Tercera tarea de Matemáticas IV (17 de octubre de 2017)

En las siguientes páginas hay ejercicios resueltos y problemas del libro. Revisar los problemas resueltos listados abajo, entenderlos y escribir las soluciones con sus palabras.
Resolver los ejercicios listados abajo.

- Ejemplos 5.6.2 y 5.6.3
- Ejercicios 5.6: 1, 2, 3, 4, 6
- Ejemplo 6.2.1
- Ejercicios 6.2: 1, 2, 3, 6, 7, 8
- Ejemplo 6.3.2
- Ejercicios 6.3: 1, 2, 3, 7
- Ejemplo 6.4.2
- Ejercicio 6.4: 1
- Ejemplo 6.5.2
- Ejercicios 6.5: 1, 2, 3

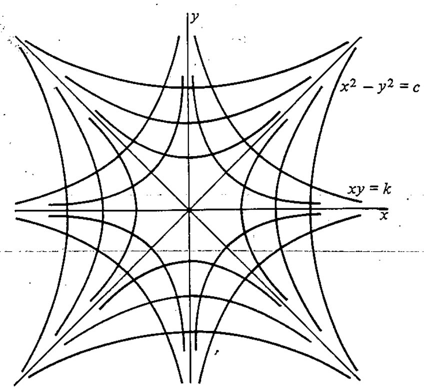

FIGURE 29

This differential equation is exact since $M_y = N_x = 1$. The solution is obviously $xy = k$, where k is an arbitrary constant. The trajectories form another family of hyperbolas. The original family and the family of orthogonal trajectories are indicated in Fig. 29. Note that $xy = 0$ is a member of the family of trajectories.

EXAMPLE 5.6.2 An object of mass m is dropped with zero initial velocity. The air resistance is assumed to be proportional to the square of the velocity. At what velocity will the object drop, and what distance will it travel in time t? The acceleration of the object is dv/dt, and m times this is to be put equal to the force of gravity mg minus kv^2, where k is a constant. Hence, if s is the distance from the starting point, the differential equation is

$$m\frac{d^2s}{dt^2} = m\frac{dv}{dt} = mg - kv^2$$

The initial conditions are at $t = 0$, $v = 0$, and $s = 0$. The differential equation is separable and we can write

$$\frac{dv}{a^2 - v^2} = b\,dt$$

where $a^2 = mg/k$ and $b = k/m$. Integrating, we have

$$\frac{1}{2a}[\ln(a + v) - \ln(a - v)] = bt + c$$

where c is arbitrary. When $t = 0$, $v = 0$ and this implies that $c = 0$. Therefore, we have

$$\ln\frac{a + v}{a - v} = 2abt$$

Solving for v we have

$$v = a\frac{e^{2abt} - 1}{e^{2abt} + 1} = a \tanh abt$$

Notice that as $t \to \infty$, $v \to a$, which is usually called the *terminal velocity*.

$$a = \sqrt{\frac{mg}{k}}$$

Finally,

$$v = \frac{ds}{dt} = a \tanh abt$$

$$s = \frac{1}{b}\ln\cosh abt$$

using the fact that $s = 0$ when $t = 0$.

EXAMPLE 5.6.3 Consider the following pursuit problem. An airplane is flying in a straight line with a constant speed of 200 miles per hour. A second plane is initially flying directly toward the first on a line perpendicular to its path. The second plane continues to pursue the first in such a way that the distance between the planes remains constant (5 miles) and the pursuing plane is always headed toward the other; that is, the tangent to the path of the pursuer passes through the other. Consider the problem in the xy plane (see Fig. 30). Let the coordinates of the pursuing plane be (x,y) and the coordinates of the other be $(s,0)$. The conditions of the problem can be stated in the following equations

$$(s - x)^2 + y^2 = 25$$

$$\frac{dy}{dx} = \frac{-y}{s - x}$$

$$s = 200t$$

We wish to find x and y as functions of t subject to the initial conditions at $t = 0$; $s = 0$, $x = 0$, $y = 5$. Differentiating the first equation, we have

$$(s - x)\left(200 - \frac{dx}{dt}\right) + y\frac{dy}{dt} = 0$$

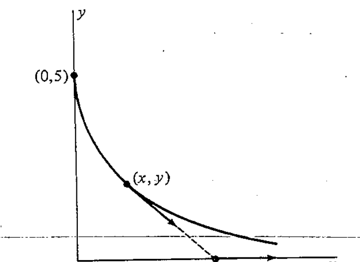

FIGURE 30

From the second equation we have

$$y\frac{dy}{dt} = \frac{-y^2}{s-x}\frac{dx}{dt} = \frac{(s-x)^2 - 25}{s-x}\frac{dx}{dt}$$

Eliminating y, we have

$$\frac{dx}{dt} = 8(s-x)^2 = 8(200t - x)^2$$

We introduce the new variable $u = 200t - x$. Then

$$\frac{dx}{dt} = 200 - \frac{du}{dt} = 8u^2$$

$$\frac{du}{dt} = 200 - 8u^2$$

This equation is separable. Hence,

$$\frac{du}{25 - u^2} = 8\,dt$$

$$\tfrac{1}{10}[\ln(5+u) - \ln(5-u)] = 8t + c$$

When $t = 0$, $u = 0$, which implies that $c = 0$. Solving for u, we have

$$x = 200t - u = 200t - 5\tanh 40t$$

Finally,

$$\frac{dy}{dt} = \frac{-y}{u}\frac{dx}{dt} = -8uy$$

$$\frac{dy}{y} = -8u\,dt = -40\tanh 40t\,dt$$

$$\ln y = -\ln\cosh 40t + k$$

When $t = 0$, $y = 5$, which gives us

$$y = 5\operatorname{sech} 40t$$

EXERCISES 5.6

1 A ball is thrown straight up with initial velocity v_0. Neglecting air resistance, determine how high the ball will rise.

