QUANTIZATION OF EINSTEIN-ROSEN WAVES COUPLED WITH MATTER

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Classical treatment

① The model

② Symmetry reduction

③ Hamiltonian formalism

④ Equations of motion

Quantization

⑤ Canonical quantization

⑥ Two point function

⑦ Position space interpretation

- ✦ Newton-Wigner states
- ✦ Radial wave function

⑧ Conclusions

ER waves + massless scalar field

Solutions of General Relativity coupled to a massless scalar field with two commuting, spacelike, hypersurface orthogonal Killing vector fields (one translational and one rotational). Twist: $\omega_a = 0$

$$
{}^4S({}^{(4)}g_{ab}, \Phi_s) = \frac{1}{16\pi G_N} \int_{\mathcal{M}\times Z} |{}^{(4)}g|^{1/2} \Big[{}^{(4)}\mathbf{R} - \frac{1}{2}{}^{(4)}g^{ab}\nabla_a\Phi_s\nabla_b\Phi_s \Big] + \frac{1}{8\pi G_N} \int_{\partial(\mathcal{M}\times Z)} (|{}^{(3)}h|^{1/2}\ {}^{(3)}K - |{}^{(3)}h^0|^{1/2}\ {}^{(3)}K^0)
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\end{aligned}
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- Infinite number of d.o.f. and radial diff. invariance
- Exact quantization (Backreaction)
- External field to study the quantum geometry

Symmetry reduction

- ❖ Geroch reduction + conformal transformation of the metric
- ❖ Equivalent action in 2+1 dimensions:

$$
{}^{3}S(g_{ab},\phi_g,\phi_s) = \frac{1}{16\pi G_3}\int_{\mathcal{M}} |g|^{1/2}\Big[\,{}^{(3)}\mathbf{R} - \frac{1}{2} g^{ab}\nabla_a\phi_g\nabla_b\phi_g - \frac{1}{2} g^{ab}\nabla_a\phi_s\nabla_b\phi_s\Big] + \frac{1}{8\pi G_3}\int_{\partial\mathcal{M}} \left(|h|^{1/2}\mathbf{K} - |h^0|^{1/2}K^0\right)
$$

$$
\phi_g = \log({}^4 g_{ab}\,\xi^a\,\xi^b)
$$

- ϕ_g encodes the gravitational degrees of freedom
- ϕ_g-term and ϕ_s-term have the same form
- ϕ_g and ϕ_s are coupled through the metric

Hamiltonian Formalism

[Ashtekar and Pierri, *J. Math. Phys.* **37**, 6250 (1996)]

❖ Solutions asymptotically flat (in 2+1)

❖ Preferred foliation (t^a Killing at infinity)

❖ Gauge fixing + Solve the constraints

❖ Reduced phase space: $\phi_g(r)\,,\phi_s(r)\,,p_g(r)\,,p_s(r)$

$$H = \frac{1}{4G_3}(1 - e^{-H_0/2})$$

$$H_0 = \sum_{i=s,g} \frac{1}{2} \int_0^\infty \mathrm{d}\rho \, \frac{1}{\rho} \Big((8G_3 p_i)^2 + \rho^2 \phi_i'^{\,2} \Big)$$

- H_0 is the Hamiltonian of two free, axially symmetric scalar fields in 2+1 dimensions

Equations of motion

$$\dot{\phi}_i = \frac{\delta H}{\delta p_i} = e^{-H_0/2}\frac{(8G_3 p_i)}{r}$$

$$\dot{p}_i = -\frac{\delta H}{\delta \phi_i} = \frac{1}{8G_3}e^{-H_0/2}(r\phi_i')'$$

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- For every solution, H_0 is a constant of motion
- Solution dependent time variable: $T = e^{-H_0/2}\, t$

$$\partial_T^2 \phi_g - {\phi_g}'' - \frac{1}{r}{\phi_g}' = 0$$

$$\partial_T^2 \phi_s - {\phi_s}'' - \frac{1}{r}{\phi_s}' = 0$$

- Equations for two free massless, axially symmetric scalar fields in 2+1 dimensions

