Cuarta tarea de Matemáticas IV (7 de noviembre de 2017)

En las siguientes páginas hay ejercicios resueltos y problemas del libro.
Revisar los problemas resueltos listados abajo, entenderlos y escribir las soluciones con sus palabras.
Resolver los ejercicios listados abajo.

- Ejemplo 8.2.2
- Ejercicios 8.2: 1a, 1d, 1e, 2, 3, 4
- Ejemplos 8.5.4 y 8.5.5
- Ejercicios 8.5: 1, 2, 3, 6
- Ejemplos 9.2.4 y 9.2.5
- Ejercicio 9.2.2
- Ejemplo 9.3.3
- Ejercicios 9.3: 1, 6, 7, 8
- Ejemplo 9.4.3
- Ejercicios 9.4: 1a, 1b, 2a, 2b, 5

where c is an arbitrary constant. We let $c = 1$, and then

$$A = \int_{x_0}^{x} u^{-2} \exp\left[-\int_{x_0}^{t} p(s)\,ds\right] dt$$

and

$$v(x) = u(x) \int_{x_0}^{x} [u(t)]^{-2} \exp\left[-\int_{x_0}^{t} p(s)\,ds\right] dt$$

If $u(t)$ is continuous and does not vanish in the interval $I = \{t \mid x_0 \le t \le x\}$, then the integration can always be carried out and we do get a solution. The functions $u(x)$ and $v(x)$ are independent, because if not, one would be a multiple of the other, which is clearly not the case. If $u(x_0) = 1$ and $u'(x_0) = 0$, then $v(x_0) = 0$ and

$$v'(x_0) = [u(x_0)]^{-1} \exp\left[-\int_{x_0}^{x_0} p(s)\,ds\right]$$

leading to the conclusion that $v'(x_0) = 1$.

EXAMPLE 8.2.2 Find the general solution of $y'' - xy' - y = 0$. Since $p(x) = -x$ and $q(x) = -1$ have Taylor expansions about $x = 0$, zero is an ordinary point of the differential equation. Assuming a solution of the form

$$y(x) = \sum_{k=0}^{\infty} c_k x^k$$

we have, upon substituting in the differential equation,

$$\sum_{k=2}^{\infty} k(k-1)c_k x^k - \sum_{k=0}^{\infty} (k+1)c_k x^k = 0$$

In the first summation we let $k = m + 2$, and in the second summation we let $k = m$. Then we have

$$\sum_{m=0}^{\infty} [(m+1)(m+2)c_{m+2} - (m+1)c_m]x^m = 0$$

Equating coefficients to zero, we obtain for $m = 0, 1, 2, \ldots$

$$c_{m+2} = \frac{c_m}{m+2}$$

If we put $c_0 = 1$ and $c_1 = 0$, we obtain the solution

$$y_1(x) = 1 + \frac{x^2}{2} + \frac{x^4}{2\cdot 4} + \frac{x^6}{2\cdot 4\cdot 6} + \cdots$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{x^2}{2}\right)^k = e^{x^2/2}$$

We can get another series solution by putting $c_0 = 0$, $c_1 = 1$, but this time let us look for a solution of the form $y_2(x) = A(x)e^{x^2/2}$. According to the above, the differential equation satisfied by A is

$$A'' + xA' = 0$$

which has an integrating factor $e^{x^2/2}$. Then

$$\frac{d}{dx}(A'e^{x^2/2}) = 0$$

$$A' = e^{-x^2/2}$$

$$y_2(x) = e^{x^2/2} \int_0^x e^{-t^2/2}\,dt$$

Notice that $y_2(0) = 0$, $y_2'(0) = 1$, so y_1 and y_2 are clearly independent.

EXERCISES 8.2

1 Show that each of the following equations has an ordinary point at $x = 0$. In each case determine the minimum radius† of convergence of the power-series solution:

(*a*) $y'' + x^2y' + xy = 0$
(*b*) $y'' + (1 - x)y' + x^2y = 0$
(*c*) $(1 - x)y'' + y' + xy = 0$
(*d*) $(1 - x^2)y'' + xy' + \dfrac{1}{2 + x}y = 0$
(*e*) $y'' - 2xy' + \lambda y = 0$ (Hermite equation), λ constant
(*f*) $y'' = xy$ (Airy equation)
(*g*) $(1 - x^2)y'' - 2xy' + \lambda y = 0$ (Legendre equation), λ constant
(*h*) $(1 - x^2)y'' - xy' + \lambda y = 0$ (Tchebysheff equation), λ constant

2 Determine two independent power-series solutions of the Airy differential equation, $y'' - xy = 0$. Where do they converge?

3 Determine two independent power-series solutions of the Hermite equation, $y'' - 2xy' + \lambda y = 0$. Where do they converge? Show that if $\lambda = 2n$, $n = 0, 1, 2, 3, \ldots$, then there is a polynomial solution. These polynomials are proportional to the Hermite polynomials.

4 Determine two independent power-series solutions of the Legendre equation, $(1 - x^2)y'' - 2xy' + \lambda y = 0$. Where do they converge? Show that if $\lambda = n(n + 1)$, $n = 0, 1, 2, \ldots$, then there is a polynomial solution. These polynomials are proportional to the Legendre polynomials.

5 Determine two independent power-series solutions of the Tchebysheff equation, $(1 - x^2)y'' - xy' + \lambda y$. Where do they converge? Show that if $\lambda = n^2$,

† Note that Theorem 8.2.1 does not preclude the possibility that the power-series solutions could converge for $|x| \ge R$.