2 A ball is thrown straight up with initial velocity v_0. Suppose air resistance is proportional to the speed (magnitude of velocity). How high will the ball rise?

3 At a point 4,000 miles from the center of the earth a rocket has expended all its fuel and is moving radially outward with a velocity v_0. Let the force on the rocket due to the earth's gravitational attraction be $10^5 m/r^2$, where m is the mass of the rocket, r is the distance to the center of the earth measured in miles, and time is measured in seconds. Find the minimum v_0 such that the rocket will not return to the earth, sometimes called the *escape velocity*. If v_0 is less than the escape velocity, find the maximum altitude attained by the rocket.

4 Find the family of orthogonal trajectories for each of the given families:

(*a*) $y^2 = cx$ (*b*) $\frac{x^2}{4} + \frac{y^2}{9} = c^2$

(*c*) $x^2 - xy + y^2 = c^2$ (*d*) $x^2 - 2cx + y^2 = 1$

5 An airplane with a constant airspeed of 200 miles per hour starts out to a destination 300 miles due east. There is a wind out of the north of 25 miles per hour. The plane always flies so that it is headed directly at the destination. Find the path of the airplane.

6 Find the curve passing through the point (3,4) such that the tangent to the curve and the line to the origin are always perpendicular.

7 Find the equation of the curve passing through (2,4) such that the segment of the tangent line between the curve and the x axis is bisected by the y axis.

8 A ball is dropped from a great height. Assuming that the acceleration of gravity is constant and that air resistance is proportional to the square root of the speed, find the terminal velocity.

5.7 NUMERICAL METHODS

Out of all the nonlinear first order differential equations with solutions, only a relatively small number can be solved in closed form. Therefore, it is very important that there be numerical methods for solving differential equations. In this section, we give a very brief introduction to a very large subject, which is playing an increasingly important role in applied mathematics as nonlinear analysis becomes more and more inescapable in modern technology. Of course this development has been aided and abetted by the invention of large-scale

EXAMPLE 6.2.1 Consider the differential equation $\dddot{y} - \dot{y} = 1$. Find a solution of this equation satisfying $y(0) = 1, \dot{y}(0) = -1, \ddot{y}(0) = 0$. In this case,

$L(y) = \dddot{y} - \dot{y}$, and the associated homogeneous equation is $\dddot{y} - \dot{y} = 0$. We shall see later that when L is an operator with constant coefficients, the homogeneous equation usually has exponential solutions. Therefore, let us substitute e^{mt} into the equation $L(y) = 0$:

$$L(e^{mt}) = m^3e^{mt} - me^{mt} = m(m-1)(m+1)e^{mt}$$

Now if $m = 0$, 1, or -1, the equation $L(y) = 0$ is satisfied by $y = e^{mt}$. We shall then take $y_1 = 1$, $y_2 = e^t$, and $y_3 = e^{-t}$. Let us check to see if these three functions are independent by computing their Wronskian:

$$W(t) = \begin{vmatrix} 1 & e^t & e^{-t} \\ 0 & e^t & -e^{-t} \\ 0 & e^t & e^{-t} \end{vmatrix} = 2$$

Therefore, 1, e^t, e^{-t} are independent and form a basis for the null space of L. Any solution of the homogeneous equation can be written in the form $c_1 + c_2e^t + c_3e^{-t}$. Next we look for a particular solution of the nonhomogeneous equation $L(y) = 1$. In this case it is fairly easy to guess a solution since a constant times t will yield a constant upon one differentiation and will yield zero upon three differentiations. Therefore, $L(kt) = -k = 1$ if $k = -1$, and a solution is $-t$. We now know that the solution of the initial-value problem can be found among functions of the form

$$y(t) = c_1 + c_2e^t + c_3e^{-t} - t$$

To evaluate the constants c_1, c_2, and c_3 we must solve the system of equations

$$\begin{aligned} y(0) &= 1 = c_1 + c_2 + c_3 \\ \dot{y}(0) &= -1 = c_2 - c_3 - 1 \\ \ddot{y}(0) &= 0 = c_2 + c_3 \end{aligned}$$

The solution is $c_1 = 1$ and $c_2 = c_3 = 0$, and $y(t) = 1 - t$ is the unique solution to the initial-value problem. Note that the coefficient matrix of the above system is

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix}$$

and its determinant is $W(0) = 2$. This guarantees that the equations determining the c's have a unique solution. This underlines the importance of having a system of solutions of the homogeneous equation with a Wronskian which is never zero in an interval where we wish to solve the initial-value problem and shows the importance of Theorem 6.2.3.

