On the observable algebra of local covariant effective field theories

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— Outline —

Context

Who should be interested in the talk?

People interested on

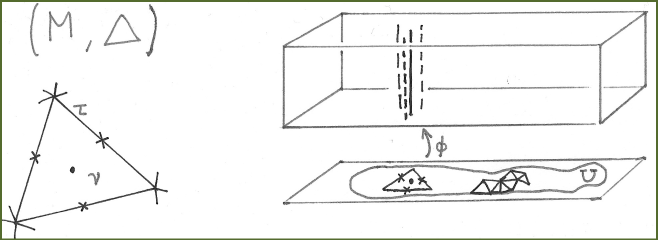
- spin foam models for field theory
- the notion of “boundary observables”
- the notion of observables
  in any local covariant field theory
- theoretical physics

Review of the geometric framework

Prologue

- The **first order effective field bundle**, $J^1 Y_\Delta$, is a finite dimensional manifold (with the str. of a fiber bundle over a simplicial complex)
- Local objects are defined on $J^1 Y_\Delta$
- Histories are local sections, among them we have "solutions"
- **Geometric structure** emerges as relations among local objects that hold when evaluated on "solutions"

Simplicial first order effective field bundle



Decimated local record of a history in 1st order format

$$\nu \overset{\tilde{\phi}}{\longmapsto} \tilde{\phi}(\nu) = (\nu, \phi_\nu \in \mathcal{F}, \{\phi_\tau \in \mathcal{F}\}_{\tau \subset \partial\nu})$$

A variation $\delta\tilde{\phi}(\nu) = \tilde{v}(\nu) = (v_\nu \in T_{\phi_\nu}\mathcal{F}, \{v_\tau \in T_{\phi_\tau}\mathcal{F}\}_{\tau \subset \partial\nu})$

Notation: (M, Δ), $\nu \in U^n_\Delta$, $\tau \in (\partial U)^{n-1}_\Delta$, or $\tau \in U^{n-1}_\Delta$,
$\tilde{\phi}(\nu) \in J^1 Y_\Delta$, $\tilde{\phi} \in \mathrm{Hists}_U$, $\tilde{v} \in T_{\tilde{\phi}}\mathrm{Hists}_U$, or $\tilde{v} \in \mathfrak{X}(J^1 Y_\Delta|_U)$

Variational principle, field eqs. and geometric structure

$$S(\tilde{\phi}) = \sum_{\nu \in U^n_\Delta} L(\tilde{\phi}(\nu))$$

$\Rightarrow$

$$dS(\tilde{\phi})[\tilde{v}] = \sum_{U - \partial U} \tilde{\phi}^* \; i_{\tilde{v}} \mathrm{E}_L + \sum_{\partial U} \tilde{\phi}^* \; i_{\tilde{v}} \Theta_L$$

where

$$\Theta_L(\cdot, \tilde{\phi}(\tau_\nu)) = \frac{\partial L}{\partial \phi_\tau}(\tilde{\phi}(\nu)) d\phi_\tau \quad \text{[1 form, n-1 cochain] on } J^1 Y_\Delta,$$

$$\mathrm{E}_L(\cdot, \tilde{\phi}(\nu)) = \frac{\partial L}{\partial \phi}(\tilde{\phi}(\nu)) d\phi_\nu + \sum_{\tau \in (\partial \nu)^{n-1}} \frac{\partial L}{\partial \phi_\tau}(\tilde{\phi}(\nu)) d\phi_\tau$$

Hamilton's principle: **(i)** field equations, **(ii)** geometric str.
Field eqs: **(i.a)** internal to each ν,
(i.b) gluing (momentum matching) at each $\tau = \nu \cap \nu'$

The (pre)multisymplectic form

$$\Omega_L(\tilde{v}(\nu), \tilde{w}(\nu), \tilde{\phi}(\tau_\nu)) \doteq -d(\Theta_L|_{\tilde{\phi}(\tau_\nu)})(\tilde{v}(\nu), \tilde{w}(\nu))$$

assigns (pre)symplectic structures to spaces of data over codimension 1 domains $\Sigma \mapsto \Omega_\Sigma$

$$\Omega_{\Sigma,\tilde{\phi}}(\tilde{v}, \tilde{w}) = \sum_{\Sigma} \tilde{\phi}^* \; i_{\tilde{w}} i_{\tilde{v}} \Omega_L$$

E.g. scalar field Σ spacelike $\Omega_{\Sigma,\tilde{\phi}}(\tilde{v}, \tilde{w}) = \frac{2k}{h} \sum_{\Sigma} d\phi_\nu \wedge d\phi_\tau(\tilde{v}, \tilde{w})$