Canonical quantization

Canonical quantization

- Two Fock Spaces: $\mathcal{F}_g\,, \mathcal{F}_s$
- Hilbert Space: $\mathcal{H} = \mathcal{F}_g \otimes \mathcal{F}_s$
- Creation and annihilation operators: $[\hat{a}_{g,s}(k), \hat{a}^\dagger_{g,s}(q)] = \delta(k,q)$

$$\hat{A}^\dagger_g(k) \equiv \hat{a}^\dagger_g(k) \otimes \mathbb{I}_s \qquad \hat{A}^\dagger_s(k) \equiv \mathbb{I}_g \otimes \hat{a}^\dagger_s(k)$$

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$$\hat{\phi}_{g,s}(R) = \sqrt{4G_3\hbar} \int_0^\infty J_0(Rk) \, [\hat{A}_{g,s}(k) + \hat{A}^\dagger_{g,s}(k)] \, \mathrm{d}k$$

$$\hat{p}_{g,s}(R) = \frac{iR}{2} \sqrt{\frac{\hbar}{4G_3}} \int_0^\infty k J_0(Rk) \, [\hat{A}^\dagger_{g,s}(k) - \hat{A}_{g,s}(k)] \, \mathrm{d}k$$

$$[\hat{\phi}_{g,s}(R_1), \hat{p}_{g,s}(R_2)] = i\hbar\delta(R_1, R_2)$$

- Vacuum state: $|\Omega\rangle = |0\rangle_g \otimes |0\rangle_s$
- One particle states: $|k\rangle_{g,s} \equiv \hat{A}^\dagger_{g,s}(k)|\Omega\rangle$

Quantum Hamiltonian:

$$\hat{H} = \frac{1}{4G_3}\left[1 - \exp\left(-4G_3\hbar\int_0^\infty k\,[\hat{A}_g^\dagger(k)\hat{A}_g(k) + \hat{A}_s^\dagger(k)\hat{A}_s(k)]\,\mathrm{d}k\right)\right]$$

- Non linear and bounded function of the sum of two free Hamiltonians
- It is an observable of the system (energy)
- The state that most closely resembles Minkowski metric is $|\Omega\rangle$

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Evolution operator:

$$\hat{U}(t,t_0) = \exp\left(-\frac{i(t-t_0)}{\hbar}\hat{H}\right) = \exp\left(-\frac{i(t-t_0)}{4G_3\hbar}\left[1 - e^{-4G_3\hbar(\hat{H}_0^g + \hat{H}_0^s)}\right]\right)$$

- There is no conversion of quanta of one type into the other
- There is no creation nor destruction of particles with the evolution
- It defines the S-matrix of the system
- It is an interacting theory, but the solucion is exact. Non perturbative.
- Length scale of the system: $4G_3\hbar \equiv 4G$. “Planck length”

Two point function

❖ Interpretation of propagation amplitudes from one spacetime event to another.

- Adimensional variables:

$$\rho_1 = \frac{R_1}{4G} \qquad \rho_2 = \frac{R_2}{4G} \qquad \tau = \frac{t_2 - t_1}{4G} \qquad q = 4Gk$$

$$\langle \Omega | \hat{\phi}_{s,g}(R_2, t_2) \hat{\phi}_{s,g}(R_1, t_1) | \Omega \rangle = \int_0^\infty J_0(\rho_1 q) J_0(\rho_2 q) \exp[-i\tau(1 - e^{-q})] \, \mathrm{d}q$$