We conclude this section with some terminology which we shall use in what is to follow. It should be clear by now that a scheme for solving a given initial-value problem is the following:

1 Find n independent solutions of the homogeneous nth order equation $L(y) = 0$. We shall refer to such a set as a *fundamental system of solutions*.
2 Form an arbitrary solution of the homogeneous equation by forming a linear combination of a fundamental system with n arbitrary constants. We shall call this a *complementary solution* and denote it by $y_c(t)$.
3 Find any particular solution of the nonhomogeneous equation $L(y) = f(t)$. We shall call this a *particular solution* and denote it by $y_p(t)$.
4 Add $y_c(t)$ and $y_p(t)$ and call the sum the *general solution*.
5 Evaluate the n constants in $y_c(t)$ so that $y(t) = y_c(t) + y_p(t)$ satisfies the initial conditions.

EXERCISES 6.2

1 Show that $y_1 = e^t$ and $y_2 = e^{-t}$ form a basis for the null space of the operator $L(y) = \ddot{y} - y$.
2 Show that $y_1 = 1$, $y_2 = e^{2t}$, $y_3 = te^{2t}$ form a basis for the null space of the operator $L(y) = \dddot{y} + 4\ddot{y} + 4\dot{y}$.
3 Show that $y_1 = \sin \omega t$, $y_2 = \cos \omega t$ form a basis for the null space of the operator $L(y) = \ddot{y} + \omega^2 y$.
4 Show that $y_1 = t$, $y_2 = t^{-1}$ form a basis for the null space of the operator $L(y) = t^2\ddot{y} + t\dot{y} - y$ on the interval $\{t \mid 0 < a \le t \le b\}$.
5 Consider the differential equation $L(y) = \ddot{y} - 5\dot{y} + 6y = 0$. Look for solutions of the form $y = e^{mt}$. Find a basis for the null space of the operator L.
6 Find the general solution of $\ddot{y} - y = 1$ (see Exercise 1).
7 Find the general solution of $\ddot{y} + 4y = 1$ (see Exercise 3).
8 Find the general solution of $\dddot{y} + 4\ddot{y} + 4\dot{y} = 1$ (see Exercise 2).
9 Find the general solution of $t^2\ddot{y} + t\dot{y} - y = 1$ on the interval $\{t \mid 1 \le t \le 2\}$.
10 Find the solution of the initial-value problem $\ddot{y} - y = 1$, $y(0) = \dot{y}(0) = 1$ (see Exercise 6).
11 Find the solution of the initial-value problem $\ddot{y} + 4y = 1$, $y(0) = 1$, $\dot{y}(0) = 0$ (see Exercise 7).
12 Find the solution of the initial-value problem $\dddot{y} + 4\ddot{y} + 4\dot{y} = 1$, $y(0) = \dot{y}(0) = \ddot{y}(0) = 1$ (see Exercise 8).
13 Find the solution of the initial-value problem $t^2\ddot{y} + t\dot{y} - y = 1$, $y(1) = \dot{y}(1) = 1$ (see Exercise 9).
14 Consider two solutions y_1 and y_2 of the differential equation $\ddot{y} + p(t)\dot{y} + q(t)y = 0$ on the interval $\{t \mid a \le t \le b\}$, where p and q are continuous. Show that the Wronskian of y_1 and y_2 satisfies the differential equation $\dot{W} + pW = 0$.

Continuing in this way, we finally have

$$y_p^{(n)} = A_1 y_1^{(n)} + A_2 y_2^{(n)} + \cdots + A_n y_n^{(n)} + \dot{A}_1 y_1^{(n-1)} + \dot{A}_2 y_2^{(n-1)} + \cdots + \dot{A}_n y_n^{(n-1)}$$

Substituting in the differential equation, we find that all the terms which involve the undifferentiated A's will drop out. This is because the functions $y_1, y_2, \ldots, y_n$ are all solutions of the associated homogeneous equation. This leaves us with the following system of equations to solve for $\dot{A}_1, \dot{A}_2, \ldots, \dot{A}_n$:

$$\begin{aligned} \dot{A}_1 y_1 + \dot{A}_2 y_2 + \cdots + \dot{A}_n y_n &= 0 \\ \dot{A}_1 \dot{y}_1 + \dot{A}_2 \dot{y}_2 + \cdots + \dot{A}_n \dot{y}_n &= 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \dot{A}_1 y_1^{(n-2)} + \dot{A}_2 y_2^{(n-2)} + \cdots + \dot{A}_n y_n^{(n-2)} &= 0 \\ \dot{A}_1 y_1^{(n-1)} + \dot{A}_2 y_2^{(n-1)} + \cdots + \dot{A}_n y_n^{(n-1)} &= \frac{f(t)}{a_n(t)} \end{aligned}$$

This is a system of n equations in n unknowns with the determinant of the coefficient matrix

$$W(t) = \begin{vmatrix} y_1 & y_2 & y_3 & \cdots & y_n \\ \dot{y}_1 & \dot{y}_2 & \dot{y}_3 & \cdots & \dot{y}_n \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ y_1^{(n-1)} & y_2^{(n-1)} & y_3^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}$$

which is the Wronskian of the fundamental system of solutions $y_1, y_2, \ldots, y_n$. This Wronskian is never zero. Therefore, we can always solve uniquely for $\dot{A}_1, \dot{A}_2, \ldots, \dot{A}_n$. In fact, for $k = 1, 2, \ldots, n$

$$\dot{A}_k = \frac{f(t)W_k(t)}{a_n(t)W(t)}$$

where $W_k(t)$ is the determinant obtained from $W(t)$ by replacing the kth column by $(0, 0, 0, \ldots, 0, 1)$. A particular solution of the nonhomogeneous equation is then

$$y_p(t) = \sum_{k=1}^{n} A_k(t) y_k(t) = \sum_{k=1}^{n} y_k(t) \int_a^t \frac{f(\tau)W_k(\tau)}{a_n(\tau)W(\tau)}\, d\tau$$

The lower limit of integration need not be a since any set of integrals of the $\dot{A}$'s will do.

