Pyramids Over the Regular 3-Tori

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SIGMAP 2018

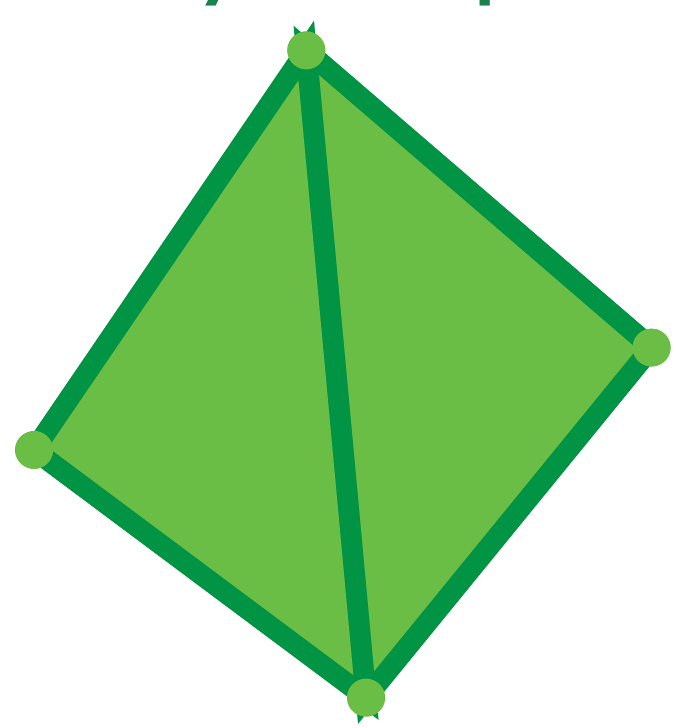
We’re really sorry.

Well… most of us.

Outline

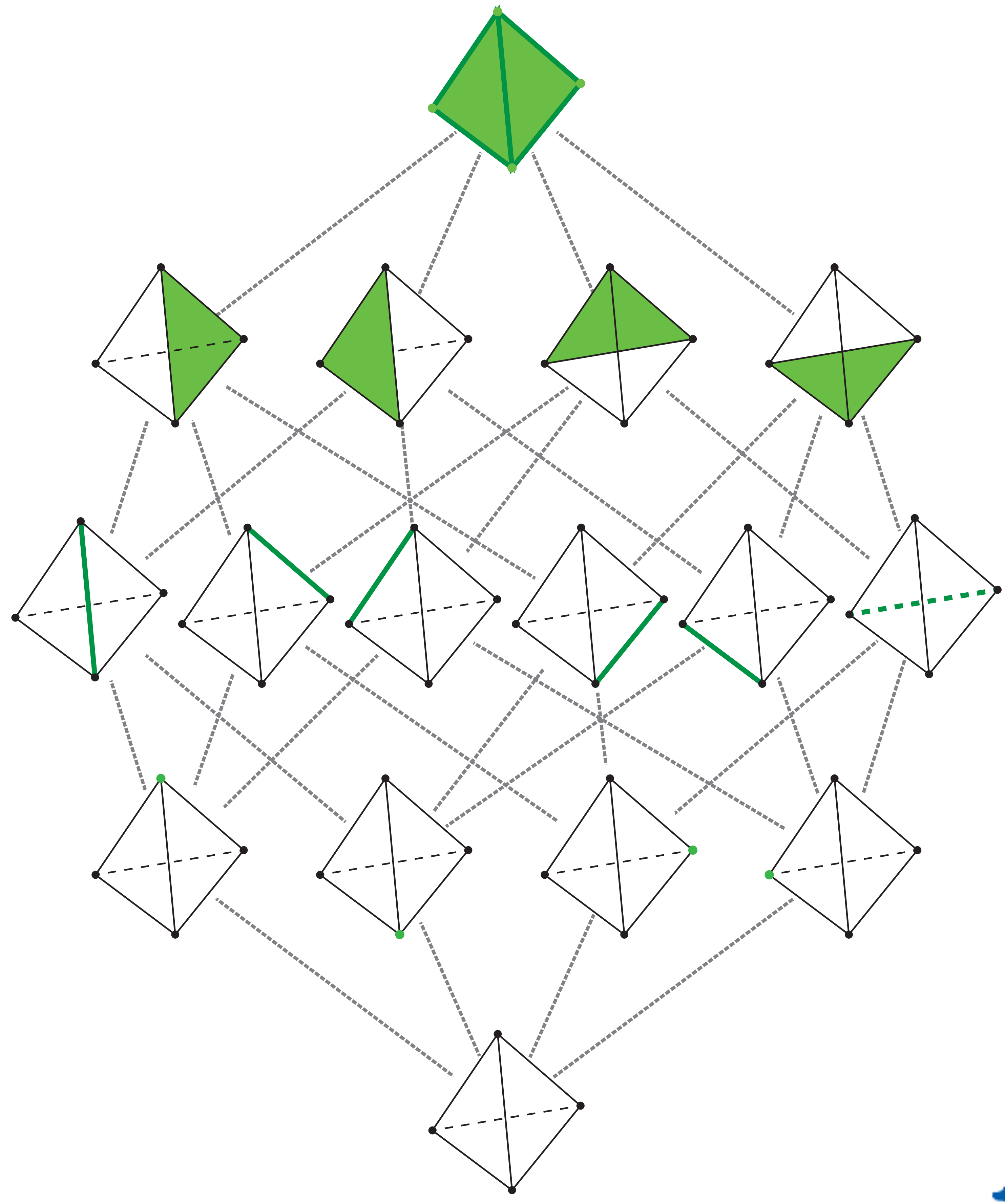
- Basic definitions
- Representing connection groups as subgroups of wreath products of automorphism groups
- An interesting infinite family of abstract polytopes

What is an Abstract Polytope?