Therefore, for nontrivial solutions $\lambda^2 = \bar{\lambda}^2$. In this case, we cannot show that $\lambda^2 > 0$. Therefore, we consider three cases.

CASE 1 $\lambda^2 = 0$ In this case $y(x) = Ax + B$, and

$$D(0) = \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} = -1$$

Therefore, $\lambda^2 = 0$ is not a characteristic value.

CASE 2 $\lambda^2 < 0$ Then if $\lambda^2 = -\mu^2$, $y(x) = Ae^{\mu x} + Be^{-\mu x}$, and

$$D(\lambda) = \begin{vmatrix} 1+\mu & 1-\mu \\ (1+\mu)e^{\mu} & (1-\mu)e^{-\mu} \end{vmatrix} = 2(\mu^2 - 1)\sinh \mu$$

Since $\sinh \mu \neq 0$, for $D(\lambda) = 0$, $\mu^2 - 1 = 0$, $\mu^2 = 1$. In this case, $\lambda^2 = -1$ is a characteristic value with a corresponding characteristic function e^{-x}.

CASE 3 $\lambda^2 > 0$ In this case, $y(x) = A\cos\lambda x + B\sin\lambda x$, and

$$D(\lambda) = \begin{vmatrix} 1 & \lambda \\ \cos\lambda - \lambda\sin\lambda & \sin\lambda + \lambda\cos\lambda \end{vmatrix} = (1 + \lambda^2)\sin\lambda$$

For $D(\lambda) = 0$, $\sin\lambda = 0$, and $\lambda = n\pi$, $n = 1, 2, 3, \ldots$. The characteristic values are $\lambda_n^2 = n^2\pi^2$ with corresponding characteristic functions $y_n(x) = n\pi\cos n\pi x - \sin n\pi x$.

It is possible to solve boundary-value problems where there is a singular point in the interval.

EXAMPLE 8.5.3 Find all characteristic values and characteristic functions for the boundary-value problem $xy'' + y' + \lambda xy = 0$, $0 \le x \le 1$, $y(0)$ finite, $y(1) = 0$. The differential equation can be written as $(xy')' + \lambda xy = 0$. If λ and y are complex-valued, then $(x\bar{y}')' + \bar{\lambda}x\bar{y} = 0$, $\bar{y}(0)$ finite, $\bar{y}(1) = 0$. Then

$$(\lambda - \bar{\lambda})\int_0^1 xy\bar{y}\,dx = \int_0^1 [(x\bar{y}')'y - (xy')'\bar{y}]\,dx = (x\bar{y}'y - xy'\bar{y})\Big|_0^1 = 0$$

Since $|y|^2 > 0$ unless $y \equiv 0$, $\lambda = \bar{\lambda}$. Next we show that $\lambda > 0$, since

$$\lambda\int_0^1 xy^2\,dx = -\int_0^1 (xy')'y\,dx = -xy'y\Big|_0^1 + \int_0^1 x(y')^2\,dx = \int_0^1 x(y')^2\,dx > 0$$

Now we make the change of variable $\xi = \sqrt{\lambda}x$. The differential equation becomes $\xi y'' + y' + \xi y = 0$, which is the Bessel equation of order zero. Since the solution must be finite at $\xi = 0$, the only possible solution is $AJ_0(\xi) = AJ_0(\sqrt{\lambda}x)$. To satisfy the other boundary condition $J_0(\sqrt{\lambda}) = 0$. It can be shown that J_0 has an infinite sequence of positive zeros $\mu_1, \mu_2, \mu_3, \ldots$, approaching infinity. Therefore, the characteristic values are $\lambda_n = \mu_n^2$, $n = 1, 2, 3, \ldots$, and the characteristic function are $y_n(x) = J_0(\sqrt{\lambda_n}x)$.

We conclude this section by considering some nonhomogeneous boundary-value problems.

EXAMPLE 8.5.4 Find all possible solutions of $y'' + \lambda^2 y = e^x$, $\lambda^2 > 0$, $0 \le x \le 1$, $y(0) = 0$, $y(1) = 1$. The general solution of the differential equation is

$$y(x) = A\cos\lambda x + B\sin\lambda x + \frac{1}{1+\lambda^2}e^x$$

To satisfy the boundary conditions we must have

$$y(0) = A + \frac{1}{1+\lambda^2} = 0$$

$$y(1) = A\cos\lambda + B\sin\lambda + \frac{1}{1+\lambda^2}e = 1$$

The determinant of the coefficients of A and B is

$$D(\lambda) = \begin{vmatrix} 1 & 0 \\ \cos\lambda & \sin\lambda \end{vmatrix} = \sin\lambda$$

If $\sin\lambda \neq 0$, or, in other words, if λ^2 is not a characteristic value of the homogeneous problem $y'' + \lambda^2 y = 0$, $y(0) = y(1) = 0$, then there is a unique solution for A and B and there is a solution of the nonhomogeneous boundary-value problem. In that case, $A = -1/(1 + \lambda^2)$, and

$$B = \left(1 - \frac{e}{1 + \lambda^2} + \frac{\cos \lambda}{1 + \lambda^2}\right) \csc \lambda$$