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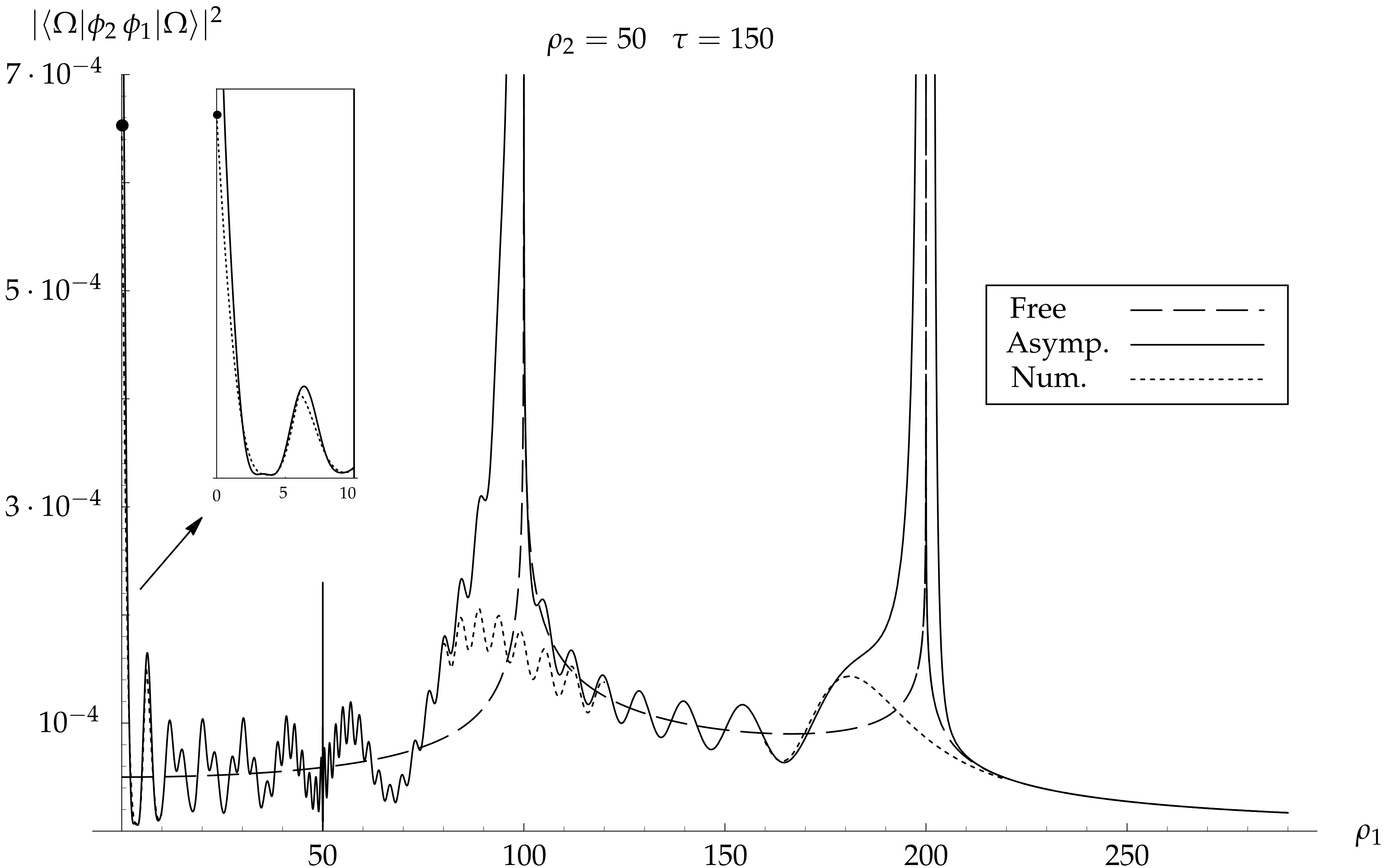
- We extract information using asymptotic techniques



✦ Large probability to find the particle near the axis

✦ Gravitational effect

Position space interpretation

❖ Newton-Wigner states:

- Orthonormal basis that mimics the ordinary position eigenstates:

$$|\psi\rangle = \int_0^\infty dR\, \psi(R)|R\rangle \qquad \langle R|\psi\rangle = \psi(R) \qquad \int_0^\infty dR|\psi(R)|^2 = 1$$

where $|R\rangle$ are the "Newton-Wigner" states:

$$|R\rangle = \int_0^\infty dk\sqrt{kR}\, J_0(kR)\, |k\rangle \qquad \langle R|R'\rangle = \delta(R - R')$$

$J_0(kR)$ is a solution of the radial part of the 2D Schrödinger equation with zero angular momentum. Orthogonality condition implies the factor $\sqrt{kR}$.