EXAMPLE 6.3.2 Find the general solution of $\dddot{y} + 3\ddot{y} + 2\dot{y} = -e^{-t}$. To find solutions of the homogeneous equation $\dddot{y} + 3\ddot{y} + 2\dot{y} = 0$ we try $y = e^{mt}$.

Substituting, we have $e^{mt}(m^3 + 3m^2 + 2m) = 0$. We shall have solutions if $m = 0, -1$, or -2. The functions $y_1 = 1$, $y_2 = e^{-t}$, $y_3 = e^{-2t}$ are independent since their Wronskian

$$W(t) = \begin{vmatrix} 1 & e^{-t} & e^{-2t} \\ 0 & -e^{-t} & -2e^{-2t} \\ 0 & e^{-t} & 4e^{-2t} \end{vmatrix} = -2e^{-3t}$$

never vanishes. To find y_p we must evaluate

$$W_1(t) = \begin{vmatrix} 0 & e^{-t} & e^{-2t} \\ 0 & -e^{-t} & -2e^{-2t} \\ 1 & e^{-t} & 4e^{-2t} \end{vmatrix} = -e^{-3t}$$

$$W_2(t) = \begin{vmatrix} 1 & 0 & e^{-2t} \\ 0 & 0 & -2e^{-2t} \\ 0 & 1 & 4e^{-2t} \end{vmatrix} = 2e^{-2t}$$

$$W_3(t) = \begin{vmatrix} 1 & e^{-t} & 0 \\ 0 & -e^{-t} & 0 \\ 0 & e^{-t} & 1 \end{vmatrix} = -e^{-t}$$

Therefore,

$$\begin{aligned} y_p &= \frac{1}{2}\int_0^t (-e^{-\tau} + 2e^{-t} - e^{\tau}e^{-2t})\, d\tau \\ &= \tfrac{1}{2}(e^{-t} - 1 + 2te^{-t} - e^{-t} + e^{-2t}) \\ &= te^{-t} + \tfrac{1}{2}e^{-2t} - \tfrac{1}{2} \end{aligned}$$

The general solution is therefore

$$y = c_1 + c_2e^{-t} + c_3e^{-2t} + te^{-t}$$

We do not include terms in the particular solution if they already appear in the complementary solution.

EXERCISES 6.3

1 Find the general solution of $\ddot{y} - y = e^{-t}$.
2 Find the general solution of $\dddot{y} + 4\ddot{y} + 4\dot{y} = 2e^t$ (see Exercise 6.2.2).
3 Find the general solution of $\ddot{y} + \omega^2 y = 2\cos\omega t$ (see Exercise 6.2.3).
4 Find the general solution of $t^2\ddot{y} + t\dot{y} - y = t$ on the interval $\{t \mid 0 < a \le t \le b\}$ (see Exercise 6.2.4).
5 Find the general solution of $\ddot{y} - 5\dot{y} + 6y = 2t + 3$.
6 Find the solution of $\ddot{y} - y = e^{-t}$ satisfying $y(0) = 1$, $\dot{y}(0) = 2$.
7 Find the solution of $\dddot{y} + 4\ddot{y} + 4\dot{y} = 2e^t$ satisfying $\ddot{y}(0) = \dot{y}(0) = y(0) = 1$.

solutions of $P(z) = 0$ and let $s_1, \bar{s}_1, s_2, \bar{s}_2, \ldots, s_p, \bar{s}_p$ be the distinct complex solutions. Then $P(D)$ can be factored as follows:

$$P(D) = (D - r_1)^{k_1} \cdots (D - r_q)^{k_q}[(D - a_1)^2 + b_1^2]^{l_1} \cdots [(D - a_p)^2 + b_p^2]^{l_p}$$

where $k_1 + k_2 + \cdots + k_q + 2l_1 + 2l_2 + \cdots + 2l_p = n$, and where $s_j = a_j + ib_j, j = 1, 2, \ldots, p$.