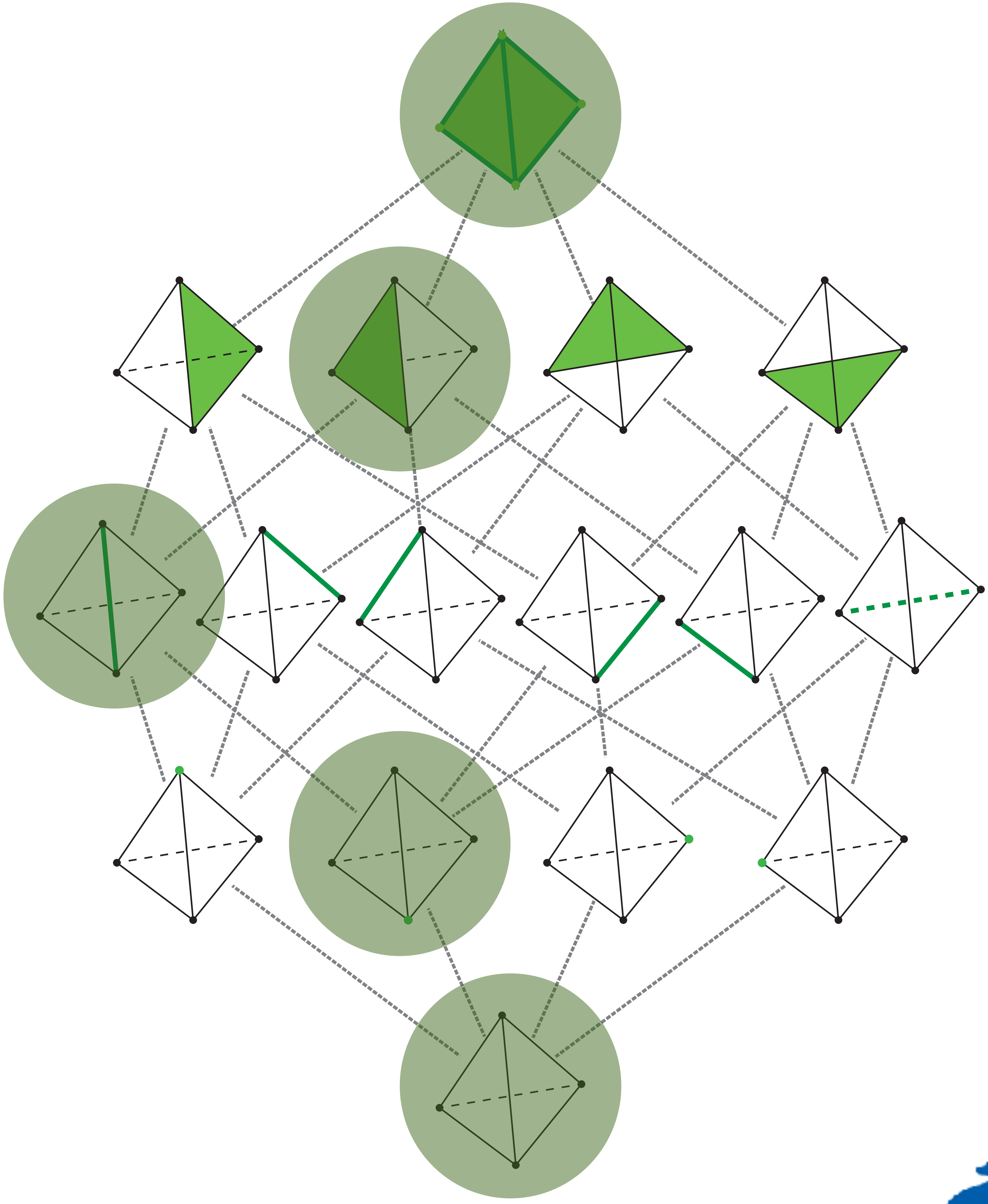


Escape Hatch

Posets as Polytopes



Posets as Polytopes



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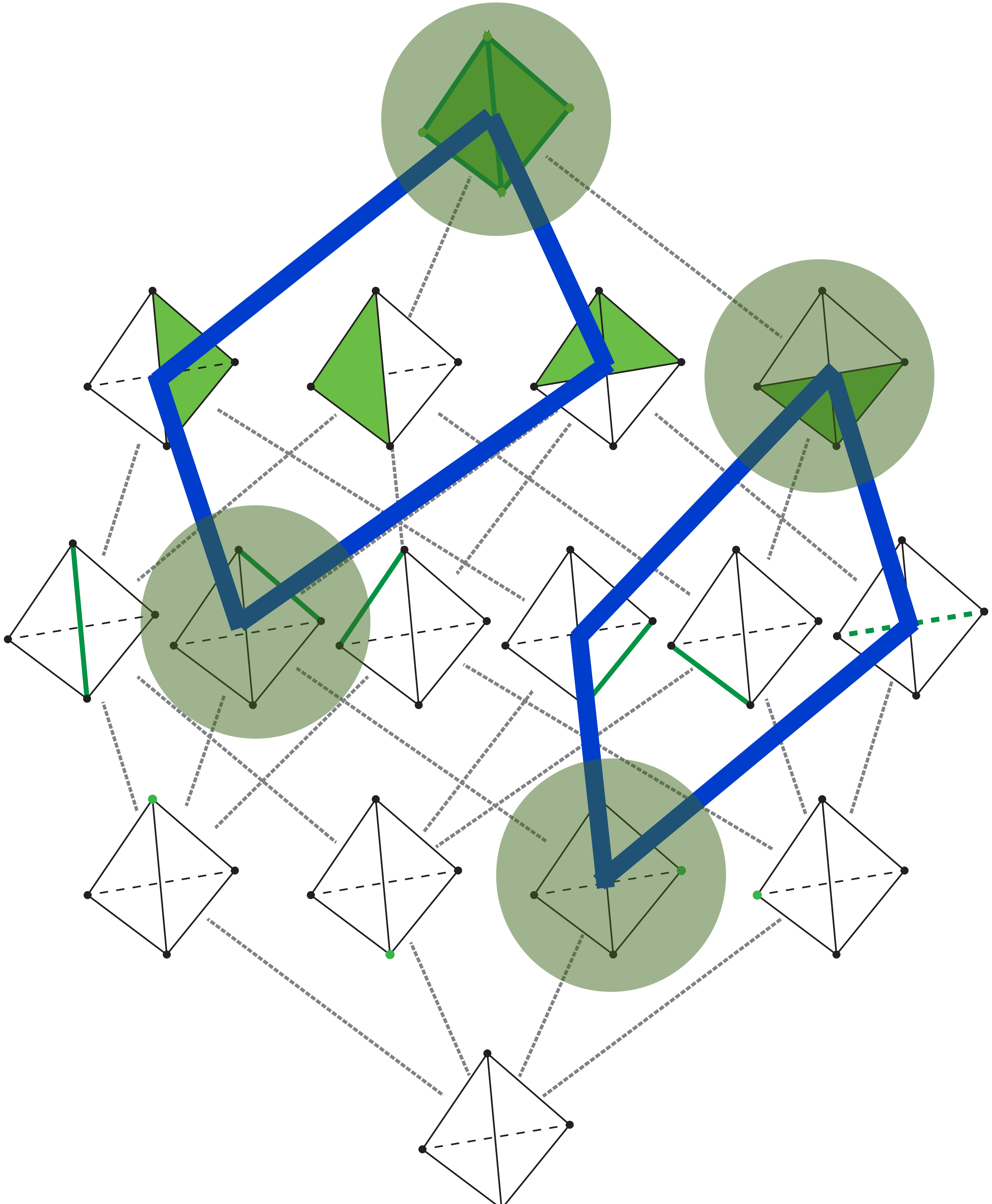
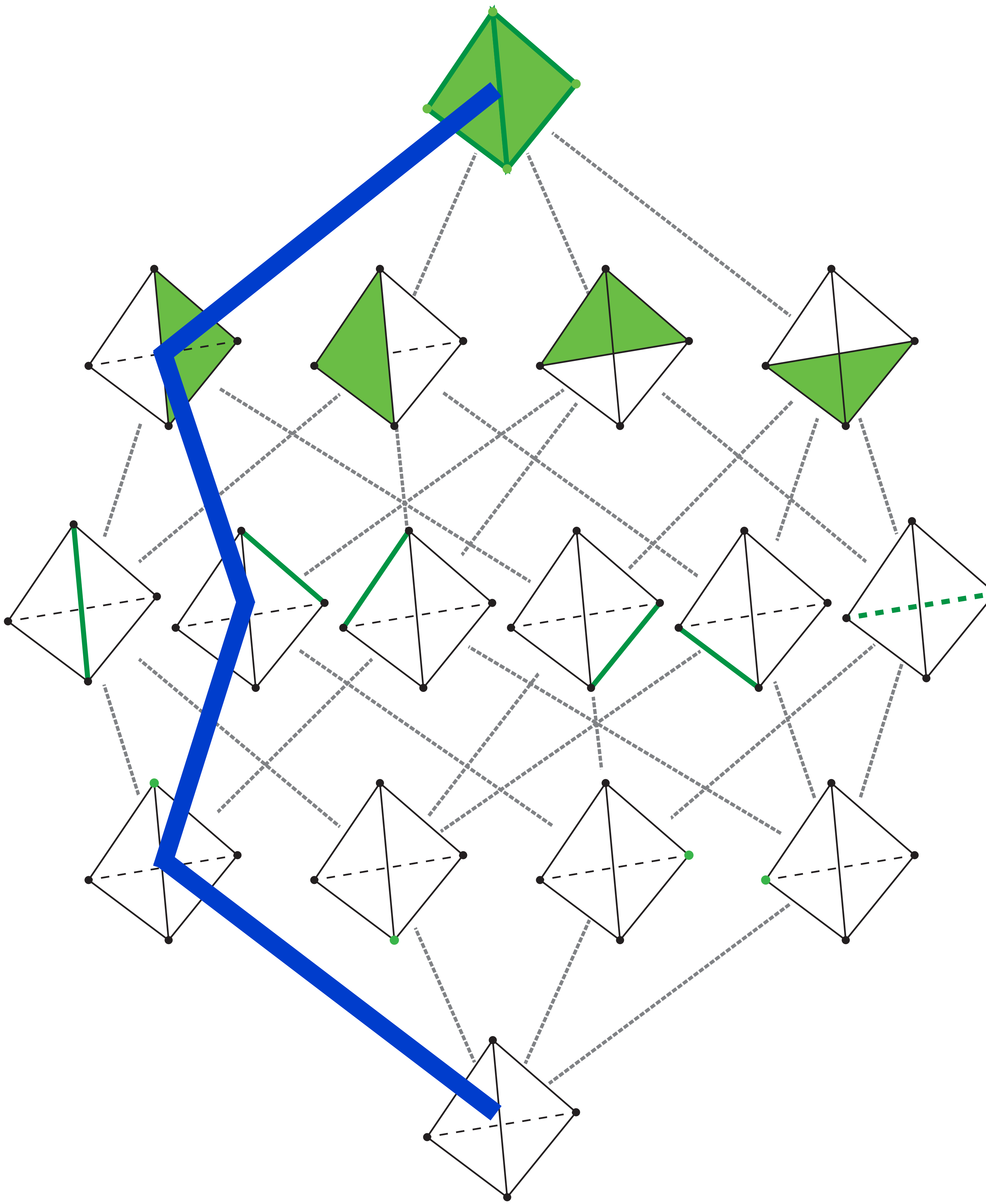


Posets as Polytopes



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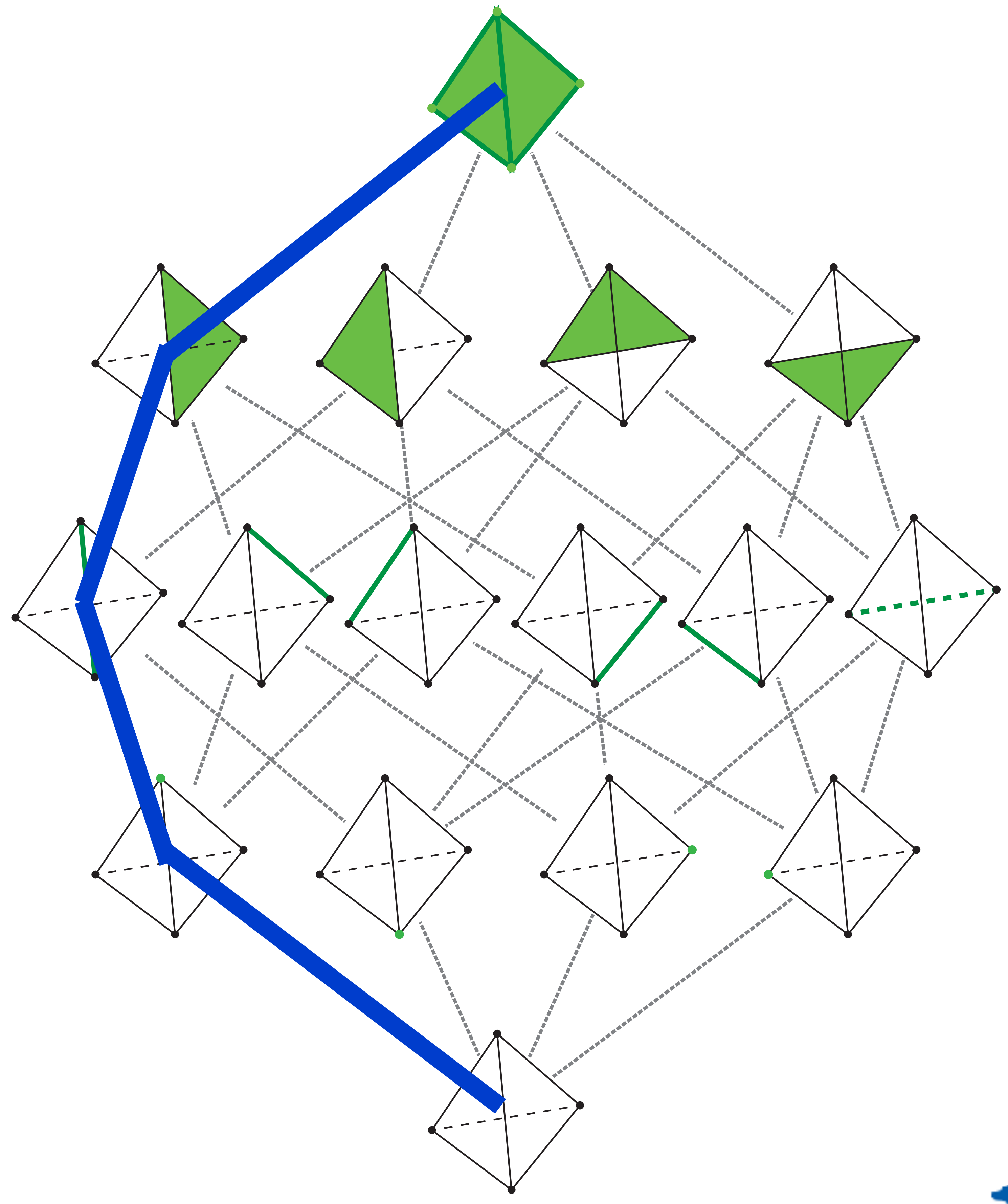