Given any $\tilde{\phi} \in \mathrm{Sols}_U$, $\tilde{v}, \tilde{w} \in T_{\tilde{\phi}}\mathrm{Sols}_U$ and $U' \subset U$
the multisymplectic formula holds:

$$\sum_{\partial U'} \tilde{\phi}^* \; i_{\tilde{w}} i_{\tilde{v}} \Omega_L = 0$$

Proof.
$0 = -ddS = -d(\sum_{\partial U} \tilde{\phi}^* \; \Theta_L) = \sum_{\partial U} \tilde{\phi}^* \; \Omega_L$

The space of first variations

Consider $\tilde{\phi} \in \mathrm{Sols}_U$.
First variations of $\tilde{\phi}$ are elements of $T_{\tilde{\phi}}\mathrm{Sols}_U \subset T_{\tilde{\phi}}\mathrm{Hists}_U$, and may be induced by vector fields on $J^1 Y_\Delta$.

- They are characterized by satisfying $\mathcal{L}_{\tilde{v}}\mathrm{E}_L = 0$
  (Recall $dS(\tilde{\phi}) = \sum_{U-\partial U} \tilde{\phi}^* \, \mathrm{E}_L + \sum_{\partial U} \tilde{\phi}^* \, \Theta_L$)
- They define a lagrangian[2] subspace of $\Omega_{\partial U', \tilde{\phi}}$ for all $U' \subset U$

[2] qua qua qua, la la la, blu blu blu qua qua qua, la la la, blu blu blu

Observable currents

$F \in \mathrm{OC}_U$ iff it is an n-1 cochain on $J^1 Y_\Delta$ ·st· $\forall\ \tilde{\phi} \in \mathrm{Sols}_U$

$$F(\tilde{\phi}(\tau_\nu)) \doteq F(\tau, \phi_\tau, \phi_\nu) = -F(\tilde{\phi}(\tau_{\nu'})) = F(-\tau, \phi_\tau, \phi_{\nu'}),$$

$$\sum_{\partial U'} \tilde{\phi}^* \ F = 0 \qquad \forall \ \ U' \subset U$$

Observables

$$Q_{F,\Sigma}(\tilde{\phi}) \doteq \sum_{\Sigma} \tilde{\phi}^* \ F$$

Notice that if Σ' is homologous to Σ and $\tilde{\phi} \in \mathrm{Sols}_U$

$$Q_{F,\Sigma'}(\tilde{\phi}) - Q_{F,\Sigma}(\tilde{\phi}) = Q_{F,\Sigma'-\Sigma}(\tilde{\phi}) = Q_{F,\partial U'}(\tilde{\phi}) = 0$$

Notice that OC_U is a vector space.

Family of examples: Noether currents *100th anniversary*

The Lie group $\mathcal{G}$ acts on $J^1 Y_\Delta$ and on histories in 1st order format

$$(\tilde{g\phi})(\nu) = (\nu, g_\nu(\phi_\nu), \{g_\tau(\phi_\tau)\}_{\tau \subset \partial\nu})$$

If $L(g\tilde{\phi}(\nu)) = L(\tilde{\phi}(\nu)) \quad \forall \quad \nu, \phi, g \implies S$ and Sols_U are $\mathcal{G}$ inv.

Thus, $\xi \in \mathfrak{g}$ induces a first variation $\tilde{v}_\xi$ of any solution $\tilde{\phi}$.
We associate to it a **Noether current**

$$N_\xi = -i_{\tilde{v}_\xi} \Theta_L$$

• Thm. (Noether)
$N_\xi \in \mathrm{OC}_U$, $\quad dN_\xi = -i_{\tilde{v}_\xi} \Omega_L$, $\quad \{N_\xi, N_\xi\} = N_{[\xi,\eta]}$

Proof. Let $\tilde{\phi} \in \mathrm{Sols}_U$ and $U' \subset U$, then

$$0 = dS|_{U'}(\tilde{\phi})[\tilde{v}_\xi] = -\sum_{\partial U'} \tilde{\phi}^* \ N_\xi \quad , \ldots$$

Can observable currents distinguish different solutions?

- OC_U has interesting elements for systems with symmetries
- Are there elements in OC_U that are not Noether currents?
- Do we get "enough" observables?
  - The functional form of obs. generated from OCs is limited *
  - Can observable currents distinguish different solutions?

    Given $\tilde{\phi} \neq \tilde{\phi}' \in \mathrm{Sols}_U$,
    look for $F \in \mathrm{OC}_U$ and a codim 1 surface $\Sigma \subset U$ such that

$$Q_{F,\Sigma}(\tilde{\phi}) \neq Q_{F,\Sigma}(\tilde{\phi}')$$

  - Can they distinguish solutions using $\Sigma \subset \partial U$ only?

