sols}_U$? **
$Q_{F,\Sigma}$ separates $\tilde{\phi}$ from nearby solutions in γ if

$$\frac{d}{ds}|_{s=0} Q_{F,\Sigma}(\gamma(s)) = \sum_{\Sigma} \tilde{\phi}^* \ dF[\tilde{w}] \neq 0$$

If the observable current has an associated hamiltonian vector field

$$dF = -i_{\tilde{v}}\Omega_L$$

(let us call such an OC a hamiltonian OC, $F \in \mathrm{HOC}_U$)
the separability condition reads

$$\sum_{\Sigma} \tilde{\phi}^* \ i_{\tilde{w}} i_{\tilde{v}} \Omega_L \neq 0.$$

Separating neighboring solutions and "pointwise measurement"

Separability measuring in the bulk

Assume Ω_L is non degenerate.
Then for any $\tilde{\phi} \in \mathrm{Sols}_U$ there is a hamiltonian OC F that can be used to separate $\tilde{\phi}$ from any neighboring solution.

Sketch of proof.
Given any non constant curve $\gamma(s) \in \mathrm{Sols}_U$ as above,
Ω_L non deg. $\Rightarrow \exists\ \tilde{v}$ and $\tau \subset U$ ·st·

$$\Omega_L(\tilde{v}, \tilde{w} = \dot{\gamma}(0), \tilde{\phi}(\tau)) \neq 0.$$

Construct F from $\tilde{v}$ (slides 22, 23). $\qquad (dF = -i_{\tilde{v}}\Omega_L)$

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** Such an F yields a "pointwise measurement" **

Separating neighboring solutions and “pointwise measurement”

Separability integrating in the boundary

Assume Ω_L satisfies a non deg. condition.
Then for any $\tilde{\phi} \in \mathrm{Sols}_U$ there is $F \in \mathrm{HOC}_U$ that separates $\tilde{\phi}$ from any neighboring solution measuring at $\Sigma \subset \partial U$.

Sketch of proof.
Ω_L non deg.' $\Rightarrow \exists\, \tilde{v}$ and $\Sigma' \subset U$ with $\partial\Sigma' \subset \partial U$ ·st·

$$\frac{d}{ds}|_{s=0} Q_{F,\Sigma'}(\gamma(s)) = -\sum_{\Sigma'} \tilde{\phi}^* \; i_{\tilde{w}} i_{\tilde{v}} \Omega_L \neq 0.$$

F may be measured at $\Sigma \subset \partial U$ ·st· $\Sigma' - \Sigma = \partial U'$.
–

** F encodes a “pointwise measurement” for a bulk “point” even when integrated at the boundary **

Locally hamiltonian vector fields

We will investigate the space of hamiltonian observable currents.

Hamiltonian (or exact) vector fields

If
$$-\, i_{\tilde v}\Omega_L = dF$$

$\tilde v$ is said to be a **hamiltonian vector field** for F.
$\tilde v \in \mathrm{Ha}(J^1 Y_\Delta|_U) \subset \mathfrak{X}(J^1 Y_\Delta|_U)$ and $F \in \mathrm{HOC}_U \subset \mathrm{OC}_U$.

Locally hamiltonian (or closed) vector fields

If $\;-i_{\tilde v}\Omega_L \doteq \sigma_{\tilde v}\;$ with

$$d\sigma_{\tilde v} = 0 \quad \text{and} \quad \sum_{\partial U'} \tilde\phi^*\; i_{\tilde w}\sigma_{\tilde v} = 0$$

for all $U' \subset U$ and $(\tilde w, \tilde\phi) \in T\mathrm{Sols}_U$,
$\tilde v$ is said to be a **locally hamiltonian vector field**.
$\tilde v \in \mathrm{LHa}(J^1 Y_\Delta|_U) \subset \mathfrak{X}(J^1 Y_\Delta|_U)$.

Conditions for a vector field to be locally hamiltonian

$$\begin{array}{rcl} d\sigma_{\tilde v} = 0 & \iff & \mathcal{L}_{\tilde v}\Omega_L = 0 \\ \sum_{\partial U'} \tilde\phi^*\, i_{\tilde w}\sigma_{\tilde v} = 0 \quad \forall U', \tilde w & \Longleftarrow & \mathcal{L}_{\tilde v}\mathrm{E}_L = 0 \\ \sum_{\partial U'} \tilde\phi^*\, i_{\tilde w}\sigma_{\tilde v} = 0 \quad \forall U', \tilde w & \Longrightarrow^{\dagger} & \mathcal{L}_{\tilde v}\mathrm{E}_L = 0 \end{array}$$

All evaluated at a $\tilde\phi \in \mathrm{Sols}_U$.