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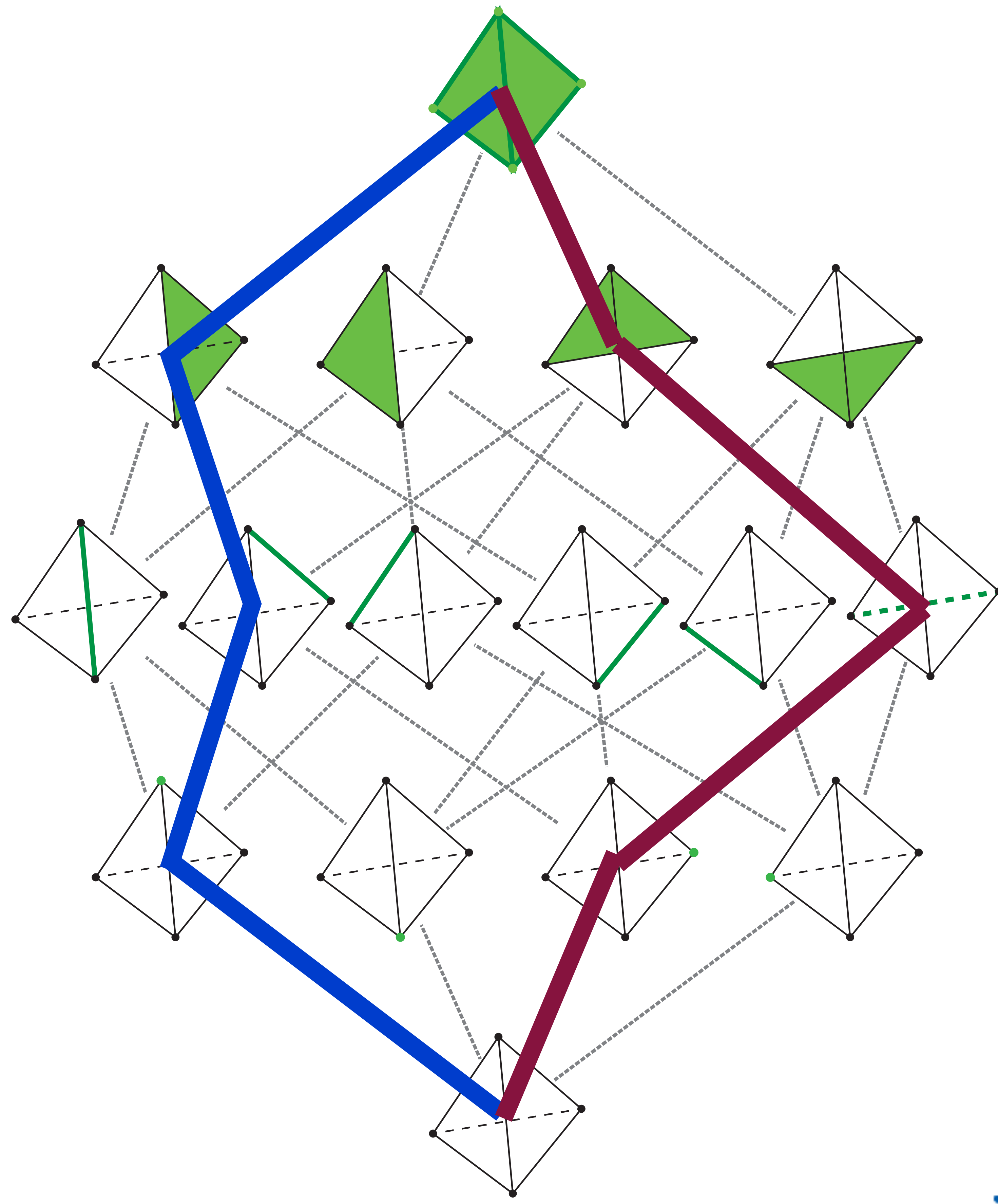

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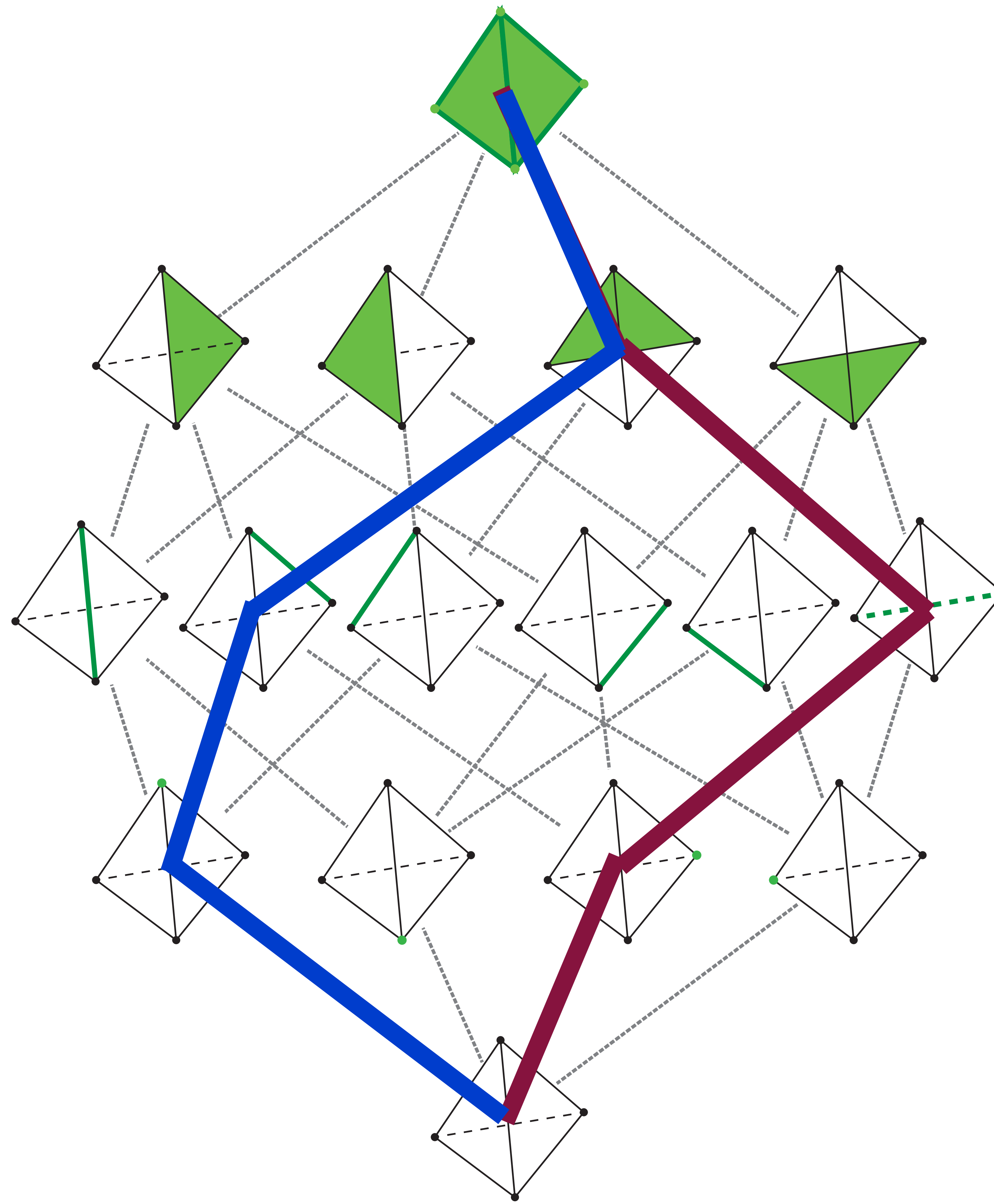

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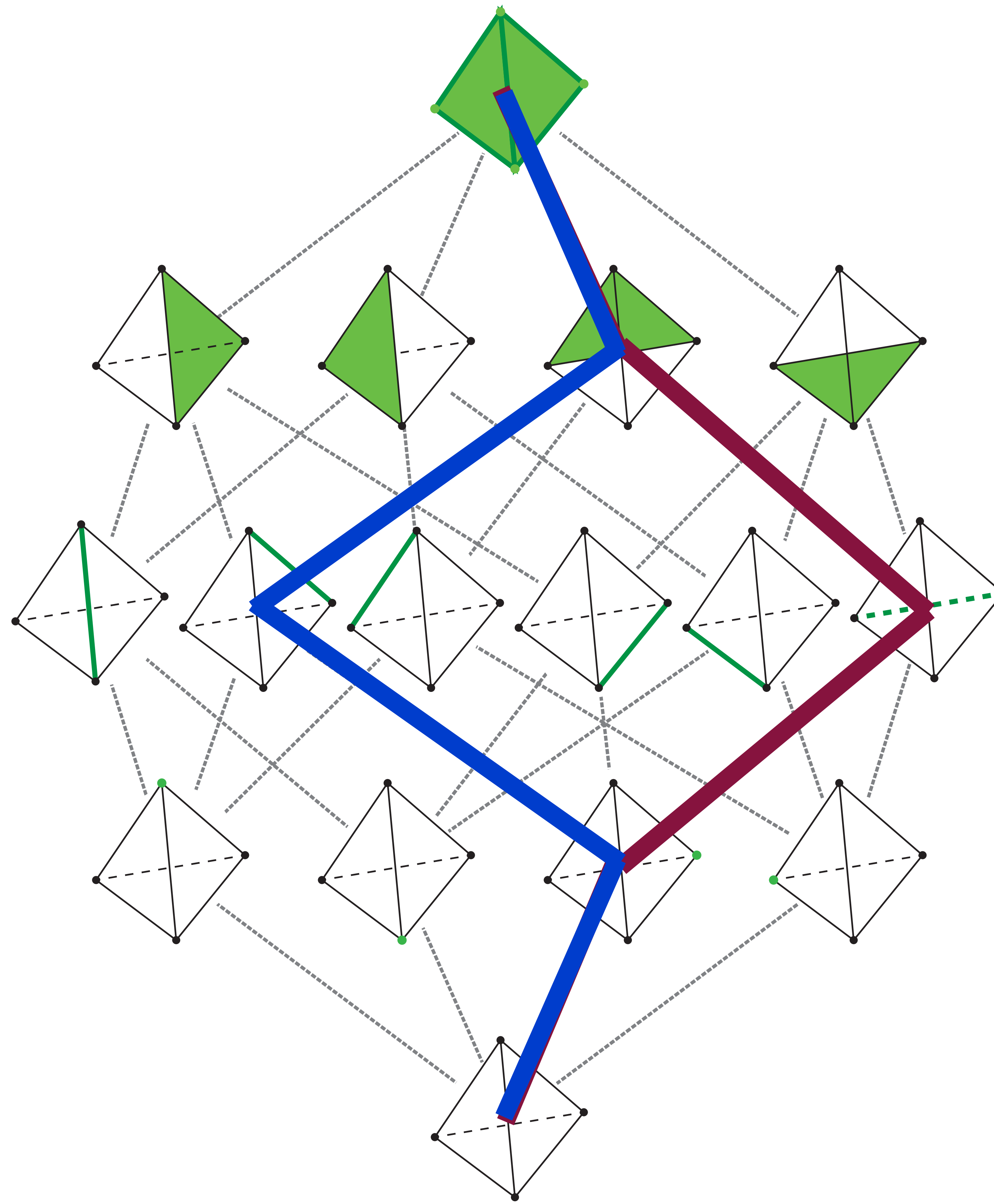