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- Propagator:

$$\langle R|U(t)|r\rangle = \sqrt{Rr}\int_0^\infty dk\,kJ_0(rk)J_0(Rk)e^{-itE(k)}$$

$$E(k) = \frac{1}{4G}(1-e^{-4Gk})$$

❖ Wave function and its evolution:

$$\psi(R,t) = \langle R|U(t)|\psi\rangle = \int_0^\infty dr\,\psi(r)\langle R|U(t)|r\rangle$$

- Specific choice:

$$\psi(r) = \sqrt{\frac{2r}{r_2^2 - r_1^2}}\,\chi_{[r_1,r_2]}(r)$$

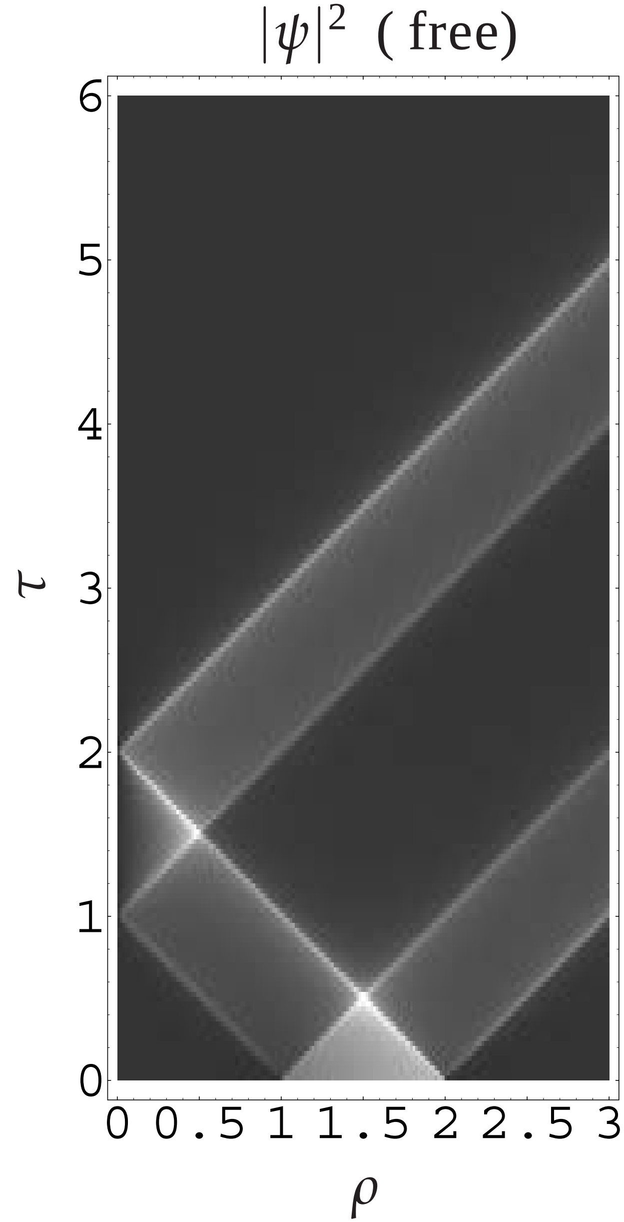
- We obtain for the wave function:

$$\psi(R,t) = \sqrt{\frac{2R}{r_2^2 - r_1^2}}\int_0^\infty dk J_0(kR)[r_2 J_1(kr_2) - r_1 J_1(kr_1)]e^{-itE(k)}$$