In this case, we can prove that the solution is unique. Suppose there were two solutions y_1 and y_2. Then if $w = y_1 - y_2$, $w'' + \lambda^2 w = 0$, $w(0) = w(1) = 0$. But $w \equiv 0$ if λ^2 is not a characteristic value. Therefore, $y_1 \equiv y_2$. If $\lambda = n\pi$, $n = 1, 2, 3, \ldots$, then we have a different situation in which $D(\lambda) = 0$. Now we know from linear algebra that there may still be solutions for A and B but they will not be unique. To determine this we could check the values of a couple of 2×2 determinants. However, there is an analytical way to proceed. We first convert the problem to one with homogeneous boundary conditions by subtracting† x from y. Let $v(x) = y(x) - x$. Then

$$v'' + \lambda^2 v = e^x + \lambda^2 x = g(x)$$

and $v(0) = v(1) = 0$. If there were a solution $v(x)$, then

$$\begin{aligned}\int_0^1 g(x) \sin \lambda x \, dx &= \int_0^1 (v'' + \lambda^2 v) \sin \lambda x \, dx \\ &= v'(x) \sin \lambda x \big|_0^1 - \lambda \int_0^1 (v' \cos \lambda x - \lambda v \sin \lambda x) \, dx \\ &= -\lambda v(x) \cos \lambda x \big|_0^1 = 0\end{aligned}$$

Therefore, if there is a solution of the nonhomogeneous boundary-value problem in the case where λ^2 is a characteristic value, then necessarily $\int_0^1 g(x) \sin \lambda x \, dx = 0$, where $\sin \lambda x$ is the corresponding characteristic function. It can be shown that this is also a sufficient condition for a solution, which, however, is not unique because if $v(x)$ exists, any multiple of $\sin \lambda x$ can be added to it. In the present example, $\int_0^1 g(x) \sin \lambda x \, dx \neq 0$, so there is no solution.

EXAMPLE 8.5.5 Find all possible solutions of the nonhomogeneous boundary-value problem $y'' + \pi^2 y = \cos \pi x$, $0 \le x \le 1$, $y(0) = y(1) = 0$. In this case, π^2 is a characteristic value of the homogeneous boundary-value problem $y'' + \lambda^2 y = 0$, $y(0) = y(1) = 0$. However,

$$\int_0^1 \sin \pi x \cos \pi x \, dx = \frac{1}{2} \int_0^1 \sin 2\pi x \, dx = 0$$

† The function $u(x) = x$ satisfies the nonhomogeneous boundary conditions but does not satisfy the differential equation.

and therefore there is a nonunique solution of the boundary-value proble
The general solution of the differential equation is

$$y(x) = A \cos \pi x + B \sin \pi x + \frac{1}{2\pi} x \sin \pi x$$

For the boundary conditions we have

$$y(0) = A = 0$$

$$y(1) = B \sin \pi + \frac{\sin \pi}{2\pi} = 0$$

and B is arbitrary. Therefore, all solutions are of the form $y(x) = B \sin \pi x$
$(1/2\pi)x \sin \pi x$.

EXERCISES 8.5

1 Find all the characteristic values and characteristic functions of the boundar value problem $y'' + \lambda^2 y = 0$, $0 \le x \le L$, $y'(0) = y'(L) = 0$.

2 Find all the characteristic values and characteristic functions of the boundar value problem $y'' + \lambda^2 y = 0$, $0 \le x \le L$, $y'(0) = y(L) = 0$.

3 Show that the characteristic functions $y_n(x)$ of the boundary-value proble $y'' + \lambda^2 y = 0$, $y(0) = y(L) = 0$ are orthogonal on the interval $\{x \mid 0 \le x \le L$ that is,

$$\int_0^L y_n(x) y_m(x) \, dx = 0 \text{ for } n \neq m$$

Hint: This can be shown directly from the differential equation and boundar conditions without explicitly knowing the characteristic functions.

4 Show that the characteristic functions $y_n(x)$ of the boundary-value proble $y'' + \lambda^2 y = 0$, $y'(0) = y'(L)$ are orthogonal on the interval $\{x \mid 0 \le x \le L\}$.

5 Find the characteristic values and characteristic functions of the boundary-valu problem $y'' + \lambda^2 y = 0$, $0 \le x \le 1$, $y(0) = y(1) + y'(1) = 0$. Show that thes characteristic functions are orthogonal on the interval $\{x \mid 0 \le x \le 1\}$.

6 Show that the characteristic functions of Example 8.5.3 are orthogonal on th interval $\{x \mid 0 \le x \le 1\}$; that is, show that

$$\int_0^1 x y_n(x) y_m(x) \, dx = 0 \text{ for } n \neq m$$

7 Find the characteristic values and characteristic functions of the boundary-value problem $(xy')' + (\lambda x - 1/x)y = 0$, $0 \le x \le 1$, $y(0)$ finite, $y(1) = 0$. Show that the characteristic functions are orthogonal on the interva $\{x \mid 0 \le x \le 1\}$; that is, show that

$$\int_0^1 x y_n(x) y_m(x) \, dx = 0 \text{ for } n \neq m$$

PROOF For simplicity consider the case where $\mathbf{F}$ and $\mathbf{Y}$ have two coordinates $f_1(t, y_1, y_2)$ and $f_2(t, y_1, y_2)$. Let $\mathbf{Y}_1 = (y_{11}, y_{12})$ and $\mathbf{Y}_2 = (y_{21}, y_{22})$ be in R_3. Then

$$\begin{aligned}\|\mathbf{F}(t,\mathbf{Y}_1) - \mathbf{F}(t,\mathbf{Y}_2)\| &\le |f_1(t,y_{11},y_{12}) - f_1(t,y_{21},y_{22})| \\ &\quad + |f_2(t,y_{11},y_{12}) - f_2(t,y_{21},y_{22})|\end{aligned}$$