Notice that if $\mathcal{L}_{\tilde v}\Omega_L = 0$ holds at Σ,
the multisymplectic formula implies that it also holds at any
$\Sigma' = \Sigma + \partial U'$ if $\mathcal{L}_{\tilde v}\mathrm{E}_L = 0$ holds inside U'.

$\Longrightarrow$ The bulk condition is $\mathcal{L}_{\tilde v}\mathrm{E}_L = 0$ (i.e. $\tilde v \in T_{\tilde\phi}\mathrm{Sols}_U$)

† If $T_{\tilde\phi}\mathrm{Sols}_U$ defines a lagrangian subspace of $\Omega_{\partial U', \tilde\phi}$ for all $U' \subset U$

Observable currents and locally hamiltonian vector fields

- Some closed 1-forms may be integrated, revealing that they are exact. This is the subject of the next slide.

$$\mathrm{LHa}(J^1 Y_\Delta|_U) \supset \mathrm{Ha}(J^1 Y_\Delta|_U)$$

- If $\Omega_L(\cdot,\cdot;\tilde{\phi}(\tau_\nu))$ is non degenerate $\forall \tau_\nu \in U$

$$0 \longrightarrow \mathrm{OC}_U \xrightarrow{\Omega_L^{-1}} \mathrm{Ha}(J^1 Y_\Delta|_U)$$

This contrasts with Multisymplectic Field Theory in the continuum, where the n+1 form Ω_L is not invertible.
The situation is closer to initial data formulations of field theory where the symplectic form is invertible.
The difference arises from the fact that in the discrete setting there is a predetermined set of codimension 1 faces on which Ω_L may be evaluated to induce a (collection of) 2 forms.

Observable currents from LHVFs

$\tilde{v} \in \mathrm{LHa}(J^1 Y_\Delta|_U)$ induces $\sigma_{\tilde{v}}$,
integration on the fibers may lead to $F_{\tilde{v},K} \in \mathrm{OC}_U$.
Integration requires the choice of
a system of integration constants K;
an allowed choice of integration constants implies

$$\sum_{\partial U'} \tilde{\phi}^* \, F_{\tilde{v},K} = 0 \qquad \forall \, \tilde{\phi} \in \mathrm{Sols}_U, \;\; U' \subset U$$

Adding a closed n-1 cochain C in U to a system of allowed integration constants K yields a new system of allowed integration constants $K' = K + C$.
$F_{\tilde{v},K} \in \mathrm{OC}_U$ and its physical meaning are determined by $\tilde{v}$ and K.

OC_U is in correspondence with $T\mathrm{Sols}_U$;
when Ω_L is non deg. the corresp. is roughly 1 to 1
making OCs capable of separating neighboring solutions.

Poisson brackets among observable currents

Given two observable currents $F_{\tilde{v},K}, G_{\tilde{w},L} \in \mathrm{OC}_U$
their Poisson bracket is another observable current

$$\{F_{\tilde{v},K}, G_{\tilde{w},L}\}(\tilde{\phi}(\tau_\nu)) = \Omega_L(\tilde{w}, \tilde{v}, \tilde{\phi}(\tau_\nu))$$

whose hamiltonian vector field is $[\tilde{v}, \tilde{w}]$.

Recall $\Omega_L = -d\Theta_L$ and $dS|_{\mathsf{Sols}} = \sum_{\partial U} \Theta_L$
Then $\{F, \cdot\}$ is related to the variation of a solution $\tilde{\phi}$ induced by
$S_U(\tilde{\phi}) \to S_{U,\lambda}(\tilde{\phi}) = S(\tilde{\phi}) + \lambda \sum_{\partial U} F\tilde{\phi}$.

Similar considerations for bulk obs. lead to Peierls' bracket.
Peierls' bracket defines an equivalence relation among bulk obs.
Using our bracket, the equivalence relation extends to bdary obs.
In fact, bdary obs. may be used to label equiv. classes of bulk obs.

Summary and remarks

- **Physical observables at scale Δ** induced by integrating over hypersurfaces
$$Q_{F,\Sigma}(\tilde{\phi}) \doteq \sum_{\nu\subset\Sigma} F(\tilde{\phi}(\nu))$$
were studied within a multisymplectic framework for discrete *local* field theories.
These physical obs. are capable of separating solutions/gauge.
- If the model at scale Δ is corrected using a model for a finer scale Δ' the observables are inherited at the finer scale; however, they have to be corrected as to retain their "Σ-independence" property, $Q_{F,\Sigma'}(\tilde{\phi}) = Q_{F,\Sigma}(\tilde{\phi})$.
- Our observables are "equivalent" to bulk observables.
- There is a Poisson bracket for observable currents at scale Δ.
- Quantization of the multisymplectic framework for discrete *local* field theories leads to spin foam models.

Thank you for your attention!