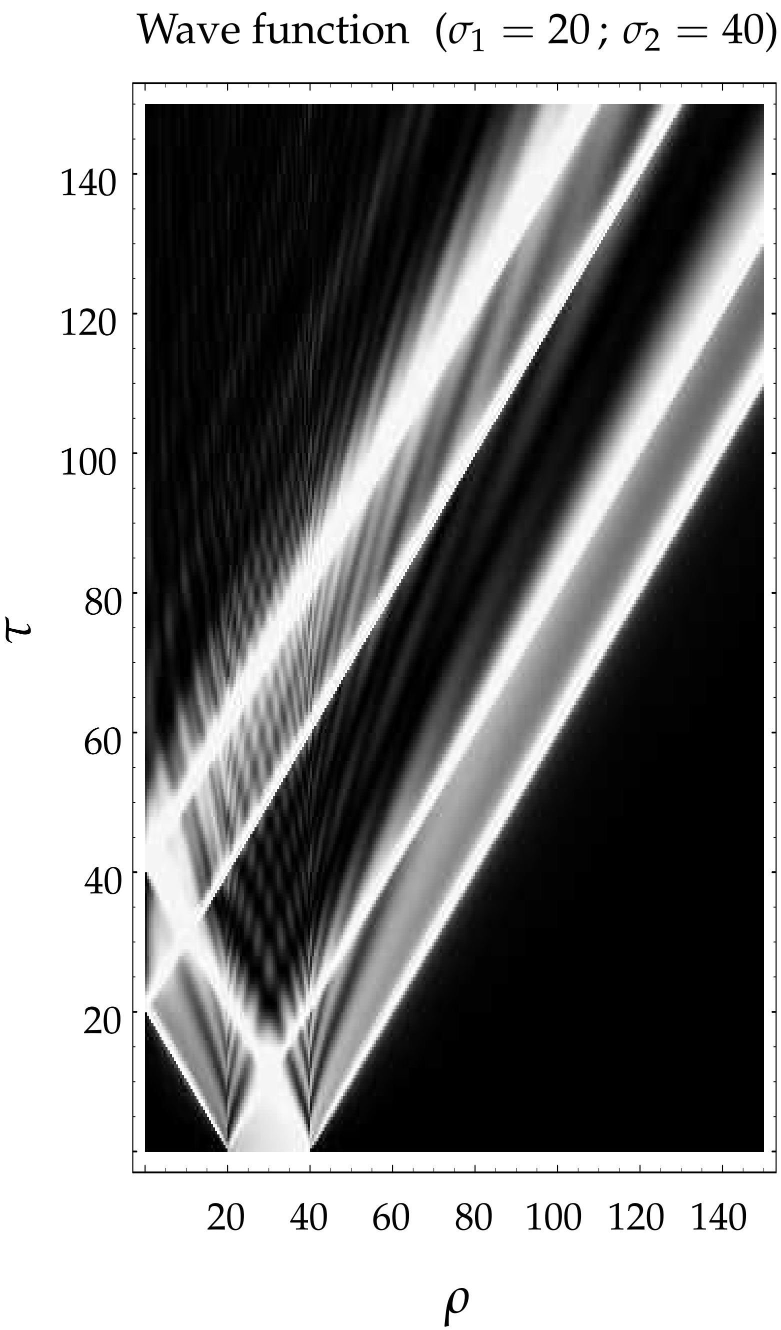
$$\rho \equiv \frac{R}{4G} \qquad \sigma_{1,2} \equiv \frac{r_{1,2}}{4G} \qquad \tau \equiv \frac{t}{4G} \qquad q \equiv 4Gk$$