Can observable currents distinguish neighboring solutions?

Consider curves of solutions $\gamma(s) \in \mathrm{Sols}_U$ with

$$\gamma(0) = \tilde{\phi} \in \mathrm{Sols}_U \ , \quad \dot{\gamma}(0) = \tilde{w} \in T_{\tilde{\phi}}\mathrm{Sols}_U.$$

** Is OC_U large enough to resolve $T_{\tilde{\phi}}\mathrm{Sols}_U$? **

$Q_{F,\Sigma}$ separates $\tilde{\phi}$ from nearby solutions in γ if

$$\frac{d}{ds}|_{s=0} Q_{F,\Sigma}(\gamma(s)) = \sum_{\Sigma} \tilde{\phi}^* \ dF[\tilde{w}] \neq 0$$

If the observable current has an associated hamiltonian vector field

$$dF = -i_{\tilde{v}}\Omega_L$$

(let us call such an OC a hamiltonian OC, $F \in \mathrm{HOC}_U$)
the separability condition reads

$$\sum_{\Sigma} \tilde{\phi}^* \ i_{\tilde{w}} i_{\tilde{v}} \Omega_L \neq 0.$$

Observable currents distinguish neighboring solutions

Separability measuring in the bulk

Assume Ω_L is non degenerate. Then for any $\tilde{\phi} \in \mathrm{Sols}_U$ there is a hamiltonian OC F that can be used to separate $\tilde{\phi}$ from any neighboring solution.

Sketch of proof.

Given any non constant curve $\gamma(s) \in \mathrm{Sols}_U$ as above,
Ω_L non deg. $\Rightarrow \exists\, \tilde{v}$ and $\tau \subset U$ ·st· $\Omega_L(\tilde{v}, \tilde{w}, \tilde{\phi}(\tau)) \neq 0$.

Separability measuring in the boundary

Assume Ω_L satisfies a non deg. condition. Then for any $\tilde{\phi} \in \mathrm{Sols}_U$ there is $F \in \mathrm{HOC}_U$ that separates $\tilde{\phi}$ from any neighboring solution measuring at $\Sigma \subset \partial U$.

Sketch of proof.

Ω_L non deg.' $\Rightarrow \exists\, \tilde{v}$ and $\Sigma' \subset U$ with $\partial\Sigma' \subset \partial U$ ·st·
$\frac{d}{ds}|_{s=0} Q_{F,\Sigma'}(\gamma(s)) = -\sum'_{\Sigma} \tilde{\phi}^* \, i_{\tilde{w}} i_{\tilde{v}} \Omega_L \neq 0$.
F may be measured at $\Sigma \subset \partial U$ ·st· $\Sigma' - \Sigma = \partial U'$.

Locally hamiltonian vector fields

We will investigate the space of hamiltonian observable currents.

Hamiltonian (or exact) vector fields

If
$$-\, i_{\tilde{v}}\Omega_L = dF$$

$\tilde{v}$ is said to be a **hamiltonian vector field** for F.
$\tilde{v} \in \mathrm{Ha}(J^1 Y_\Delta|_U) \subset \mathfrak{X}(J^1 Y_\Delta|_U)$ and $F \in \mathrm{HOC}_U \subset \mathrm{OC}_U$.

Locally hamiltonian (or closed) vector fields

If $\;-i_{\tilde{v}}\Omega_L \doteq \sigma_{\tilde{v}}\;$ with

$$d\sigma_{\tilde{v}} = 0 \quad \text{and} \quad \sum_{\partial U'} \tilde{\phi}^* \; i_{\tilde{w}}\sigma_{\tilde{v}} = 0$$

for all $U' \subset U$ and $(\tilde{w}, \tilde{\phi}) \in T\mathrm{Sols}_U$,
$\tilde{v}$ is said to be a **locally hamiltonian vector field**.
$\tilde{v} \in \mathrm{LHa}(J^1 Y_\Delta|_U) \subset \mathfrak{X}(J^1 Y_\Delta|_U)$.

Conditions for a vector field to be locally hamiltonian

$$
\begin{aligned}
d\sigma_{\tilde v} = 0 \quad & \iff & \mathcal{L}_{\tilde v}\Omega_L = 0 \\
\sum_{\partial U'} \tilde\phi^* \; i_{\tilde w}\sigma_{\tilde v} = 0 \quad \forall U', \tilde w \quad & \Longleftarrow & \mathcal{L}_{\tilde v}\mathrm{E}_L = 0 \\
\sum_{\partial U'} \tilde\phi^* \; i_{\tilde w}\sigma_{\tilde v} = 0 \quad \forall U', \tilde w \quad & \Longrightarrow^{\dagger} & \mathcal{L}_{\tilde v}\mathrm{E}_L = 0
\end{aligned}
$$

All evaluated at a $\tilde\phi \in \mathrm{Sols}_U$.