We shall solve the homogeneous equation $[(D - a)^2 + b^2]^l y = 0$, where a and b are real and l is a positive integer. The operator can be factored as follows:

$$[(D - a)^2 + b^2]^l = (D - a - ib)^l(D - a + ib)^l$$

Therefore, there are complex solutions of the form

$$e^{(a+ib)t}, te^{(a+ib)t}, t^2e^{(a+ib)t}, \ldots, t^{l-1}e^{(a+ib)t}$$
$$e^{(a-ib)t}, te^{(a-ib)t}, t^2e^{(a-ib)t}, \ldots, t^{l-1}e^{(a-ib)t}$$

By the linearity of the equation we can add and subtract solutions and obtain solutions. Hence,

$$\frac{e^{(a+ib)t} + e^{(a-ib)t}}{2} = e^{at} \cos bt$$
$$\frac{e^{(a+ib)t} - e^{(a-ib)t}}{2i} = e^{at} \sin bt$$

are both solutions. Similarly

$$te^{at} \cos bt, t^2e^{at} \cos bt, \ldots, t^{l-1}e^{at} \cos bt$$
$$te^{at} \sin bt, t^2e^{at} \sin bt, \ldots, t^{l-1}e^{at} \sin bt$$

are all solutions. It is possible to show that these $2l$ solutions are independent.

In the general case, where we have p operators of the form

$$[(D - a_j)^2 + b_j^2]^{l_j}$$

$j = 1, 2, \ldots, p$, corresponding to the distinct pairs of complex numbers $s_j = a_j + ib_j$ and $\bar{s}_j = a_j - ib_j$, we shall have $2l_j$ independent solutions for each operator as follows:

$$e^{a_1t} \cos b_1t, e^{a_1t} \sin b_1t, \ldots, t^{l_1-1}e^{a_1t} \cos b_1t, t^{l_1-1}e^{a_1t} \sin b_1t$$
$$e^{a_2t} \cos b_2t, e^{a_2t} \sin b_2t, \ldots, t^{l_2-1}e^{a_2t} \cos b_2t, t^{l_2-1}e^{a_2t} \sin b_2t$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$e^{a_pt} \cos b_pt, e^{a_pt} \sin b_pt, \ldots, t^{l_p-1}e^{a_pt} \cos b_pt, t^{l_p-1}e^{a_pt} \sin b_pt$$

This accounts for $2l_1 + 2l_2 + \cdots + 2l_p$ solutions, and, of course, the real numbers $r_1, r_2, \ldots, r_q$ account for $k_1 + k_2 + \cdots + k_q$ solutions. As we have seen from above,

$$k_1 + k_2 + \cdots + k_q + 2l_1 + 2l_2 + \cdots + 2l_p = n$$

so we have achieved our goal of finding a fundamental system of n independent solutions of the homogeneous equation. We can complete the task of finding the general solution of the nonhomogeneous equation by using the method of variation of parameters. In the next section, we shall take up another method for finding particular solutions when the right-hand side of the equation has the form of a solution of some homogeneous linear differential equation with constant coefficients.

EXAMPLE 6.4.2 Find the general solution of

$$(D^5 + D^4 - D^3 - 3D^2 + 2)y = 0$$

To solve this equation we must find all the distinct solutions of $P(z) = z^5 + z^4 - z^3 - 3z^2 + 2 = 0$. There is a theorem† from algebra which says that if a polynomial equation with integer coefficients has a rational solution $r = p/q$, where p and q are integers, then p divides the constant term and q divides the coefficient of the highest power of z. In this case, the only possible rational roots are $-1, 1, -2, 2$. By substituting we find that $P(1) = P(-1) = 0$. Therefore,

$$P(z) = (z - 1)(z + 1)(z^3 + z^2 - 2)$$

Let $Q(z) = z^3 + z^2 - 2$. Then we find that $Q(1) = 0$. Hence,

$$P(z) = (z - 1)^2(z + 1)(z^2 + 2z + 2) = (z - 1)^2(z + 1)[(z + 1)^2 + 1]$$

and the operator $P(D)$ can be written

$$P(D) = (D - 1)^2(D + 1)[(D + 1)^2 + 1]$$

and the general solution is

$$y = c_1e^{-t} + c_2e^t + c_3te^t + c_4e^{-t} \cos t + c_5e^{-t} \sin t$$

EXERCISES 6.4

1 Let y be any twice-differentiable function. Show that $(D - a)(D - b)y = (D - b)(D - a)y = [D^2 - (a + b)D + ab]y$ where a and b are constants.

2 Let y be any three-times-differentiable function. Show that

$$(D - a)[(D - b)(D - c)]y = [(D - a)(D - b)](D - c)y$$

where a, b, and c are constants.

† The reader will be asked to verify this in Exercise 6.4.5.

Hence, $y_p = c_1 + c_2e^t + c_3e^{-t} + bt + at^2$. Omitting the first three terms because they are already in the complementary solution, we have $y_p = at^2 + bt$, which is the form we know works.

EXAMPLE 6.5.2 Find the general solution of $(D^2 - 5D + 6)y = e^{2t} + \cos t$. Let $f_1 = e^{2t}$, and consider $(D^2 - 5D + 6)y_1 = e^{2t}$ or $(D-2)(D-3)y_1 = e^{2t}$. An annihilation operator for e^{2t} is $D - 2$. Hence

$$(D-2)^2(D-3)y_1 = (D-2)e^{2t} = 0$$

$$y_1 = c_1e^{2t} + c_2e^{3t} + ate^{2t}$$

Omitting the first two terms because they are already in the complementary solution, we have $y_1 = ate^{2t}$. Then $Dy_1 = ae^{2t} + 2ate^{2t}$, $D^2y_1 = 4ae^{2t} + 4ate^{2t}$, and substituting gives