By the mean-value theorem

$$f_1(t,y_{11},y_{12}) - f_1(t,y_{21},y_{22}) = \frac{\partial f_1}{\partial y_1}(y_{11} - y_{12}) + \frac{\partial f_1}{\partial y_2}(y_{21} - y_{22})$$

$$f_2(t,y_{11},y_{12}) - f_2(t,y_{21},y_{22}) = \frac{\partial f_2}{\partial y_1}(y_{11} - y_{12}) + \frac{\partial f_2}{\partial y_2}(y_{21} - y_{22})$$

where the partial derivatives are evaluated, in each case, at a point between $\mathbf{Y}_1$ and $\mathbf{Y}_2$ which is in R_3. Since the partial derivatives are all continuous in R_3, let M stand for the maximum of the absolute values of the four first partial derivatives in R_3. Then

$$\begin{aligned}\|\mathbf{F}(t,\mathbf{Y}_1) - \mathbf{F}(t,\mathbf{Y}_2)\| &\le 2M|y_{11} - y_{22}| + 2M|y_{21} - y_{22}| \\ &\le 4M \max\,[|y_{11} - y_{12}|, |y_{21} - y_{22}|] \\ &\le 4M\|\mathbf{Y}_1 - \mathbf{Y}_2\|\end{aligned}$$

This completes the proof in the simple case. The general case follows by an obvious extension of these ideas.

In Sec. 9.6, we shall prove the following theorem.

Theorem 9.2.2 If $\mathbf{F}(t,\mathbf{Y})$ is continuous and satisfies a Lipschitz condition in

$$\begin{aligned}R_{n+1} = \{(t, y_1, y_2, \ldots, y_n) \mid |t - t_0| \le r_0, \\ |y_1 - a_1| \le r_1, \ldots, |y_n - a_n| \le r_n\}\end{aligned}$$

$r_0, r_1, r_2, \ldots, r_n$ all positive, then there exists a unique solution of the initial-value problem $\dot{\mathbf{Y}} = \mathbf{F}(t,\mathbf{Y})$, $\mathbf{Y}(t_0) = \mathbf{A} = (a_1, a_2, \ldots, a_n)$ for $|t - t_0| \le h = \min\,[r_0, r_1/M, r_2/M, \ldots, r_n/M]$ where $M = \max \|\mathbf{F}(t,\mathbf{Y})\|$ in R_{n+1}.

EXAMPLE 9.2.3 Show that there exists a unique solution of the initial-value problem $\dot{y}_1 = y_1 + y_2$, $\dot{y}_2 = y_1^2 + y_2^2$, $y_1(0) = 0$, $y_2(0) = 0$. According to Example 9.2.1, $\mathbf{F} = (y_1 + y_2, y_1^2 + y_2^2)$ satisfies a Lipschitz condition in $R_3 = \{(t,y_1,y_2) \mid |t| \le 1, |y_1| \le 1, |y_2| \le 1\}$ which contains the point (0,0,0). According to Theorem 9.2.2, there exists a unique solution to the initial-value problem. Clearly the constant function $\mathbf{Y} = (0,0)$ satisfies all the conditions. Therefore, this is the unique solution.

EXAMPLE 9.2.4 Show that the initial-value problem $\dot{y}_1 = y_1$, $\dot{y}_2 = \sqrt{y_2}$, $y_1(0) = 0$, $y_2(0) = 0$, has a nonunique solution. Clearly $y_1 = 0$, $y_2 = 0$ satisfies all the conditions. However, so does the solution $y_1 = 0$, $y_2 = t^2/4$. Notice that $\mathbf{F} = (y_1, \sqrt{y_2})$ does not satisfy a Lipschitz condition in

$$R_3 = \{(t,y_1,y_2) \mid |t| \le 1, |y_1| \le 1, 0 \le y_2 \le 1\}$$

EXAMPLE 9.2.5 Show that the initial-value problem $\dot{y}_1 = a_{11}(t)y_1 + a_{12}(t)y_2 + b_1(t)$, $\dot{y}_2 = a_{21}(t)y_1 + a_{22}(t)y_2 + b_2(t)$, $y_1(t_0) = c_1$, $y_2(t_0) = c_2$, where $a_{11}, a_{12}, a_{21}, a_{22}, b_1$, and b_2 are all continuous for $|t - t_0| \le r_0$, has a unique solution in the entire interval $I = \{t \mid |t - t_0| \le r_0\}$. Consider the "rectangle" $R_3 = \{(t,y_1,y_2) \mid |t - t_0| \le r_0, |y_1 - c_1| \le r_1, |y_2 - c_2| \le r_2\}$. Let $f_1 = a_{11}y_1 + a_{12}y_2 + b_1$, $f_2 = a_{21}y_1 + a_{22}y_2 + b_2$. Since $\partial f_1/\partial y_1 = a_{11}(t)$, $\partial f_1/\partial y_2 = a_{12}(t)$, $\partial f_2/\partial y_1 = a_{21}(t)$, $\partial f_2/\partial y_2 = a_{22}(t)$ are all continuous for $|t - t_0| \le r_0$ for all y_1 and y_2, $\mathbf{F} = (f_1, f_2)$ satisfies a Lipschitz condition for arbitrary positive r_1 and r_2. It can be shown (Exercise 9.2.1) that $M = \max \|\mathbf{F}\| \le \alpha r_1 + \beta r_2 + \gamma$, where α, β, and γ are positive constants. Therefore, $\min\,[r_0, r_1/M, r_2/M]$ is either r_0 or some fixed positive number independent of where the initial conditions are taken in I. Hence, Theorem 9.2.2 gives us existence and uniqueness in the entire interval or gives us a minimum positive distance away from the initial point where a unique solution can be obtained. In the latter case, a finite sequence of initial-value problems can be solved to continue the solution uniquely throughout the interval I. The situation covered in this example is typical of the general linear first order system, which will be discussed in the next two sections.