Notice that if $\mathcal{L}_{\tilde v}\Omega_L = 0$ holds at Σ,
the multisymplectic formula implies that it also holds at any
$\Sigma' = \Sigma + \partial U'$ if $\mathcal{L}_{\tilde v}\mathrm{E}_L = 0$ holds inside U'.

$\Longrightarrow$ The bulk condition is $\mathcal{L}_{\tilde v}\mathrm{E}_L = 0$ (i.e. $\tilde v \in T_{\tilde\phi}\mathrm{Sols}_U$)

† If $T_{\tilde\phi}\mathrm{Sols}_U$ defines a lagrangian subspace of $\Omega_{\partial U', \tilde\phi}$ for all $U' \subset U$

Observable currents and locally hamiltonian vector fields

- Some closed 1-forms may be integrated, revealing that they are exact. This is the subject of the next slide.

$$\mathrm{LHa}(J^1 Y_\Delta|_U) \supset \mathrm{Ha}(J^1 Y_\Delta|_U)$$

- If $\Omega_L(\cdot,\cdot;\tilde{\phi}(\tau_\nu))$ is non degenerate $\forall \tau_\nu \in U$

$$0 \longrightarrow \mathrm{OC}_U \xrightarrow{\Omega_L^{-1}} \mathrm{Ha}(J^1 Y_\Delta|_U)$$

This contrasts with Multisymplectic Field Theory in the continuum, where the n+1 form Ω_L is not invertible.
The situation is closer to initial data formulations of field theory where the symplectic form is invertible.
The difference arises from the fact that in the discrete setting there is a predetermined set of codimension 1 faces on which Ω_L may be evaluated to induce a (collection of) 2 forms.

Observable currents from LHVFs

$\tilde{v} \in \mathrm{LHa}(J^1 Y_\Delta|_U)$ induces $\sigma_{\tilde{v}}$,
integration on the fibers may lead to $F_{\tilde{v},K} \in \mathrm{OC}_U$.
Integration requires the choice of
a system of integration constants K;
an allowed choice of integration constants implies

$$\sum_{\partial U'} \tilde{\phi}^* \, F_{\tilde{v},K} = 0 \qquad \forall\, \tilde{\phi} \in \mathrm{Sols}_U,\ U' \subset U$$

Adding a closed n-1 cochain C in U to a system of allowed integration constants K yields a new system of allowed integration constants $K' = K + C$.
$F_{\tilde{v},K} \in \mathrm{OC}_U$ and its physical meaning are determined by $\tilde{v}$ and K.

OC_U is in correspondence with $T\mathrm{Sols}_U$;
when Ω_L is non deg. the corresp. is roughly 1 to 1
making OCs capable of separating neighboring solutions.

Tailored locally hamiltonian fields

- On a simplicial lattice data over corners is free complete data $\tilde{v}_{\tilde{\phi}} \in J^1 Y_\Delta|_{\text{corner}}$.
  (each n dim atom has n+1 corners storing 1st order data)
- The data may be extended to $\tilde{v} \in \mathrm{LHa}(J^1 Y_\Delta|_{\text{corner}})$.
  Kinematical. Here there is a lot of freedom;
  we choose a "simple extension" to each vector.
- The field equations and the linearized field eqs. may be solved to find $\tilde{\phi} \in \mathrm{Sols}_\nu$, $\tilde{v} \in T_{\tilde{\phi}}\mathrm{Sols}_\nu$ which extend
  $\tilde{v} \in \mathrm{LHa}(J^1 Y_\Delta|_{\text{corner}})$ (for each $\tilde{v}_{\tilde{\phi}} \in J^1 Y_\Delta|_{\text{corner}}$).
- The generated vector field in $J^1 Y_\Delta|_\nu$ is also locally hamiltonian in the rest of the faces of ν. **Dynamical**
  - The multisymplectic formula implies $d\sigma_{\tilde{v}}(\cdot, \tilde{\phi}(\tau_\nu)) | T_{\tilde{\phi}}\mathrm{Sols}_\nu = 0$
  - Since evaluation of 1st variations span $T_{\tilde{\phi}(\nu)|_\tau} J^1 Y_\Delta|_{\tau_\nu}$
    $d\sigma_{\tilde{v}}(\cdot, \tilde{\phi}(\tau_\nu)) = 0$

Example: Scalar field

- The phase space at each τ_ν is R^2 with the canonical symplectic structure
- The steps of the algorithm sketched above are very simple
- The linearized field equations are simple linear algebraic formulas correlating the phase spaces at the boundary of atoms
- There are enough OCs in the sense that they can distinguish solutions to the boundary value problem using only OCs evaluated at the boundary because
  "They can measure the value of the field or the field's momentum at any $\tau \subset U$"
- For comments on the functional form of the corresponding observables wait ...