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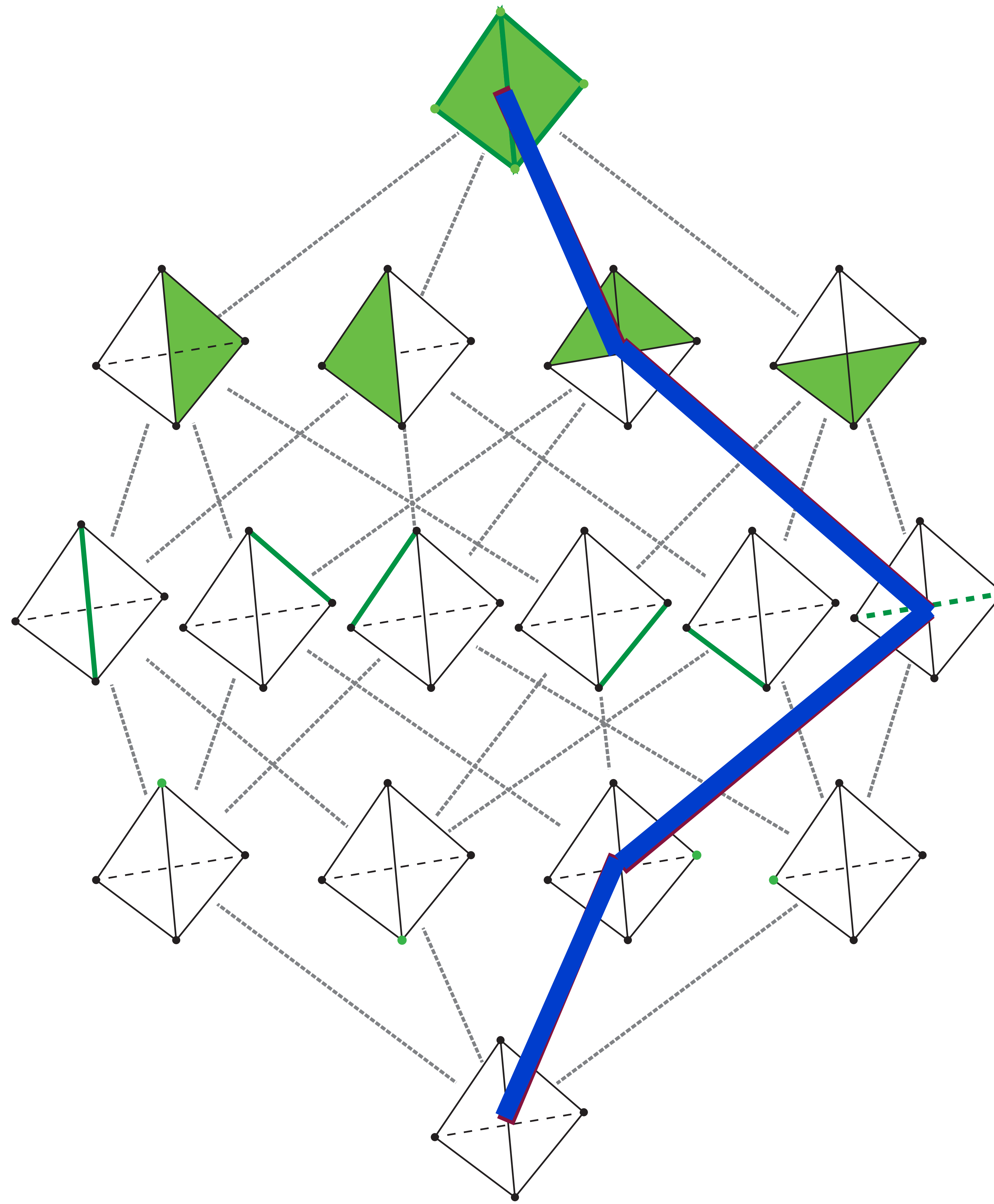

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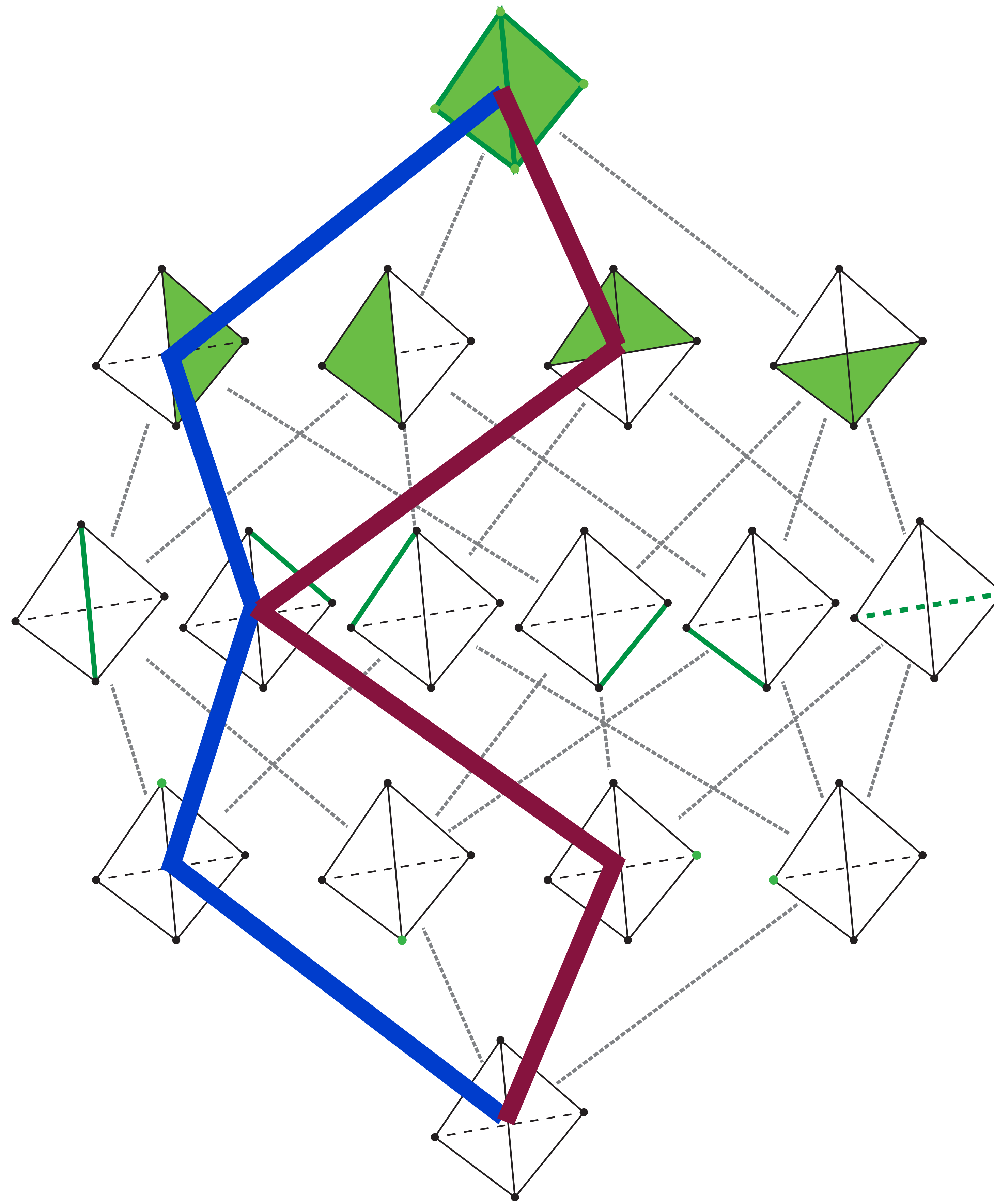

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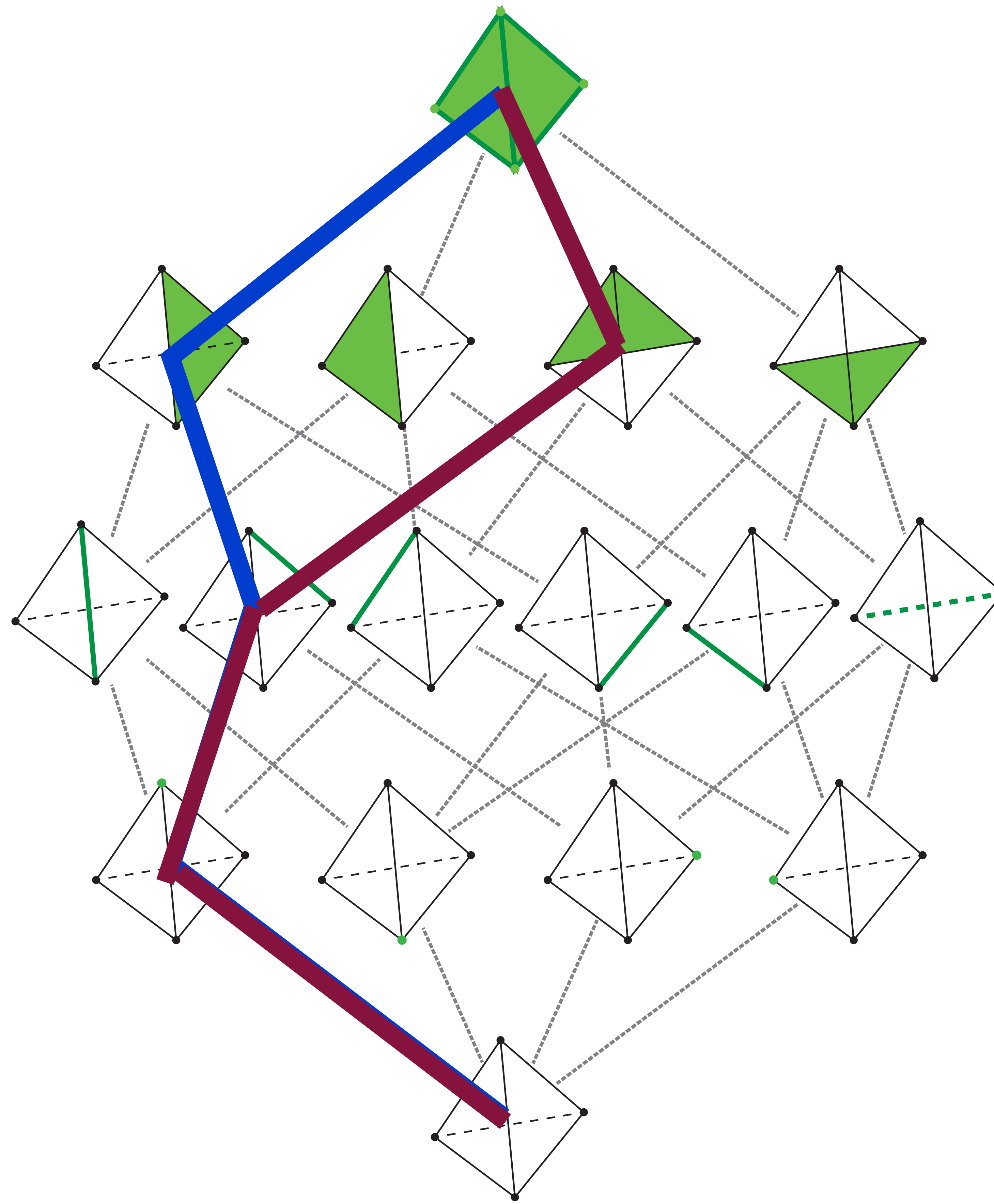