EXERCISES 9.2

1 Referring to Example 9.2.5, show that $M = \max \|\mathbf{F}\| \le \alpha r_1 + \beta r_2 + \gamma$, where α, β, and γ are positive constants. (α and β depend only on $a_{11}, a_{12}, a_{21}, a_{22}$.)

2 Find a first order system equivalent to the second order equation $\ddot{y} + 5\dot{y} + 6y = e^t$. Use Theorem 9.2.2 to prove that the initial-value problem $y(0) = y_0$, $\dot{y}(0) = \dot{y}_0$ has a unique solution.

which would require that $|M - \lambda I| = 0$ and that $\mathbf{X}$ be a characteristic vector. The condition that $W(t) \neq 0$ is exactly the condition that $\mathbf{X}_1$ and $\mathbf{X}_2$ be independent. So we come back to the same question raised in Chap. 4, namely when does an $n \times n$ matrix of constants have n independent characteristic vectors? We shall look at linear first order systems with constant coefficients from this point of view in the next section.

EXAMPLE 9.3.2 Find the general solution of the system

$$\begin{aligned} \dot{y}_1 &= 9y_1 - 3y_1 \\ \dot{y}_2 &= -3y_1 + 12y_2 - 3y_3 \\ \dot{y}_3 &= -3y_2 + 9y_3 \end{aligned}$$

We look for solutions of the form $\mathbf{Y} = \mathbf{X}e^{\lambda t}$. Substituting, we have

$$(\lambda\mathbf{X} - M\mathbf{X})e^{\lambda t} = \mathbf{0}$$

A necessary condition for nontrivial solutions is

$$|M - \lambda I| = \begin{vmatrix} 9-\lambda & -3 & 0 \\ -3 & 12-\lambda & -3 \\ 0 & -3 & 9-\lambda \end{vmatrix} = -(\lambda - 6)(\lambda - 9)(\lambda - 15)$$

The characteristic values are $\lambda_1 = 6$, $\lambda_2 = 9$, $\lambda_3 = 15$. The corresponding characteristic vectors are

$$\mathbf{X}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad \mathbf{X}_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \qquad \mathbf{X}_3 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

The general solution is therefore

$$\mathbf{Y} = c_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{6t} + c_2 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} e^{9t} + c_3 \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} e^{15t}$$

where c_1, c_2, and c_3 are arbitrary constants.

If the linear system is nonhomogeneous and we have a fundamental system of solutions $\mathbf{Y}_1, \mathbf{Y}_2, \ldots, \mathbf{Y}_n$ of the homogeneous system, then it is always possible to find a solution of the nonhomogeneous system by the method of *variation of parameters*. We write the system as

$$\dot{\mathbf{Y}} - M\mathbf{Y} = \mathbf{B}(t)$$

and look for a solution in the form

$$\mathbf{Y} = A_1(t)\mathbf{Y}_1 + A_2(t)\mathbf{Y}_2 + \cdots + A_n(t)\mathbf{Y}_n$$

Differentiating, we have

$$\dot{\mathbf{Y}} = A_1\dot{\mathbf{Y}}_1 + A_2\dot{\mathbf{Y}}_2 + \cdots + A_n\dot{\mathbf{Y}}_n + \dot{A}_1\mathbf{Y}_1 + \dot{A}_2\mathbf{Y}_2 + \cdots + \dot{A}_n\mathbf{Y}_n$$

Substituting and using the fact that $\dot{\mathbf{Y}}_k - M\mathbf{Y}_k = \mathbf{0}$, $k = 1, 2, \ldots, n$, gives

$$\dot{A}_1\mathbf{Y}_1 + \dot{A}_2\mathbf{Y}_2 + \cdots + \dot{A}_n\mathbf{Y}_n = \mathbf{B}(t)$$

This is a system of linear algebraic equations for the unknowns $\dot{A}_1, \dot{A}_2, \ldots, \dot{A}_n$. The determinant of the coefficient matrix is the Wronskian of $\mathbf{Y}_1, \mathbf{Y}_2, \ldots, \mathbf{Y}_n$, which is never zero. Hence, we can always solve for $\dot{A}_1, \dot{A}_2, \ldots, \dot{A}_n$. Integrating, we obtain $A_1, A_2, \ldots, A_n$, giving us the required solution.