So, are there enough observable currents in general?

I am not sure.

It seems that for a degenerate Ω_L
only classes of solutions will be separable,
which is what intend in those cases.

I have started to study other cases only recently.

Poisson brackets among observable currents

Given two observable currents $F_{\tilde{v},K}, G_{\tilde{w},L} \in \mathrm{OC}_U$
their Poisson bracket is another observable current

$$\{F_{\tilde{v},K}, G_{\tilde{w},L}\}(\tilde{\phi}(\tau_\nu)) = \Omega_L(\tilde{w}, \tilde{v}, \tilde{\phi}(\tau_\nu))$$

whose hamiltonan vector field is $[\tilde{v}, \tilde{w}]$.

On the Peierls bracket of observables

Peierls' bracket arises from the following formula

$$dF_{\tilde{v}_f,\Sigma}[\tilde{w}] = \int_\Sigma \tilde{\phi}^* \sigma_{\tilde{v}_f}[\tilde{w}] = -\int_\Sigma \tilde{\phi}^* \Omega_L(\tilde{v}_f, \tilde{w}) = \int_{R^4} f[\tilde{w}]$$

where f is a source and the causal Green function is the responsible for $\tilde{v} = \tilde{v}(f) = \tilde{v}_f$.

The corresponding formula for $F_{\tilde{v}_f,\Sigma}(\psi)$ is

$$F_{\tilde{v}_f,\Sigma}(\psi) = \int_{R^4} f[\psi]$$

Boundary smeared field formulas for observable currents

For smearing in codimension 1 we consider a source for the gluing equations. We return to the boundary value problem at ∂U asking for solutions with a given momentum flux density

$$f[\tilde{w}] = \tilde{\phi}_f^* i_{\tilde{w}} \Theta_L \quad \forall \tilde{w} \quad \text{with } \tilde{\phi}_f \text{ as unknown}$$

Let us assume that the linearized problem has been solved and to each f we have a corresponding $\tilde{v} = \tilde{v}(f) = \tilde{v}_f$. Then

$$\begin{aligned}
\sigma_{\tilde{v}_f}(\tilde{w}, \tilde{\phi}(\tau_\nu)) &= -\Omega_L(\tilde{v}_f, \tilde{w}, \tilde{\phi}(\tau_\nu)) \\
&= \tilde{v}_f[\Theta_L(\tilde{w}, \tilde{\phi}(\tau_\nu))] - \tilde{w}[\Theta_L(\tilde{v}_f, \tilde{\phi}(\tau_\nu))] - \ldots \\
&= f[\tilde{w}](\tau) - \tilde{w}[\Theta_L(\tilde{v}_f, \tilde{\phi}(\tau_\nu))] - \Theta_L([\tilde{v}_f, \tilde{w}], \tilde{\phi}(\tau_\nu))
\end{aligned}$$

which can be interpreted as a local smearing formula for $\sigma_{\tilde{v}_f}$ with corrections (that applies when acting on chains $\tau \subset \partial U$).

Coarse graining

The geometric framework includes coarse graining and correction of models at coarser scales from models at finer scales.

Observable currents at coarser scale should induce observable currents at finer scales.
In the boundary smearing field formula the correction terms need to be adjusted using the finer multisymplectic structure.

Gluing

Gluing is interesting and simple for observable currents.

In the construction in terms of
$\tilde{v} \in \mathrm{LHa}(J^1 Y_\Delta|_U) \subset \mathfrak{X}(J^1 Y_\Delta|_U)$ we

- glue vector fields using the linearized gluing field equations
- glue integration constants for compatibility at boundaries with shared dof

Conclusions and outlook

- Observable currents are interesting
- In the discrete formalism described here they are simpler than in the continuum

- The study of OCs for fields of other types is within reach
- The study of differential forms of other degrees in the continuum has lead to interesting results (Kanatchikov). Wilson loops and fluxes are clear candidates to study in discrete gauge theories.
- In the continuum there is a product among currents of different degrees (Kanatchikov).
  We are considering modifications of this product.

Thank you for your attention!