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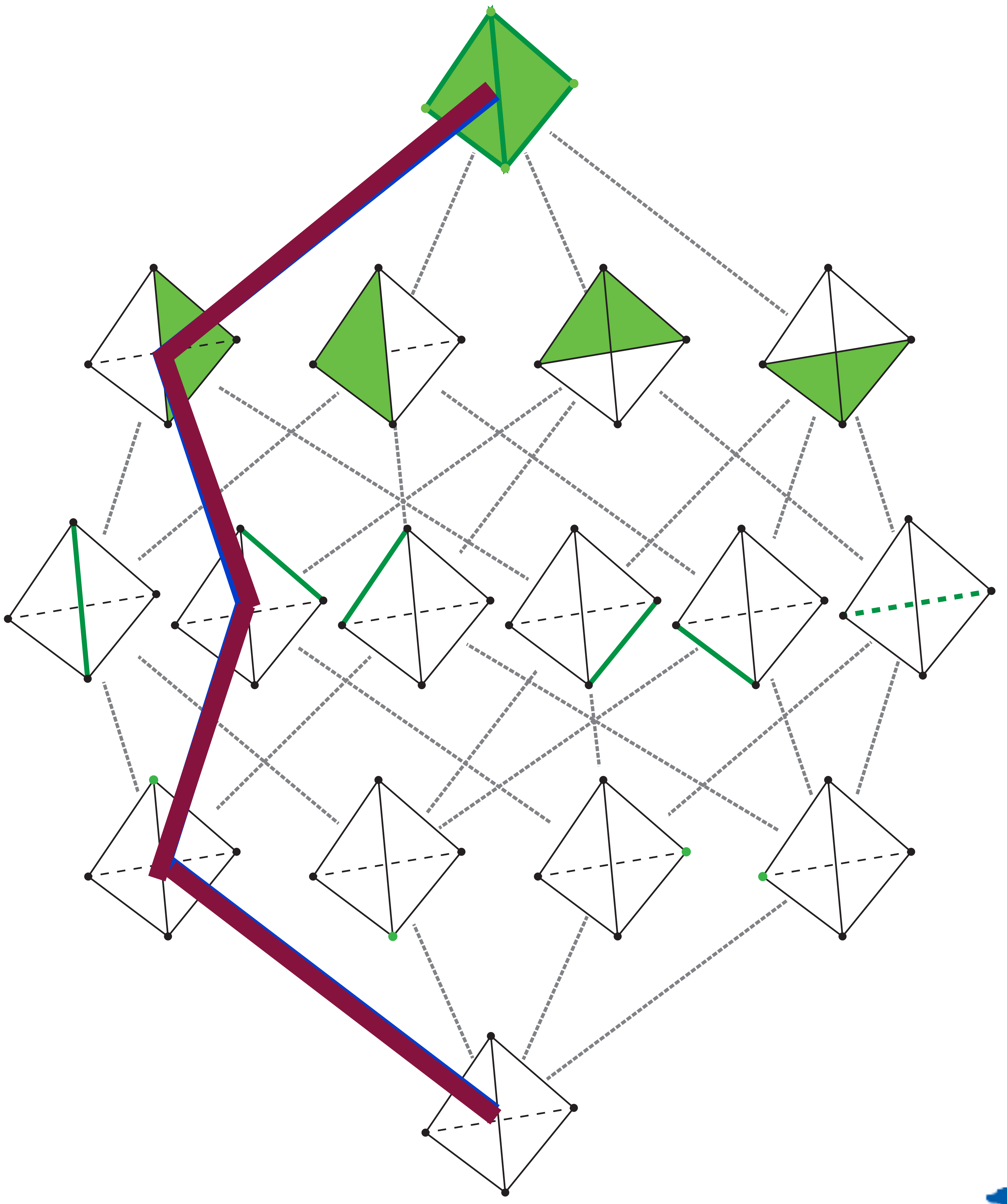
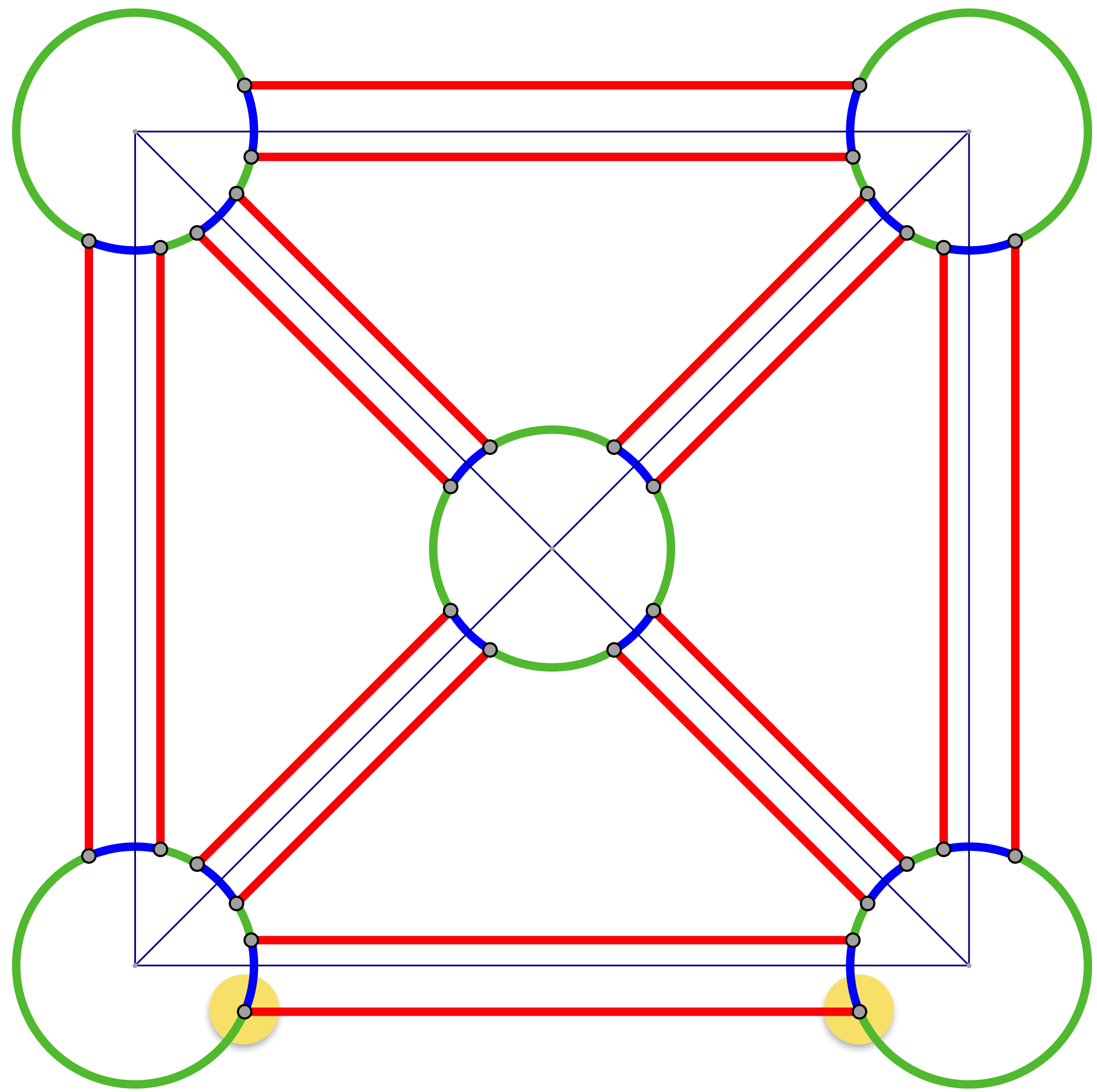


The Flag Graph

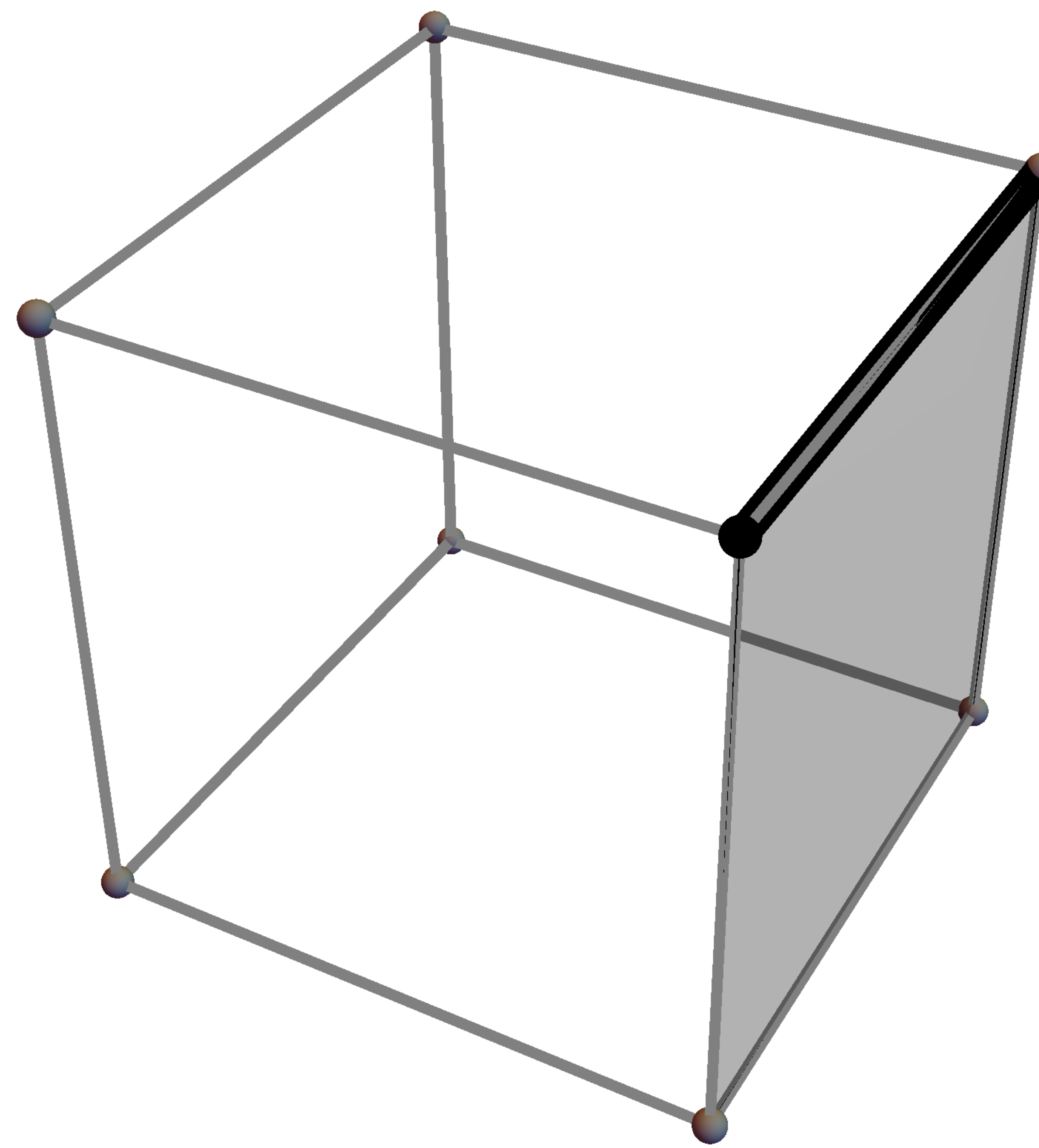
- Nodes: *Flags* = maximal chains
- Edges: Connected *adjacent* flags, i.e., flags differing by exactly one element.



