Open problems in k-orbit polytopes

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Open problems in polytopes that are neither regular nor chiral (and some open problems even about chiral polytopes)

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Abstract polytopes

A ranked poset $\mathcal{P}$ is an abstract n-polytope if:

1. It has a maximal face of rank n and a minimal face of rank -1,
2. All flags (maximal chains) of $\mathcal{P}$ have $n+2$ faces,
3. It is strongly connected, and
4. It satisfies the Diamond Condition: Whenever $F < H$ with $\text{rank}\, H = \text{rank}\, F + 2$, then there are exactly two faces G_1 and G_2 such that $F < G_i < H$.

Polytopes and flag graphs

The *flag graph* of an abstract n-polytope $\mathcal{P}$ is an n-valent connected graph where:

- Each node corresponds to a flag of $\mathcal{P}$, and
- Flags Φ and Ψ are connected by an edge labeled i if they differ only in their i-face.

(In fact, the flag graph captures all of the information about a polytope!)

A *maniplex* is essentially a graph that almost looks like the flag graph of a polytope. (See *Maniplexes: Part 1* by Steve Wilson, *Polytopality of Maniplexes* by Jorge Garza-Vargas and Isabel Hubard)

Regular and k-orbit polytopes

The automorphism group $\Gamma(\mathcal{P})$ of a polytope $\mathcal{P}$ is the group of color-preserving graph automorphisms of the flag graph.

- If the action is transitive on the flags, then $\mathcal{P}$ is *regular*.
- In general, if there are k orbits on the flags, then $\mathcal{P}$ is a *k-orbit polytope*.
- If $\mathcal{P}$ is 2-orbit, and adjacent flags always lie in different orbits, then $\mathcal{P}$ is chiral.

The algebraic approach

There is a well-known correspondence between regular polytopes and *string C-groups*. Most of what we know about regular polytopes comes from working with groups.

Similarly, there is a correspondence between chiral polytopes and a certain type of group.

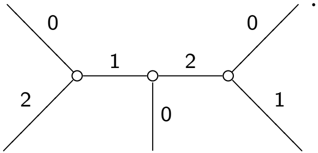
How can we do the same thing for k-orbit polytopes?

Classifying k-orbit polytopes

We classify k-orbit polytopes by their *symmetry type graph*, which is the quotient of the flag graph by the automorphism group. (C., Del Río Francos, Hubard, and Toledo, 2015)

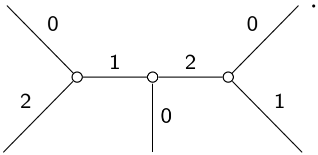
In other words, the nodes of the symmetry type graph correspond to flag-orbits, and the edges show you how you move from one orbit to the next as you walk around in the flag graph.

Example: The symmetry type graph of a triangular prism is:



Automorphism group of a k-orbit polytope

If you know the symmetry type graph of $\mathcal{P}$, you can describe a generating set for the automorphism group.
For $k \geq 3$, the generating sets depend on the choice of base flag and on the choice of a spanning tree of the symmetry type graph.



$$\Gamma(\mathcal{P}, \Phi_1) = \langle \alpha_0,\, \alpha_2,\, \alpha_{1,0,1},\, \alpha_{1,2,1,2,1} \rangle,$$
$$\Gamma(\mathcal{P}, \Phi_2) = \langle \alpha_0,\, \alpha_{1,0,1},\, \alpha_{1,2,1},\, \alpha_{2,1,2} \rangle,$$
$$\Gamma(\mathcal{P}, \Phi_3) = \langle \alpha_0,\, \alpha_1,\, \alpha_{2,1,0,1,2},\, \alpha_{2,1,2,1,2} \rangle,$$

where $\alpha_{i_1,\dots,i_m}$ is the automorphism that sends the base flag Φ_j to $\Phi_j^{i_1\cdots i_m}$.

Building k-orbit polytopes from groups

In order to build a correspondence between a particular class of k-orbit polytope and a class of groups, we would need to know:

1. How can we build a k-orbit polytope or maniplex from a group with a certain set of generators? (Using a coset geometry)
2. What kind of "intersection conditions" ensure that you get a polytope, instead of a maniplex?

(Stay tuned for the next talk!...)