EXAMPLE 9.3.3 Find the general solution of the system $\dot{y}_1 = y_1 + 2y_2 + e^t$, $\dot{y}_2 = 3y_1 + 2y_2 - e^{-t}$. This system is nonhomogeneous so we use the method of variation of parameters. We have already determined a fundamental system of solutions of the homogeneous equations, namely $\mathbf{Y}_1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix} e^{4t}$ and $\mathbf{Y}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t}$. We seek a solution of the nonhomogeneous system of the form

$$\mathbf{Y}(t) = A_1(t)\mathbf{Y}_1 + A_2(t)\mathbf{Y}_2$$

According to the above, we must solve

$$\dot{A}_1\mathbf{Y}_1 + \dot{A}_2\mathbf{Y}_2 = \mathbf{B}(t) = \begin{pmatrix} e^t \\ -e^{-t} \end{pmatrix}$$

or

$$\begin{aligned} 2\dot{A}_1e^{4t} + \dot{A}_2e^{-t} &= e^t \\ 3\dot{A}_1e^{4t} - \dot{A}_2e^{-t} &= -e^{-t} \end{aligned}$$

Solving for $\dot{A}_1$ and $\dot{A}_2$, we have

$$\begin{aligned} \dot{A}_1 &= \tfrac{1}{5}e^{-3t} - \tfrac{1}{5}e^{-5t} \\ \dot{A}_2 &= \tfrac{3}{5}e^{2t} + \tfrac{2}{5} \end{aligned}$$

or

$$\begin{aligned} A_1 &= -\tfrac{1}{15}e^{-3t} + \tfrac{1}{25}e^{-5t} \\ A_2 &= \tfrac{3}{10}e^{2t} + \tfrac{2}{5}t \end{aligned}$$

and the general solution is

$$\mathbf{Y}(t) = c_1\mathbf{Y}_1 + c_2\mathbf{Y}_2 + (\tfrac{1}{25}e^{-5t} - \tfrac{1}{15}e^{-3t})\mathbf{Y}_1 + (\tfrac{3}{10}e^{2t} + \tfrac{2}{5}t)\mathbf{Y}_2$$

EXERCISES 9.3

1. Let M be a constant $n \times n$ matrix. Show that the system $\dot{\mathbf{Y}} = M\mathbf{Y}$ has fundamental system of solutions of the form $\mathbf{X}_1 e^{\lambda_1 t}, \mathbf{X}_2 e^{\lambda_2 t}, \ldots, \mathbf{X}_n e^{\lambda_n t}$ if M has n independent characteristic vectors $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n$ with corresponding characteristic values $\lambda_1, \lambda_2, \ldots, \lambda_n$.
2. Let M be a constant $n \times n$ matrix. Show that the system $\dot{\mathbf{Y}} = M\mathbf{Y}$ has a fundamental system of solutions of the form $\mathbf{X}_1 e^{\lambda_1 t}, \mathbf{X}_2 e^{\lambda_2 t}, \ldots, \mathbf{X}_n e^{\lambda_n t}$, where $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n$ are characteristic vectors of M, if the characteristic values $\lambda_1, \lambda_2, \ldots, \lambda_n$ are distinct.
3. Let M be a constant $n \times n$ matrix. Show that the system $\dot{\mathbf{Y}} = M\mathbf{Y}$ has a fundamental system of solutions of the form $\mathbf{X}_1 e^{\lambda_1 t}, \mathbf{X}_2 e^{\lambda_2 t}, \ldots, \mathbf{X}_n e^{\lambda_n t}$, where $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n$ are characteristic vectors of M, if M is real and symmetric.
4. Find the general solution of the system $2\dot{y}_1 = y_1 + y_2$, $2\dot{y}_2 = y_1 + y_2$.
5. Find the particular solution of the system in Exercise 4 satisfying $y_1(0) = 1$, $y_2(0) = -1$.
6. Find the general solution of the system $2\dot{y}_1 = y_1 + y_2 + e^t$, $2\dot{y}_2 = y_1 + y_2 - t$.
7. Find the general solution of the system $\dot{y}_1 = 8y_1 + 9y_2 + 9y_3$, $\dot{y}_2 = 3y_1 + 2y_2 + 3y_3$, $\dot{y}_3 = -9y_1 - 9y_2 - 10y_3$.
8. Find the particular solution of the system in Exercise 7 satisfying $y_1(0) = 1$, $y_2(0) = 0$, $y_3(0) = -1$.
9. Find the general solution of the system $\dot{y}_1 = 8y_1 + 9y_2 + 9y_3 + e^{-t}$, $\dot{y}_2 = 3y_1 + 2y_2 + 3y_3 - t$, $\dot{y}_3 = -9y_1 - 9y_2 - 10y_3 + e^{2t}$.
10. Find the general solution of $\dot{y}_1 = y_1 + 2y_2 + 3y_3$, $\dot{y}_2 = 2y_2 + 3y_3$, $\dot{y}_3 = 2y_3$. *Hint:* Solve the third equation, then the second, then the first. Compare this solution with that obtained using characteristic vectors.
11. Find the general solution of the system $\dot{y}_1 = y_1 - 2y_2$, $\dot{y}_2 = y_1 - y_2$. Express the answer using real-valued functions only.
12. Consider the linear first order system $\dot{\mathbf{Y}} = M\mathbf{Y}$, with a fundamental system of solutions $\mathbf{Y}_1, \mathbf{Y}_2, \ldots, \mathbf{Y}_n$. Show that $\dot{W} = (m_{11} + m_{22} + \cdots + m_{nn})W$, where W is the Wronskian. This is another way to show that either $W \equiv 0$ or $W \neq 0$. Why?