$$4ae^{2t} + 4ate^{2t} - 5ae^{2t} - 10ate^{2t} + 6ate^{2t} = -ae^{2t} = e^{2t}$$

and $a = -1$. Next let $f_2 = \cos t$, and consider $(D^2 - 5D + 6)y_2 = \cos t$. An annihilation operator for $\cos t$ is $D^2 + 1$. Hence,

$$(D^2+1)(D^2 - 5D + 6)y_2 = (D^2+1)\cos t = 0$$

$$y_2 = c_1e^{2t} + c_2e^{3t} + a\cos t + b\sin t$$

Omitting the first two terms because they are in the complementary solution of the equation, we have $y_2 = a\cos t + b\sin t$ and $Dy_2 = -a\sin t + b\cos t$, $D^2y_2 = -a\cos t - b\sin t$. Substituting, we have

$$-a\cos t - b\sin t + 5a\sin t - 5b\cos t + 6a\cos t + 6b\sin t = \cos t$$

$$(5a - 5b)\cos t + (5a + 5b)\sin t = \cos t$$

Therefore, $5a - 5b = 1$ and $5a + 5b = 0$, or $a = -b = \frac{1}{10}$. The general solution to the problem is then

$$y = c_1e^{2t} + c_2e^{3t} - te^{2t} + \tfrac{1}{10}\cos t - \tfrac{1}{10}\sin t$$

EXAMPLE 6.5.3 Find the general solution of $(D^2 + 2D + 2)y = t\cos 2t + \sin 2t$. This time the right-hand side is annihilated by $(D^2+4)^2$.

Therefore,

$$(D^2+4)^2[(D+1)^2+1]y_p = (D^2+4)^2(t\cos 2t + \sin 2t) = 0$$

$$y_p = c_1e^{-t}\cos t + c_2e^{-t}\sin t + a\cos 2t + b\sin 2t + ct\cos 2t + dt\sin 2t$$

We omit the first two terms because they are in the complementary solution. Hence, we assume

$$y_p = a\cos 2t + b\sin 2t + ct\cos 2t + dt\sin 2t$$

and

$$Dy_p = (c + 2b)\cos 2t + (d - 2a)\sin 2t + 2dt\cos 2t - 2ct\sin 2t$$

$$D^2y_p = (4d - 4a)\cos 2t + (-4b - 4c)\sin 2t - 4ct\cos 2t - 4dt\sin 2t$$

$$\begin{aligned}(D^2 + 2D + 2)y_p = {} & (-2a + 4b + 2c + 4d)\cos 2t \\ & + (-4a - 2b - 4c + 2d)\sin 2t \\ & + (-2c + 4d)t\cos 2t + (-4c - 2d)t\sin 2t\end{aligned}$$

We must solve

$$\begin{aligned} -2a + 4b + 2c + 4d &= 0 \\ -4a - 2b - 4c + 2d &= 1 \\ -2c + 4d &= 1 \\ -4c - 2d &= 0 \end{aligned}$$

The solution is $a = \frac{1}{50}$, $b = -\frac{7}{50}$, $c = -\frac{1}{10}$, $d = \frac{1}{5}$, and the general solution is

$$y = c_1e^{-t}\cos t + c_2e^{-t}\sin t + \tfrac{1}{50}\cos 2t - \tfrac{7}{50}\sin 2t - \tfrac{1}{10}t\cos 2t + \tfrac{1}{5}t\sin 2t$$

EXERCISES 6.5

1 Find annihilation operators for each of the following functions:
(*a*) $2t^2 + 3t - 5$ (*b*) $(t^2 + 2t + 1)e^t$
(*c*) $te^{2t}\cos t + e^{2t}\sin t$ (*d*) $t^3e^{-t}\sin 3t + t^2e^{-t}\cos 3t$

2 Find the general solution of each of the following differential equations:
(*a*) $(D^2 + 2D + 1)y = 3e^t$
(*b*) $(D^2 - 5D + 6)y = 2e^{3t} + \cos t$
(*c*) $(D^3 - 1)y = te^t$
(*d*) $(D^2 + 4)y = t\cos 2t + \sin 2t$
(*e*) $(D^4 + 4D^2 + 4)y = \cos 2t - \sin 2t$
(*f*) $(D^2 - 2D + 5)y = te^t\sin 2t$

3 Solve the initial-value problem $(D^2 - 2D + 1)y = e^t$, $y(0) = \dot{y}(0) = 1$.

4 Solve the initial-value problem $(D^3 + D)y = te^t$, $y(0) = \dot{y}(0) = 0$, $\ddot{y}(0) = 1$.

5 If r is not a solution of $P(z) = 0$, show that $y_p = e^{rt}/P(r)$ is a particular solution of $P(D)y = e^{rt}$.

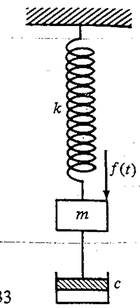

FIGURE 33

6.6 APPLICATIONS

There are many applications of linear differential equations. We shall illustrate with one from the theory of mechanical vibrations and one from the theory of electric networks. It will turn out that both problems lead to the same basic differential equation. This will illustrate the unification of two quite different fields of science through the study of a common differential equation.