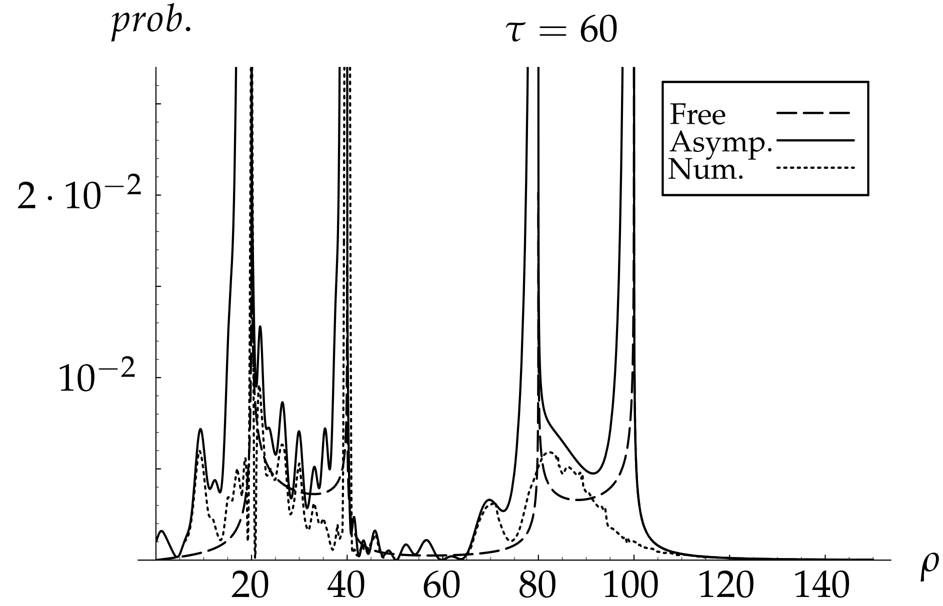
$$\psi(\rho,\tau) = \sqrt{\frac{2\rho}{4G(\sigma_2^2 - \sigma_1^2)}}\,e^{-i\tau}\int_0^\infty dq J_0(q\rho)[\sigma_2 J_1(q\sigma_2) - \sigma_1 J_1(q\sigma_1)]e^{i\tau e^{-q}}$$