9.4 LINEAR FIRST ORDER SYSTEMS WITH CONSTANT COEFFICIENTS

In this section, we consider only first order systems of the form $\dot{\mathbf{Y}} = M\mathbf{Y}$, where M is a constant $n \times n$ matrix. If M has n distinct characteristic values $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n$, then, according to Theorem 4.5.3, M has n independent characteristic vectors $\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3, \ldots, \mathbf{X}_n$. In this case, a fundamental system

of solutions of the differential equations $\dot{\mathbf{Y}} = M\mathbf{Y}$ is $\mathbf{Y}_1 = \mathbf{X}_1 e^{\lambda_1 t}$, $\mathbf{Y}_2 = \mathbf{X}_2 e^{\lambda_2 t}$, $\mathbf{Y}_3 = \mathbf{X}_3 e^{\lambda_3 t}, \ldots, \mathbf{Y}_n = \mathbf{X}_n e^{\lambda_n t}$. The Wronskian of $\mathbf{Y}_1, \mathbf{Y}_2, \ldots, \mathbf{Y}_n$ is

$$W(t) = |\mathbf{X}_1 \quad \mathbf{X}_2 \quad \cdots \quad \mathbf{X}_n| \exp{(\lambda_1 + \lambda_2 + \cdots + \lambda_n)t}$$

which is not zero by the independence of the characteristic vectors. The general solution of the system $\dot{\mathbf{Y}} = M\mathbf{Y}$ is therefore

$$\mathbf{Y}(t) = c_1\mathbf{X}_1 e^{\lambda_1 t} + c_2\mathbf{X}_2 e^{\lambda_2 t} + \cdots + c_n\mathbf{X}_n e^{\lambda_n t}$$

where $c_1, c_2, \ldots, c_n$ are arbitrary constants. There are cases where the characteristic values are not distinct but where we can still get a complete set of independent characteristic vectors.† The general solution is as shown, but $\lambda_1, \lambda_2, \ldots, \lambda_n$ are not distinct.

If M is real and the initial values $\mathbf{Y}(t_0)$ are real, then the solution should be real and it should be possible to express the general solution with real-valued functions and constants $c_1, c_2, \ldots, c_n$. However, one or more of the characteristic values may be complex. If $\lambda = \alpha + i\beta$ is a complex characteristic value, then $\bar{\lambda} = \alpha - i\beta$ is also a characteristic value. This is because the coefficients in the characteristic equation are real. Also, the equation for the characteristic vector $(M - \lambda)\mathbf{X} = \mathbf{0}$ implies that $(M - \bar{\lambda})\overline{\mathbf{X}} = \mathbf{0}$ and hence that $\overline{\mathbf{X}}$ is a characteristic vector. In this case, $\mathbf{X}$ and $\overline{\mathbf{X}}$ are independent because they correspond to different characteristic values since $\beta \neq 0$. Let $\varphi = \text{Re}\,(\mathbf{X})$ and $\psi = \text{Im}\,(\mathbf{X})$. Then

$$\text{Re}\,(\mathbf{X}e^{\lambda t}) = e^{\alpha t}(\varphi \cos \beta t - \psi \sin \beta t)$$
$$\text{Im}\,(\mathbf{X}e^{\lambda t}) = e^{\alpha t}(\varphi \sin \beta t + \psi \cos \beta t)$$

are real-valued solutions of the system. They are independent because if a linear combination

$$A\,\text{Re}\,(\mathbf{X}e^{\lambda t}) + B\,\text{Im}\,(\mathbf{X}e^{\lambda t}) = A\,\frac{\mathbf{X}e^{\lambda t} + \overline{\mathbf{X}}e^{\bar{\lambda} t}}{2} + B\,\frac{\mathbf{X}e^{\lambda t} - \overline{\mathbf{X}}e^{\bar{\lambda} t}}{2i} = \mathbf{0}$$

then

$$\frac{A - iB}{2}\mathbf{X}e^{\lambda t} + \frac{A + iB}{2}\overline{\mathbf{X}}e^{\bar{\lambda} t} = \mathbf{0}$$

implies that $A - iB = 0$ and $A + iB = 0$. But this implies that $A = B = 0$. Therefore, if there are n distinct characteristic values with some of them complex, we may express the general solution of the system in terms of real-valued functions.

† For example, if M is real and symmetric or M is hermitian.

The matrix M is

$$M = \begin{pmatrix} 2 & 1 & 0 & 1 \\ 1 & 3 & -1 & 3 \\ 0 & 1 & 2 & 1 \\ 1 & -1 & -1 & -1 \end{pmatrix}$$

and its characteristic equation is

$$|M - \lambda I| = \lambda(\lambda - 2)^3 = 0$$

The characteristic values are $\lambda_1 = 0$ with multiplicity 1, and $\lambda_2 = 2$ with multiplicity 3. There is a characteristic vector $\mathbf{X}_1 = (0,1,0,-1)$ corresponding to λ_1 and a characteristic vector $\mathbf{X}_2 = (1,0,1,0)$ corresponding to λ_2, but the null space of $M - \lambda_2 I$ is of dimension 1, so we cannot find more independent characteristic vectors. Instead, we look for a solution of the form

$$\mathbf{Y} = \mathbf{Z}_2 e^{2t} + \mathbf{X}_2 t e^{2t}$$

The equation for $\mathbf{Z}_2$ is $(M - 2I)\mathbf{Z}_2 = \mathbf{X}_2$. Solving, we have $\mathbf{Z}_2 = (0,\frac{3}{2},0,-\frac{1}{2})$. Next we look for a solution of the form

$$\mathbf{Y} = \mathbf{W}_2 e^{2t} + \mathbf{Z}_2 t e^{2t} + \mathbf{X}_2 \frac{t^2}{2} e^{2t}$$