Consider the following problem (see Fig. 33). A mass of m slugs is hanging on a spring with spring constant k pounds per foot. The motion of the spring is impeded by a dashpot which exerts a force counter to the motion and proportional to the velocity. The constant of proportionality is c pounds per foot per second. There is a variable force of $f(t)$ pounds driving the mass. If we pick a coordinate $Y(t)$ measured in feet from the position of natural length of the spring (unstretched), where $Y(t)$ is positive downward, then the forces on the mass are as follows:

mg = weight of mass
$-kY$ = restoring force of spring
$-c\dot{Y}$ = resistive force of dashpot
$f(t)$ = driving force

The sum of these forces gives the mass times the acceleration $\ddot{Y}$. Hence, the differential equation is

$$m\ddot{Y} = mg - kY - c\dot{Y} + f(t)$$

Let us introduce a new coordinate $y(t) = Y(t) - mg/k$, which is the displacement measured from the equilibrium position (recall Sec. 5.2). In terms of y the differential equation becomes

$$m\ddot{y} + c\dot{y} + ky = f(t)$$

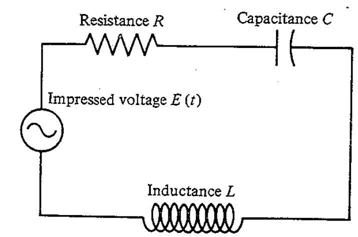

FIGURE 34

This is a nonhomogeneous second order differential equation with constant coefficients. An appropriate initial-value problem is to specify the initial displacement $y(0)$ and the initial velocity $\dot{y}(0)$.

Now let us consider the following electric network (see Fig. 34). Kirchhoff's law states that the impressed voltage is equal to the sum of the voltage drops around the circuit. If there is an instantaneous charge on the capacitor of Q coulombs, then the current flowing in the circuit is $I = \dot{Q}$ amperes and the voltage drops are as follows:

RI = voltage drop across resistance of R ohms
Q/C = voltage drop across capacitance of C farads
$L\dot{I}$ = voltage drop across inductance of L henrys

The appropriate equation is then

$$L\dot{I} + RI + \frac{Q}{C} = E(t)$$

In terms of Q this equation becomes

$$L\ddot{Q} + R\dot{Q} + \frac{Q}{C} = E(t)$$

and if we assume that R, C, and L do not change with time (a reasonable assumption in most cases), then we again have a linear second order equation with constant coefficients. If $E(t)$ is differentiable, then we can write an equation for the current I,

$$L\ddot{I} + R\dot{I} + \frac{I}{C} = \dot{E}(t)$$

which is again of the same type. An appropriate initial-value problem for the first equation is to specify the initial charge $Q(0)$ and initial current $I(0) = \dot{Q}(0)$. For the second equation we should specify the initial current $I(0)$ and the initial derivative $\dot{I}(0)$.

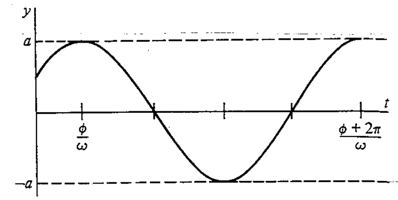

FIGURE 35

We shall study in detail the mechanical vibration problem, but the reader should keep in mind that the remarks apply equally well to the corresponding electrical problems. The cases are considered in order of increasing complexity.

1 Simple Harmonic Motion

Here we assume no damping ($c = 0$) and no forcing [$f(t) \equiv 0$]. The differential equation is $\ddot{y} + \omega^2 y = 0$, where $\omega^2 = k/m$. The general solution is

$$y = A \cos \omega t + B \sin \omega t$$

where A and B are arbitrary constants. Alternatively we can write

$$y = \sqrt{A^2 + B^2}\left(\frac{A}{\sqrt{A^2 + B^2}} \cos \omega t + \frac{B}{\sqrt{A^2 + B^2}} \sin \omega t\right) = a \cos (\omega t - \phi)$$

where $a = \sqrt{A^2 + B^2}$ is called the *amplitude* and $\phi = \tan^{-1}(B/A)$ is called the *phase angle*. This solution is plotted in Fig. 35. The *period*, which is the time required for the motion to go through one complete cycle, is

$$\tau = \frac{2\pi}{\omega}$$

Let us assume that the initial position $y(0)$ and initial velocity $\dot{y}(0)$ are given. Then $A = y(0)$ and $B = \dot{y}(0)/\omega$, and we have the following constants of the motion:

$$a = \sqrt{[y(0)]^2 + \left[\frac{\dot{y}(0)}{\omega}\right]^2}$$

$$\phi = \tan^{-1} \frac{\dot{y}(0)}{\omega y(0)}$$

$$\tau = \frac{2\pi\sqrt{m}}{\sqrt{k}}$$

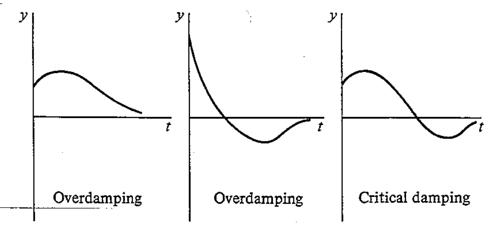

FIGURE 36

For ϕ we shall take the minimum nonnegative angle such that $\sin \phi = \dot{y}(0)/a\omega$ and $\cos \phi = y(0)/a$.