The Groups of Abstract Polytopes

The Automorphism Group

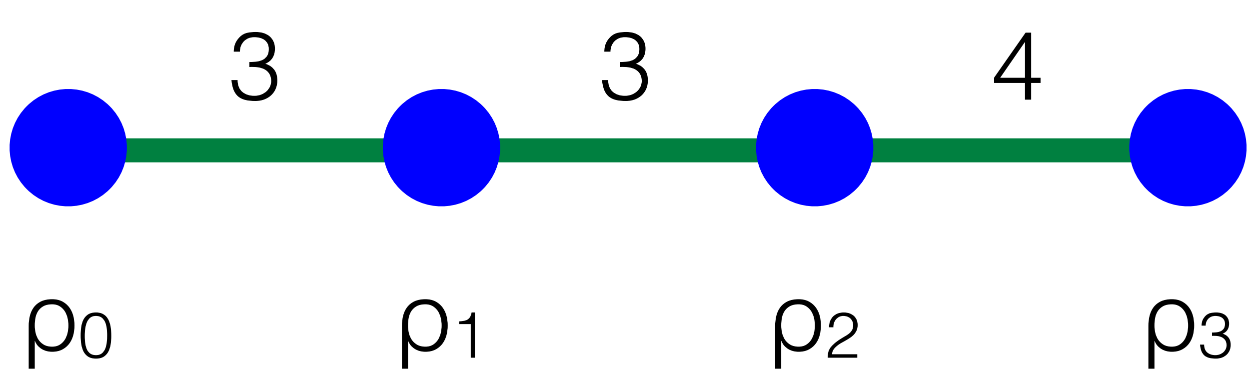
- An *automorphism* of a polytope P is an inclusion preserving permutation of its faces.
- The automorphisms form a group Aut(P), called the *automorphism group* of P.
- A polytope P is *regular* if its automorphism group acts transitively on the flags.
- Induces an automorphism group of the flag graph that preserves the edge colors.



Automorphism Groups of Regular Abstract Polytopes

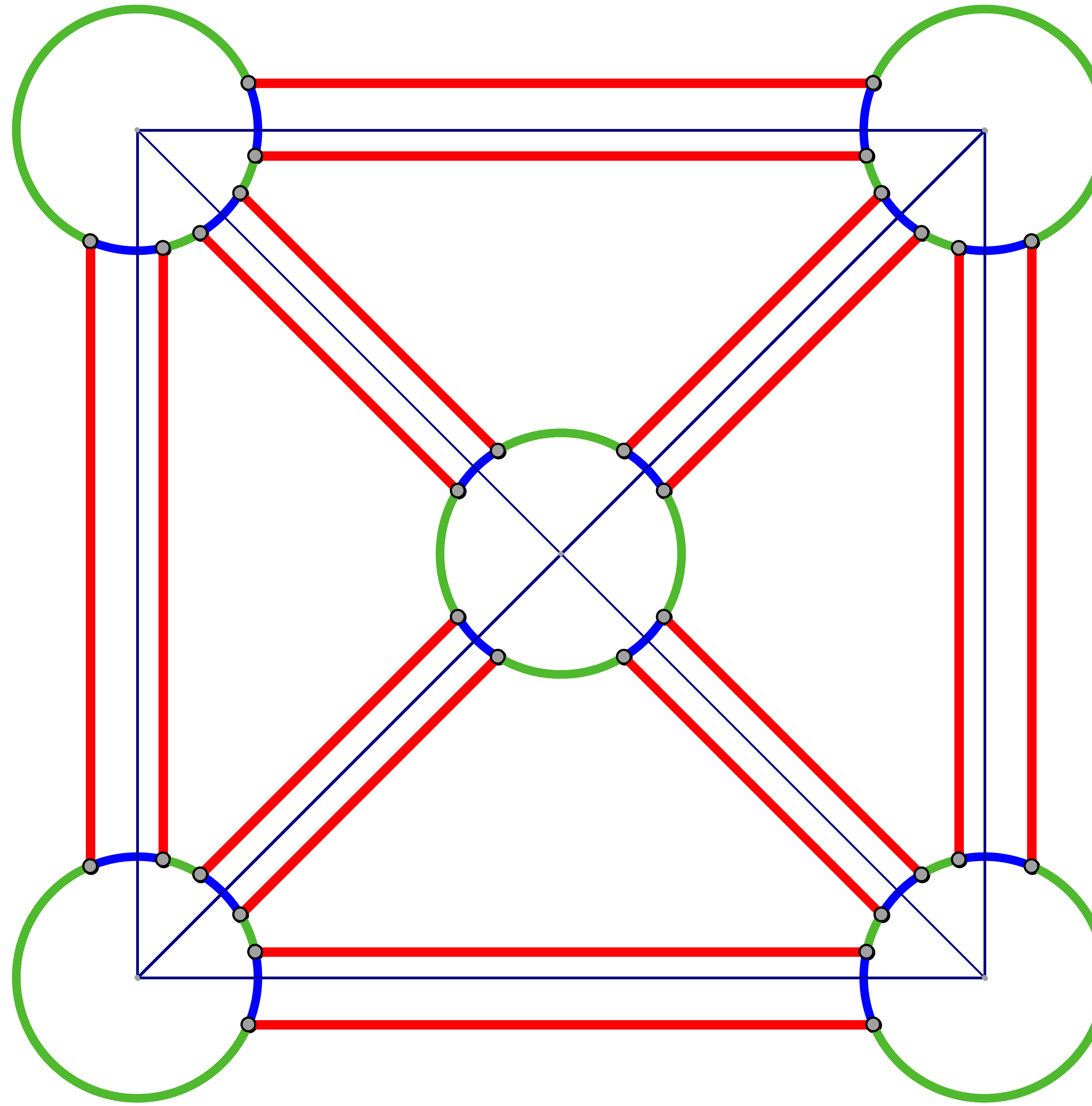
Known as *String C-groups*.

- Equipped with a privileged set of involutory generators.
- Arise as quotients of string Coxeter groups.
- Must satisfy the *intersection condition*.
- Are an example of an *sggi*, or *string group generated by involutions*.



The Connection Group

- A permutation group on the flags determined by flag adjacency.
- Also an sggi.
- Aut(P) and Con(P) distinct subgroups of the symmetric group on the flags.
    - Isomorphic when P regular

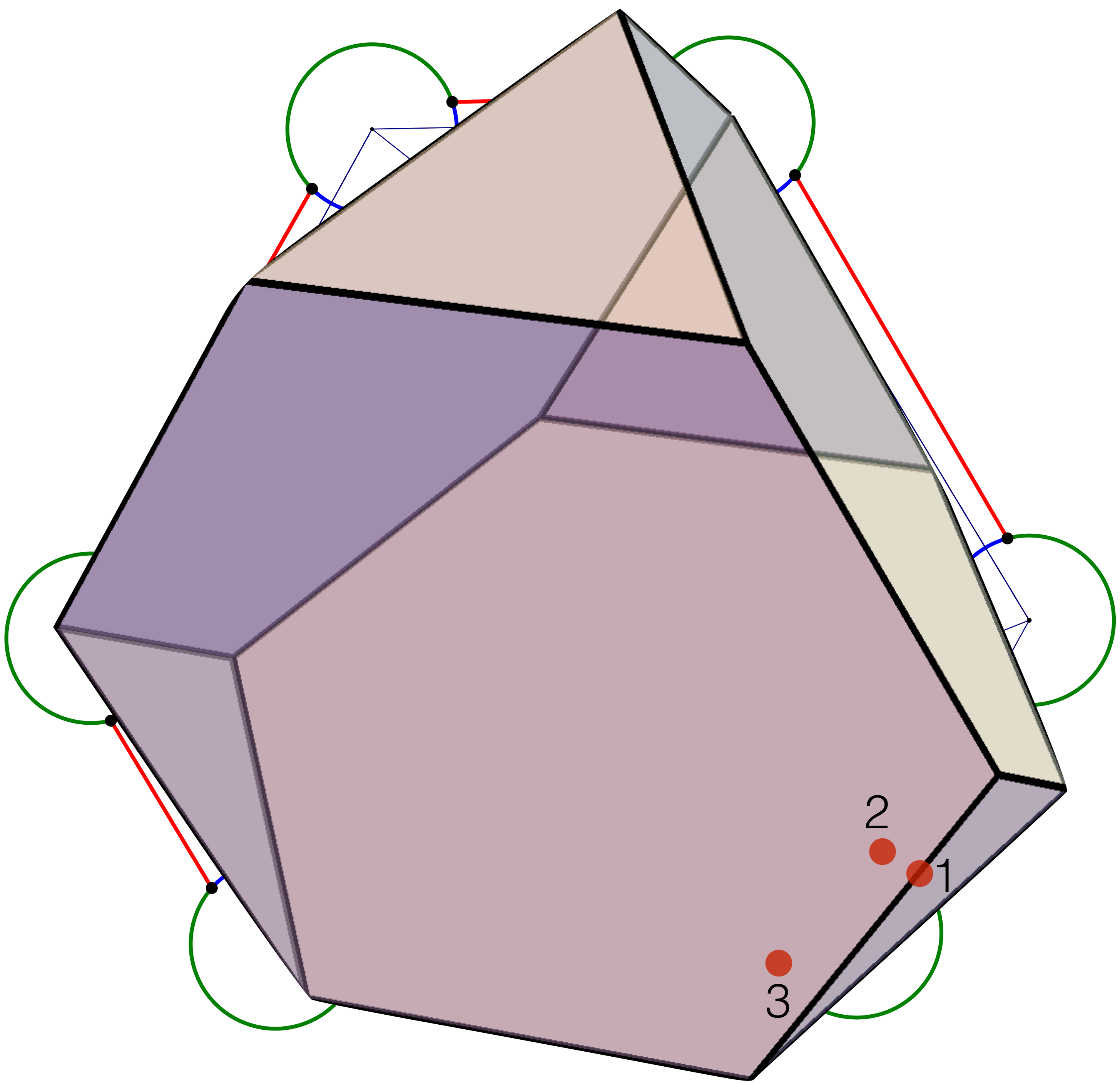


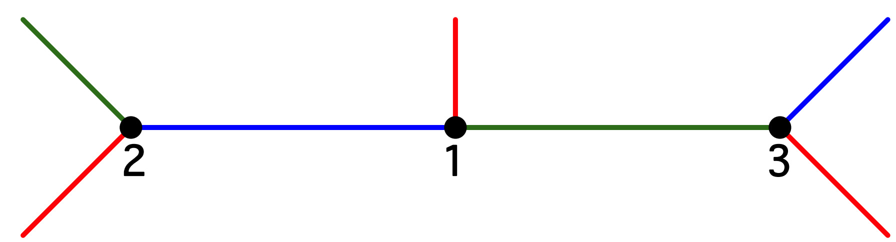
An Alternative Representation for the Connection Group

- k:= # of flag orbits for P
- $\Delta := S_k \ltimes (\mathrm{Aut}(P))^k = S_k \wr_{[k]} \mathrm{Aut}(P)$
- $(\sigma,\alpha)(\tau,\beta)=(\sigma\tau,\phi_\tau(\alpha)\cdot\beta)$ where $\phi_\tau(\alpha)=[\alpha_{\tau(1)},\alpha_{\tau(2)},\ldots,\alpha_{\tau(n)}]$
- Then Con(P) can be embedded in Δ.
- Strictly speaking, we are working with

$$\mathrm{Norm}_{\mathrm{Con}(P)}(\mathrm{Stab}_{\mathrm{Con}(P)}(\Phi))/\mathrm{Stab}_{\mathrm{Con}(P)}(\Phi)$$