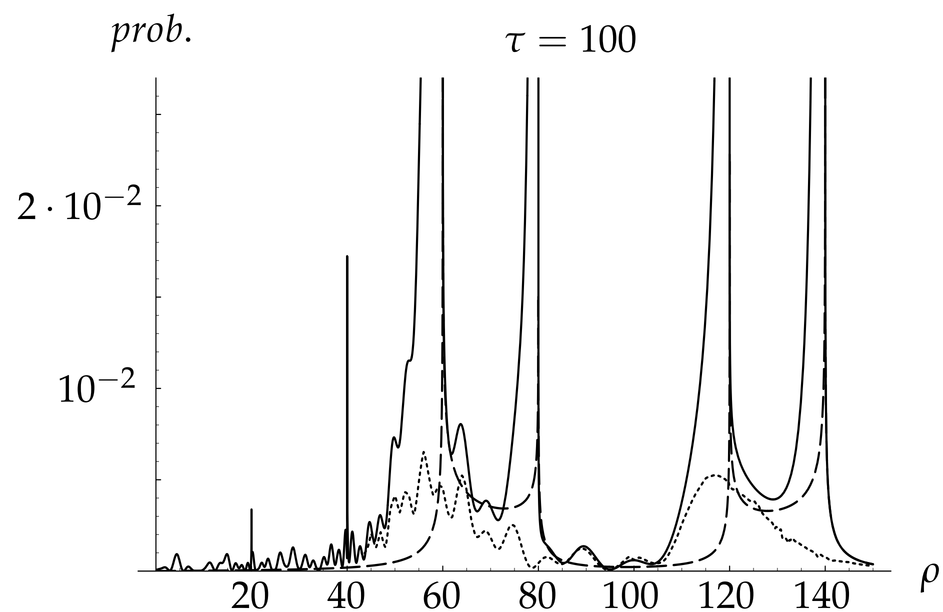
Free wave function



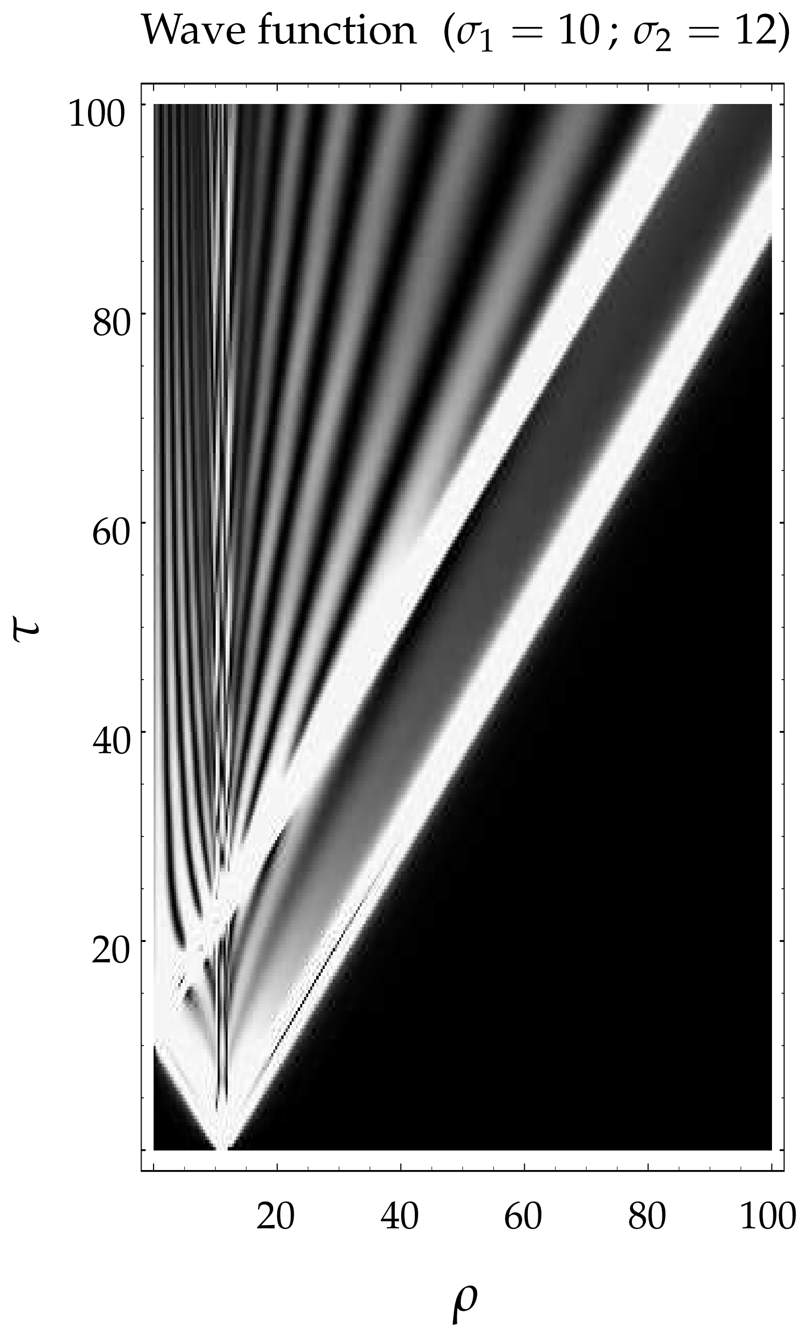
Wide support → "classical" behavior

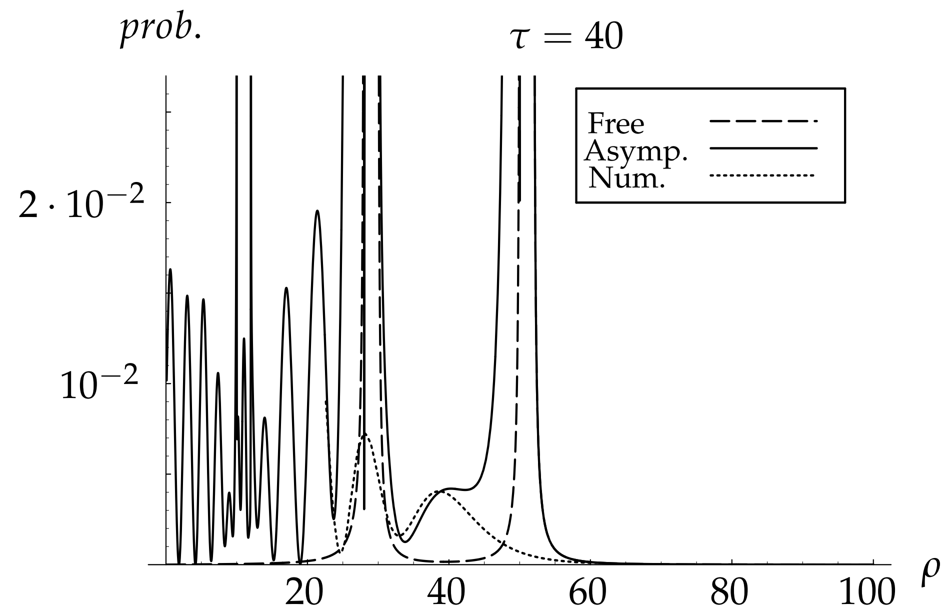


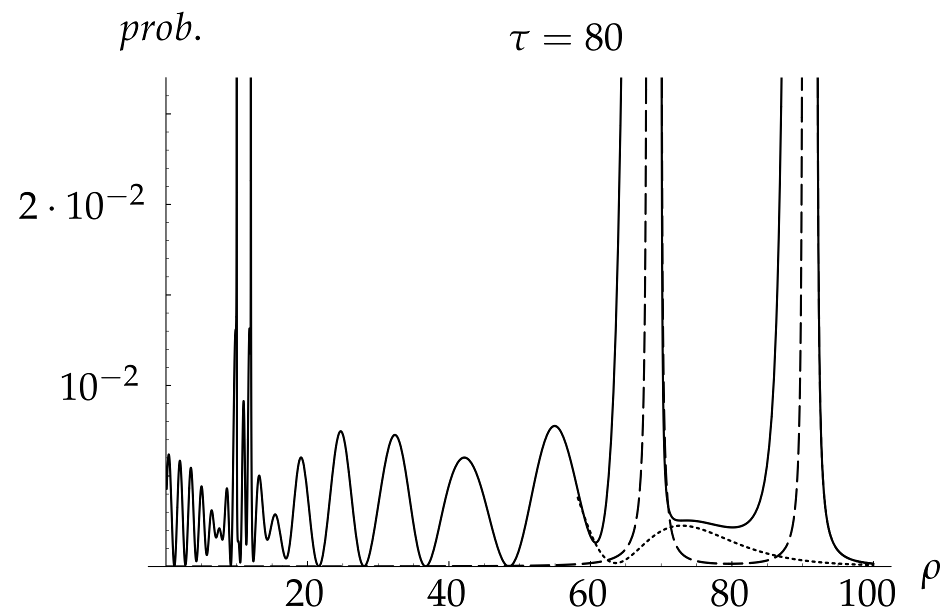




Narrow support → quantum-gravity effects







Conclusions

❖ Einstein-Rosen Waves

- ✦ Genuine field theory
- ✦ Radial diff. invariance
- ✦ Can be solved exactly
- ✦ Rich enough to display interesting behavior

❖ Adding matter

- ✦ Quantum fields and their particle-like excitations
- ✦ Obtain information about the metric in an operational way
- ✦ Try to see in which regime a classical description arises
- ✦ External probe of quantum geometry

❖ Physical effects

- ✦ Interesting physical effects at the symmetry axis
- ✦ Persistence of the amplitudes in the initial support
- ✦ Geodesics (null) of an emergente metric
- ✦ …

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the end