2 Free Vibration with Damping

In this case, we have $k > 0$ and $c > 0$ and $f(t) \equiv 0$. The differential equation is

$$(mD^2 + cD + k)y = 0$$

and the associated polynomial equation is $P(z) = mz^2 + cz + k = 0$. The solutions are

$$r_1, r_2 = \frac{-c \pm \sqrt{c^2 - 4km}}{2m}$$

The independent solutions of the differential equation depend on the value of $c^2 - 4km$ according to the following:

a Overdamping:

$$c^2 - 4km > 0 \qquad y = Ae^{r_1 t} + Be^{r_2 t} \qquad r_2 < r_1 < 0$$

b Critical damping:

$$c^2 - 4km = 0 \qquad y = (A + Bt)e^{r_1 t} \qquad r_1 = -\frac{c}{2m}$$

c Underdamping:

$$c^2 - 4km < 0 \qquad y = e^{-(c/2m)t}(A \cos \omega t + B \sin \omega t) \qquad \omega = \frac{\sqrt{4km - c^2}}{2m}$$

Some typical motions in cases *a* and *b* are illustrated in Fig. 36. In case *c*, we can write alternatively

$$y = ae^{-(c/2m)t} \cos (\omega t - \phi)$$

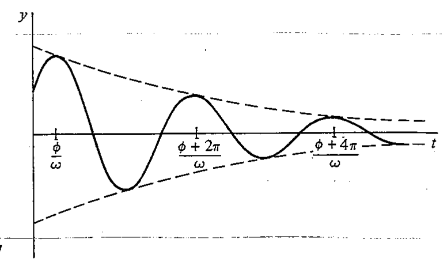

FIGURE 37

where $a = \sqrt{A^2 + B^2}$ and $\phi = \tan^{-1}(B/A)$. The motion is illustrated in Fig. 37. The motion is oscillatory, as in the case of simple harmonic motion, but the amplitude is diminishing according to the exponential factor $e^{-(c/2m)t}$. The motion is not periodic, but the time between successive peaks is $\tau = 2\pi/\omega$. If the damping constant c is very small compared with k and m, then $c/2m$ is very small and the exponential factor $e^{-(c/2m)t}$ is near 1 for reasonably small values of t. In this case the motion is very nearly simple harmonic motion, and τ is approximately a period.

For forcing we shall consider two cases.

3 Forced Vibrations without Damping

In this case $c = 0$, and for definiteness we shall take $f(t) = f_0 \cos \omega_0 t$, where f_0 is a constant and $\omega_0 \neq \omega = (k/m)^{1/2}$. The differential equation is

$$\ddot{y} + \omega^2 y = \frac{f_0}{m} \cos \omega_0 t$$

The complementary solution is

$$y_c = A \cos \omega t + B \sin \omega t$$

To find a particular solution we use the method of undetermined coefficients. We assume a solution of the form

$$y_p = C \cos \omega_0 t + D \sin \omega_0 t$$

Then

$$\ddot{y}_p + \omega^2 y_p = C(\omega^2 - \omega_0^2) \cos \omega_0 t + D(\omega^2 - \omega_0^2) \sin \omega_0 t = \frac{f_0}{m} \cos \omega_0 t$$

Therefore, $D = 0$ and

$$C = \frac{f_0}{m(\omega^2 - \omega_0^2)}$$

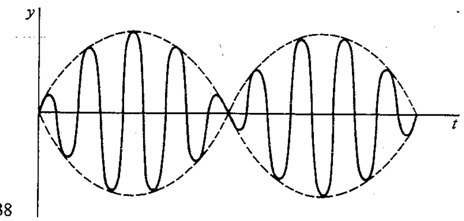

FIGURE 38

Suppose initially $y(0) = \dot{y}(0) = 0$. Then the solution is

$$y(t) = \frac{f_0}{m(\omega^2 - \omega_0^2)} (\cos \omega_0 t - \cos \omega t)$$

$$= \frac{2f_0}{m(\omega^2 - \omega_0^2)} \sin \frac{(\omega - \omega_0)t}{2} \sin \frac{(\omega + \omega_0)t}{2}$$

This motion is illustrated in Fig. 38. This phenomenon is known as *beating*. It is especially pronounced when ω_0 is approximately ω. Then one of the sine terms is slowly varying with a frequency of $(\omega - \omega_0)/4\pi$ while the other two terms are varying rapidly with a frequency of $(\omega + \omega_0)/4\pi$. This phenomenon is the basis for a technique known as *amplitude modulation* in electronics.

If the forcing term in this example had been $f_0 \cos \omega t$, then a different sort of solution would have been obtained. The reader will be asked to study this case in the exercises.

4 Forced Vibrations with Damping

In this case $k > 0$, $c > 0$, and for definiteness we shall again take $f(t) = f_0 \cos \omega_0 t$. The differential equation is

$$\ddot{y} + \frac{c}{m} \dot{y} + \omega^2 y = \frac{f_0}{m} \cos \omega_0 t$$

where $\omega^2 = k/m$. Depending on the value of $c^2 - 4km$, the complementary solution is one of three functions listed under case 2 above. To find a particular solution we assume

$$y_p = C \cos \omega_0 t + D \sin \omega_0 t$$