The equation for $\mathbf{W}_2$ is $(M - 2I)\mathbf{W}_2 = \mathbf{Z}_2$. Solving, we have $\mathbf{W}_2 = (0,-\frac{1}{2},-\frac{1}{2},\frac{1}{2})$. The general solution is therefore

$$\mathbf{Y}(t) = c_1\mathbf{X}_1 + c_2\mathbf{X}_2 e^{2t} + c_3(\mathbf{Z}_2 e^{2t} + \mathbf{X}_2 t e^{2t}) + c_4\left(\mathbf{W}_2 e^{2t} + \mathbf{Z}_2 t e^{2t} + \mathbf{X}_2 \frac{t^2}{2} e^{2t}\right)$$

If some of the repeated characteristic values are complex, then we can carry out the above procedure with these complex values, but we should expect the vectors $\mathbf{Z}$, $\mathbf{W}$, $\mathbf{V}$, etc., to be complex even if M is real. In the end we can express the general solution in terms of real-valued functions. We should expect these solutions to involve functions like $te^{\alpha t}\cos\beta t$, $te^{\alpha t}\sin\beta t$, $t^2e^{\alpha t}\cos\beta t$, $t^2e^{\alpha t}\sin\beta t$, etc.

We conclude this section by considering the solution of first order systems with constant coefficients by the use of Laplace transforms. The idea here is that in transforming the differential equations we change the system into a system of algebraic equations. We solve these algebraic equations for the transforms of the unknowns and then invert to find the solution.

EXAMPLE 9.4.3 Find the solution of the system $\dot{x} = 5x - y$, $\dot{y} = 3x + y$ satisfying $x(0) = x_0$, $y(0) = y_0$. Transforming each equation we have, letting $X(s) = \mathscr{L}[x(t)]$, $Y(s) = \mathscr{L}[y(t)]$,

$$sX - x_0 = 5X - Y$$
$$sY - y_0 = 3X + Y$$

Solving for X and Y, we have

$$X = \frac{x_0 s - x_0 - y_0}{s^2 - 6s + 8} = \frac{1}{2}\frac{y_0 - x_0}{s - 2} + \frac{1}{2}\frac{3x_0 - y_0}{s - 4}$$
$$Y = \frac{y_0 s - 5y_0 + 3x_0}{s^2 - 6s + 8} = \frac{3}{2}\frac{y_0 - x_0}{s - 2} + \frac{1}{2}\frac{3x_0 - y_0}{s - 4}$$

Inverting the transforms, we obtain

$$x(t) = \tfrac{1}{2}(y_0 - x_0)e^{2t} + \tfrac{1}{2}(3x_0 - y_0)e^{4t}$$
$$y(t) = \tfrac{3}{2}(y_0 - x_0)e^{2t} + \tfrac{1}{2}(3x_0 - y_0)e^{4t}$$

EXERCISES 9.4

1 Find the general solution of each of the following systems:

(*a*) $\dot{y}_1 = 2y_1 + y_2$, $\dot{y}_2 = y_1 + 2y_2$

(*b*) $\dot{y}_1 = y_1 - 2y_2$, $\dot{y}_2 = y_1 - y_2$

(*c*) $\dot{y}_1 = y_1 + y_2$, $\dot{y}_2 = -4y_1 + y_2$

(*d*) $2\dot{y}_1 = 3y_1 + y_2$, $2\dot{y}_2 = -y_1 + y_2$

2 Find the general solution of each of the following systems:

(*a*) $\dot{y}_1 = 3y_1 + y_2$, $\dot{y}_2 = y_1 + 3y_2$, $\dot{y}_3 = 2y_3$

(*b*) $\dot{y}_1 = 5y_1 + y_2 + y_3$, $\dot{y}_2 = -3y_1 + y_2 - 3y_3$, $\dot{y}_3 = -2y_1 - 2y_2 + 2y_3$

(*c*) $\dot{y}_1 = -8y_1 + 5y_2 + 4y_3$, $\dot{y}_2 = 5y_1 + 3y_2 + y_3$, $\dot{y}_3 = 4y_1 + y_2$

(*d*) $\dot{y}_1 = y_1 + y_2 + y_3$, $\dot{y}_2 = 2y_1 + y_2 - y_3$, $\dot{y}_3 = -y_2 + y_3$

3 Find the general solution of $\dot{y}_1 = y_1 - 2y_2 - \sin t$, $\dot{y}_2 = y_1 - y_2 + \cos t$.

4 Find the general solution of $\dot{y}_1 = y_1 + y_2 + y_3 - 3e^{-t}$, $\dot{y}_2 = 2y_1 + y_2 - y_3 + 6e^{-t}$, $\dot{y}_3 = -y_2 + y_3$.

5 Find the general solution of the system:

$$\dot{y}_1 = -7y_1 - 4y_4$$
$$\dot{y}_2 = -13y_1 - 2y_2 - y_3 - 8y_4$$
$$\dot{y}_3 = 6y_1 + y_2 + 4y_4$$
$$\dot{y}_4 = 15y_1 + y_2 + 9y_4$$

Express the solution in terms of real-valued functions.