Symmetry Type Graph

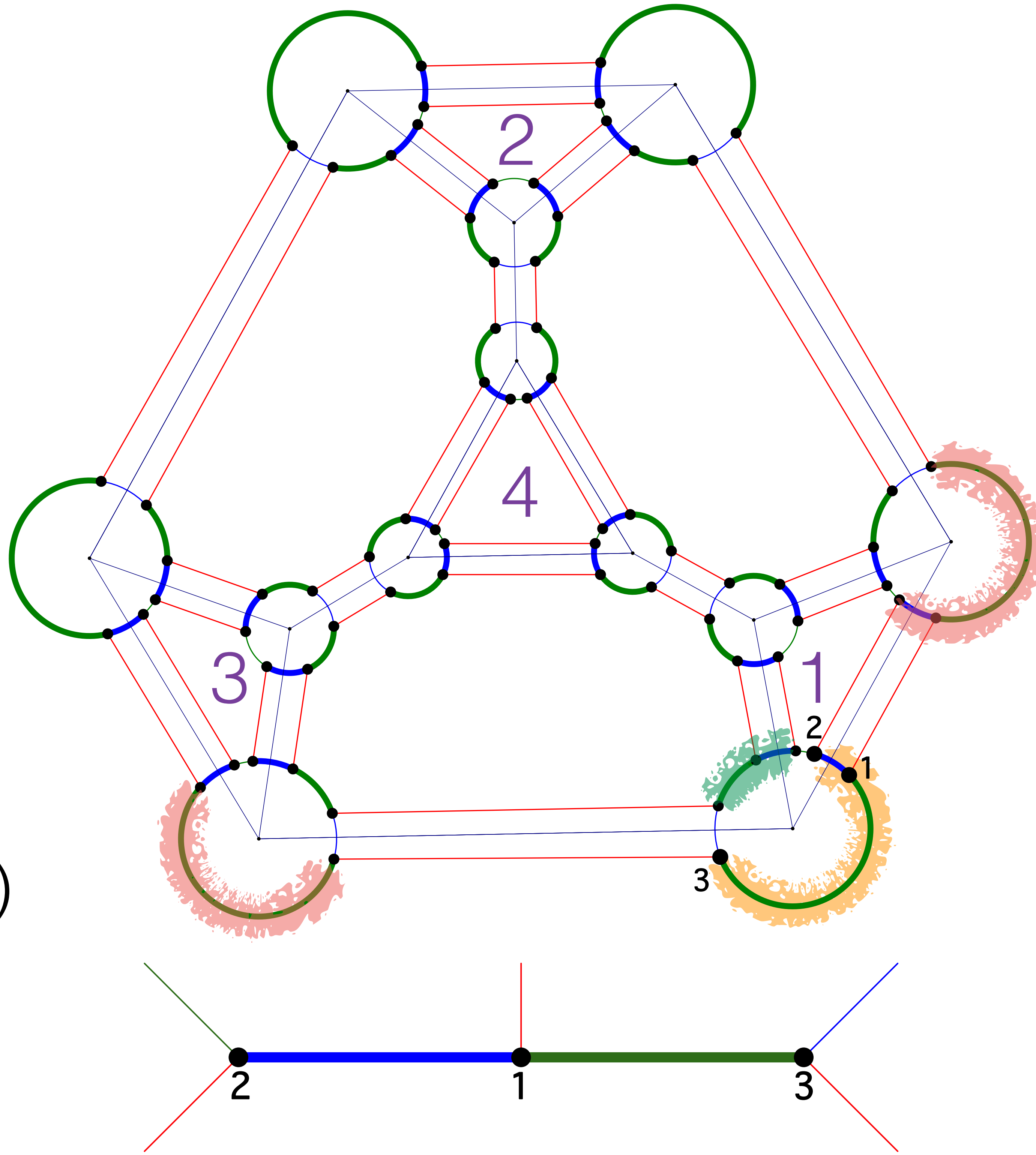




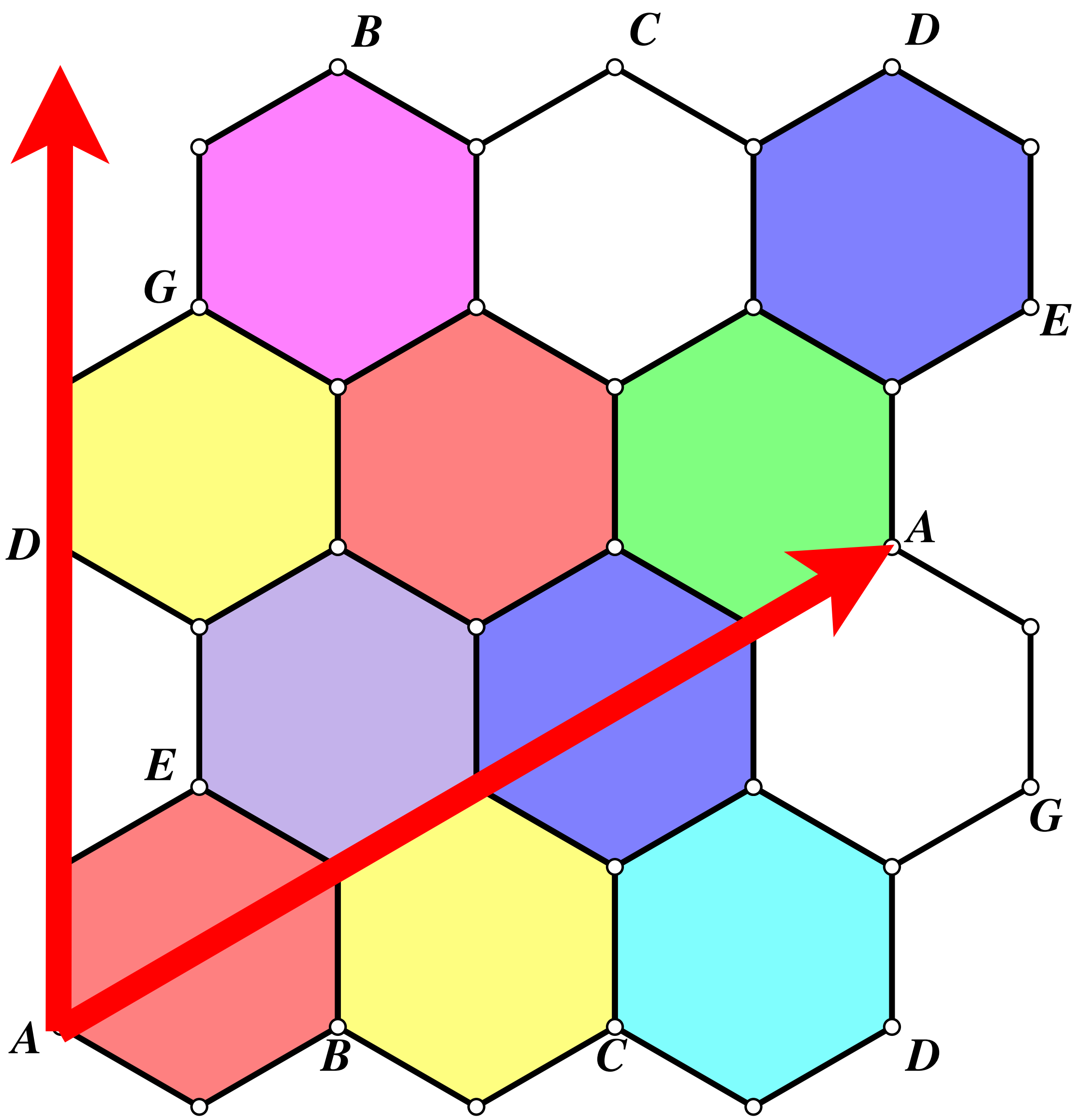
Generating the Subgroup of Δ

- Pick a spanning tree T in the symmetry type graph
- Lifts of T are *chunks*
- Automorphisms in representation relate chunks
- e.g.,
  - $\iota(r_0)$:=((),[(2,3),(2,3),(1,3)])
  - $\iota(r_1)$:=((1,3),[(),(2,4),()])
  - $\iota(r_2)$:=((1,2),[(),(),(2,4)])
  - So $\iota(r_1 r_2)$=

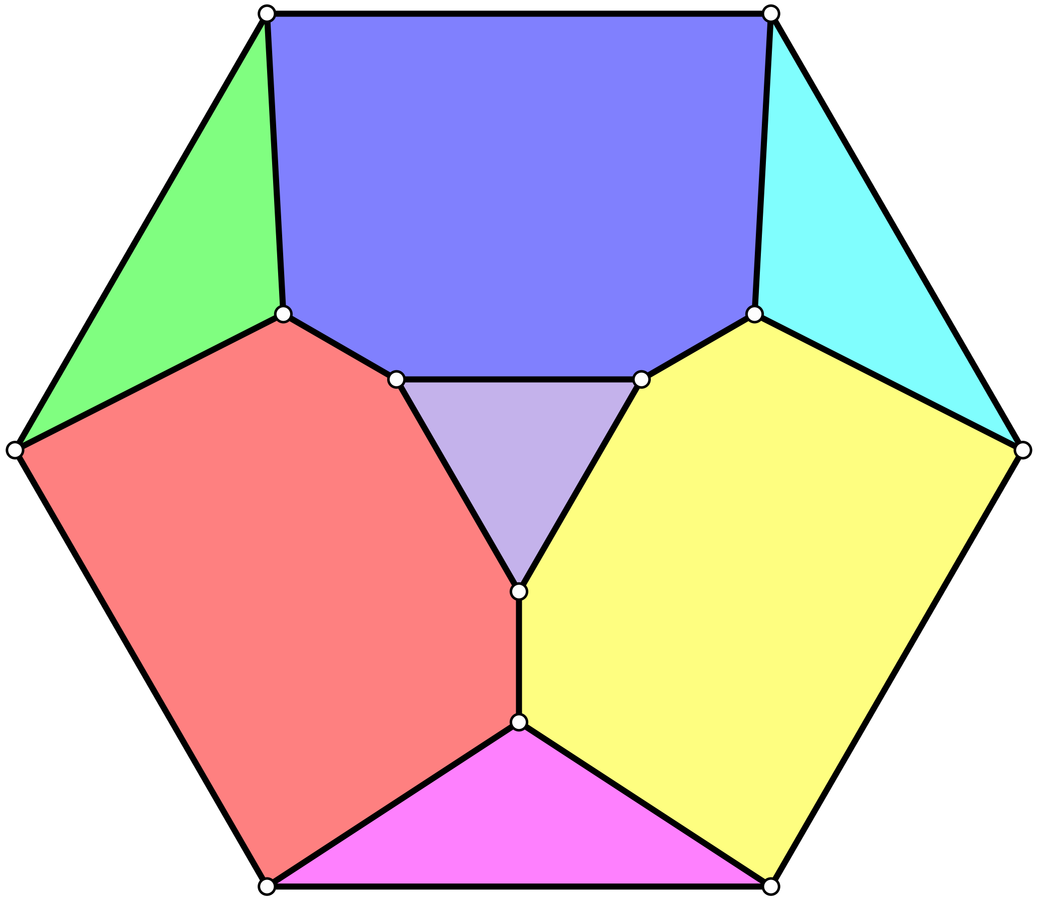
((1,3,2),[(2,4),(),()]·[(),(),(2,4)])
=((1,3,2),[(2,4),(),(2,4)])



A Regular Cover for the Truncated Tetrahedron



$\{6,3\}_{(2,2)}$



(3.6.6)

Minimal Covers

Let Q be an abstract polytope.