Coverings

We say that $\mathcal{P}$ *covers* $\mathcal{Q}$ if there is a color-preserving graph epimorphism from the flag graph of $\mathcal{P}$ to the flag graph of $\mathcal{Q}$.

Given a k-orbit polytope $\mathcal{Q}$, can we find a small regular polytope $\mathcal{P}$ that covers $\mathcal{Q}$?

Regular covers

- (Hartley, 1999) Every polytope has a regular cover.
- (Monson and Schulte, 2012) Every finite polytope has a finite regular cover.
- (2009-2018) Many papers have studied the minimal regular covers of various convex polytopes. (Berman, Hartley, Mixer, Monson, Oliveros, Pellicer, Williams)
- (Monson, Pellicer, and Williams, 2014) Every 3-polytope has a unique *minimal* regular cover.
- (Monson, Pellicer, and Williams, 2012) There is a 4-polytope (the *Tomotope*) with infinitely many minimal regular covers.

How can we find minimal regular covers?

Connection groups

We can encode the flag graph of an n-polytope using its ~~*monodromy*~~ *connection group* $\mathrm{Con}(\mathcal{P}) = \langle r_0, \ldots, r_{n-1} \rangle$, where r_i is the permutation induced by edges of color i.

It turns out that $\mathrm{Con}(\mathcal{P})$ is the automorphism group of the minimal regular *maniplex* that covers $\mathcal{P}$.

If $\mathrm{Con}(\mathcal{P})$ is a string C-group, then $\mathcal{P}$ has a unique minimal (polytopal) regular cover.

Regular cover questions

1. When is Con($\mathcal{P}$) a string C-group?
2. How can we determine the size and structure of Con($\mathcal{P}$)? (See "Pyramids over 3-Tori" by Pellicer, Williams 2018)
3. Which polytopes have infinitely many minimal regular covers?
4. Is there a polytope with exactly 2 minimal regular covers?

k-orbit convex polytopes

Nicholas Matteo has classified the convex polytopes that are geometrically k-orbit for $k = 2, 3, 4$ and those that are combinatorially 2-orbit.

Theorem (Matteo, 2015)

For each $k \geq 2$, there is a number N_k such that if $d \geq N_k$, then there are no convex d-polytopes that are geometrically k-orbit.

- $N_2 = 4,\ N_3 = 9,\ N_4 = 8$
- $k + 2 \leq N_k \leq 2^{k-3}9$

Can we improve the bounds on N_k? Are there 'holes' in the dimension spectrum for k-orbit polytopes?

k-orbit skeletal polytopes

A *skeletal polytope* consists of a set of points, connected by line segments, and then built inductively by designating certain collections of $(i-1)$-faces to be the i-faces (with some restrictions).

- (Grünbaum, Dress, 1977+) Classification of regular polyhedra in $\mathbb{E}^3$
- (McMullen and Schulte, 1997) New techniques for classifying regular polyhedra in $\mathbb{E}^3$
- (McMullen, 2004) Classification of finite regular d-polytopes and infinite regular $(d+1)$-polytopes in $\mathbb{E}^d$.
- (Schulte, 2004-2005) Classification of chiral 3-polytopes in $\mathbb{E}^3$.
- (Bracho, Hubard, and Pellicer, 2014) There is a finite chiral 4-polytope in $\mathbb{E}^4$.
- (Pellicer, 2017) Classification of infinite chiral 4-polytopes in $\mathbb{E}^3$.

k-orbit skeletal polytopes

Problems:

- Classify the finite chiral d-polytopes and infinite chiral $(d+1)$-polytopes in $\mathbb{E}^d$.
- Classify the k-orbit polyhedra in $\mathbb{E}^3$.

Combinatorial vs. geometric symmetry

In 2011, Cutler and Schulte classified the "regular polyhedra of index 2": skeletal polyhedra that are combinatorially regular and geometrically 2-orbit.

In general: what are the combinatorially regular polytopes in $\mathbb{E}^d$ with k geometric flag-orbits?

N more open questions

What is the smallest
abstract polytope / skeletal polytope / convex polytope
of rank n with
k_1 combinatorial orbits and/or k_2 geometric orbits?

Combinatorial constructions

Is there a combinatorial version of...

- The Quotient Criterion?
- The Flat Amalgamation Property?
- Universal Extensions?

Thank you!