A regular abstract polytope P is a *minimal cover* of Q if:

1. P covers Q
2. for all regular polytopes R such that P covers R and R covers Q

then $P = R$.

i.e., if $P \searrow R \searrow Q$, then P minimal $\iff$ $P{=}R$

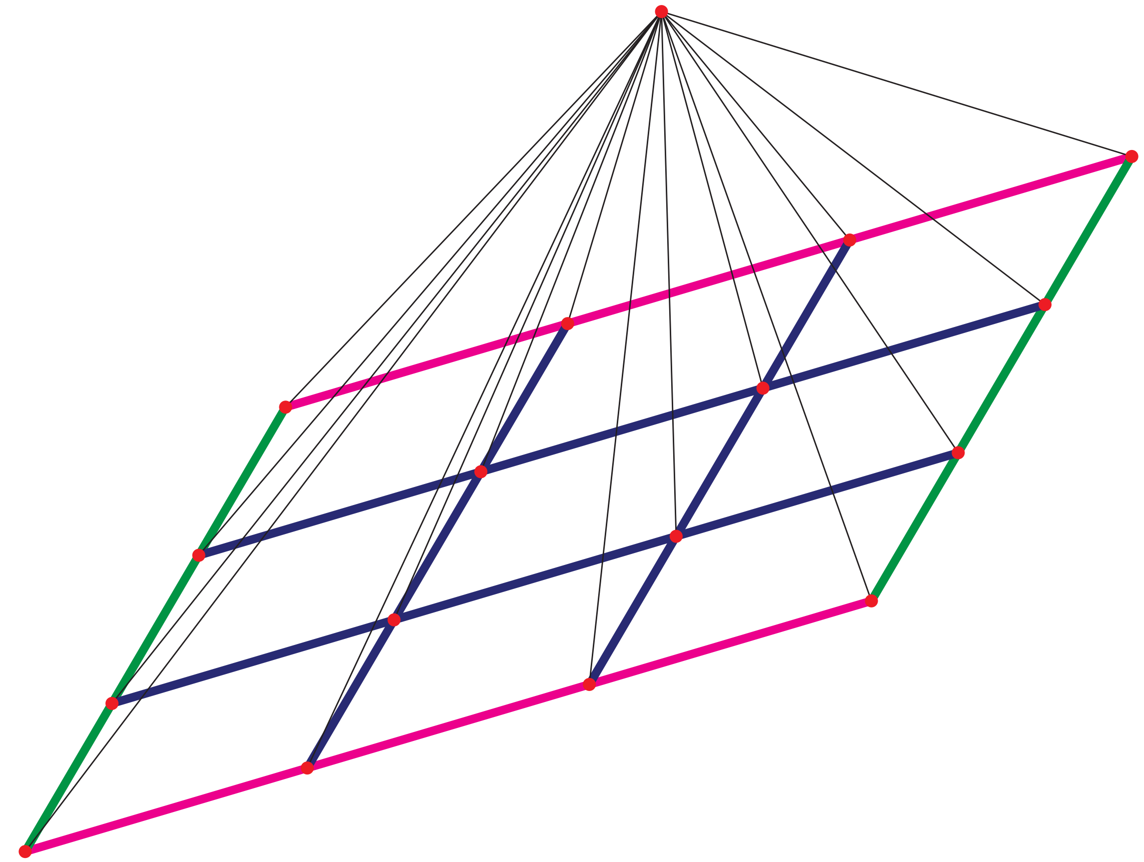
Connection Groups and Minimal Covers

If the **connection group of Q** is a string C-group, then it corresponds to the unique minimal regular cover of Q.

In every studied example, if the **connection group of Q** is *not* a string C-group, then it has more than one minimal regular cover.

$$\begin{tikzcd} P \arrow[dr] \arrow[rr, phantom, "\ncong"] & & R \arrow[dl] \\ & Q & \end{tikzcd}$$

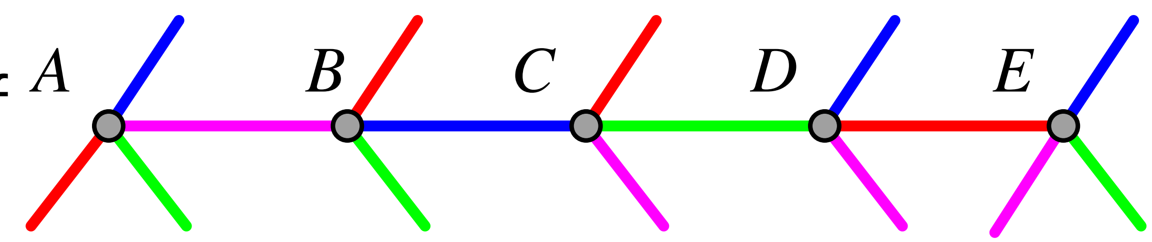
Pyramids Over 3-Tori

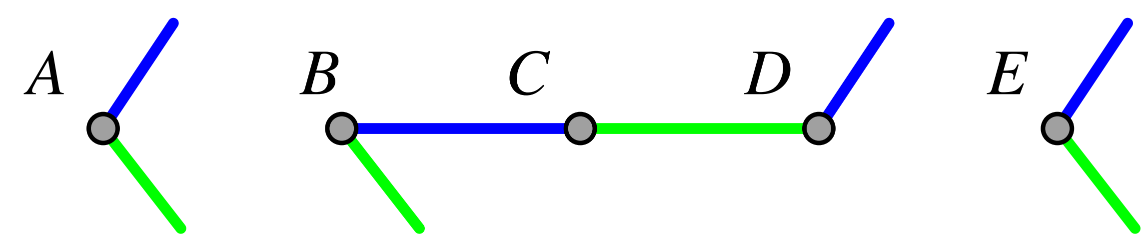


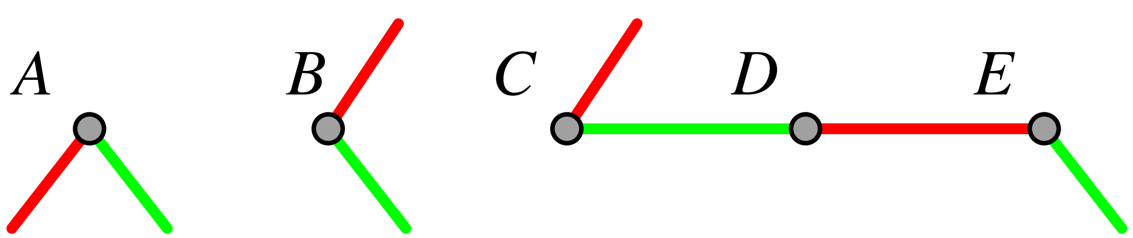
We start with the regular tori of type $\{4,4\}_{(n,0)}$, $\{4,4\}_{(n,n)}$, $\{6,3\}_{(n,0)}$, $\{6,3\}_{(n,0)}$, $\{3,6\}_{(n,0)}$, $\{3,6\}_{(n,n)}$.

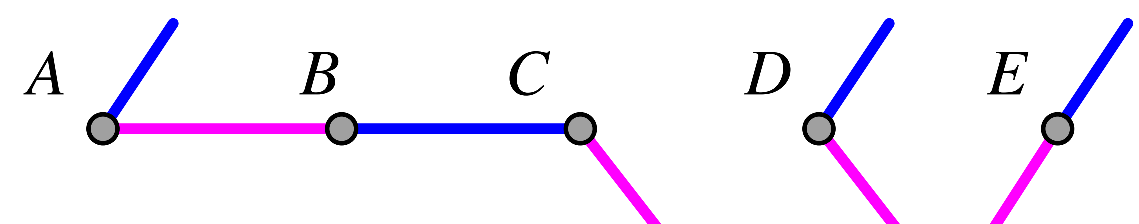
Observations and Results:

1. 5 symmetry classes of flags
2. Elements in the normal closure of $(r_0r_1)^3$, $(r_1r_2)^3$, and $(r_2r_3)^3$ fix all flag orbits.
3. Con(P) is a string C-group except when P=Pyr($\{4,4\}_{(n,0)}$).









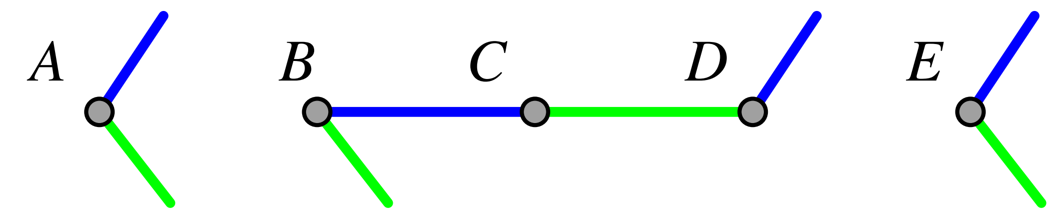
Proving that if $n\geq 3$ is **odd**, then Con(Pyr($\{4,4\}_{(n,0)}$)) is *not* a string C-group

Proposition: Let $S(n)$ be the set of elements of $C(n)$=Con(Pyr($\{4,4\}_{(n,0)}$)) that fix all flag orbits. Then $|S(n)\cap C(n)_0\cap C(n)_3|=4j$ for some $j\in\mathbb{N}$.

Also, $|S(n)\cap C(n)_0\cap C(n)_3|>4$ iff $C(n)_0\cap C(n)_3\neq<r_1,r_2>$.

So it suffices to show there is an element of $C(n)_0\cap C(n)_3$ not in $<r_1,r_2>$.

- The element $m_3=r_1r_0r_2r_1r_0(r_1r_2)^3r_0r_1r_2r_0r_1(r_1r_0)^3$ acts trivially on the pyramidal facets and like a half turn on the base.
- $\iota(m_3)=((),[(\rho_1\rho_2)^2, id, id, id, id])\in S\cap C(n)_3$
- $\iota((r_1r_2)^3)=((),[\rho_1\rho_2, id, id, id, \rho_1\rho_0])$
  - So m_3 is not in $<r_1,r_2>$, since it stabilizes flag orbits and isn't a power or inverse of $(r_1r_2)^3$.



- Pyr($\{4,4\}_{(n,0)}$) is self dual.
  - The dual element of m_3 is $d_3\in C(n)_0$ and

$$\iota(d_3)=((),[id, id, id, id, (\rho_1\rho_2)^2])$$

- Thus $\iota(d_3(\rho_1\rho_2)^6)=$

$((),[id, id, id, id, (\rho_0\rho_1)^2])\cdot((),[(\rho_1\rho_2)^2, id, id, id, (\rho_1\rho_0)^2])$

$=((),[(\rho_1\rho_2)^2, id, id, id, id])=\iota(m_3)\in C(n)_0$.

- So m_3 is in $C(n)_0\cap C(n)_3$, **but not** in $<r_1,r_2>$, as desired.

Connections and Questions

Monson and Schulte, 2014: Found two minimal regular covers of Pyr($\{4,4\}_{(3,0)}$) of order $2^{13}3^{11}5$.

We found a new one of order $2^{16}3^{11}5$ — the unique minimal regular cover of Pyr($\{4,4\}_{(6,0)}$).

- How many more are there?
- What sizes are possible?
- And for Pyr($\{4,4\}_{(2k+1,0)}$)?

Relationship to Connection Group

Does there exist $\mathcal{Q}$ s.t. $\mathrm{Con}(\mathcal{Q})$ is not a string C-group, but the number of minimal regular covers is finite?

Conjecture: *If the connection group of a polytope is not a string C-group, then it has infinitely many minimal regular covers.*

¡Gracias por su atención!

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